SHIGA UNIVERSITY

CRR WORKING PAPER SERIES　B

Working Paper No. B-2

General Equilibrium Analysis in Security Markets with Infinite Dimensional Martingale Generator

Koji Kusuda

October 2004

Center for Risk Research
Faculty of Economics
SHIGA UNIVERSITY

1-1-1 BANBA, HIKONE,
SHIGA 522-8522, JAPAN

CRR WORKING PAPER SERIES B

Working Paper No. B-2

General Equilibrium Analysis in Security Markets with Infinite Dimensional Martingale Generator

Koji Kusuda

October 2004

Center for Risk Research
Faculty of Economics
SHIGA UNIVERSITY

1-1-1 BANBA, HIKONE,
SHIGA 522-8522, JAPAN

General Equilibrium Analysis in Security Markets with Infinite Dimensional Martingale Generator

by

Koji Kusuda
Faculty of Economics
Shiga University
1-1-1 Banba
Hikone, Shiga 522-8522
kusuda@biwako.shiga-u.ac.jp

October 2004

Abstract. Jump-diffusion security market models with infinite dimensional martingale generator have been intensively studied in Finance and Financial Economics. Recently, the author's companion paper (Kusuda [35]) has shown that a generalized security market equilibrium in an "approximately complete security market" (Björk *et al.* [9]) economy with infinite dimensional martingale generator can be identified with an Arrow-Debreu equilibrium in the corresponding Arrow-Debreu economy. This paper presents (1) a sufficient condition for the existence of the Arrow-Debreu equilibria in the case of stochastic differential utilities, and (2) sufficient conditions for the existence, uniqueness, and local uniqueness of Arrow-Debreu equilibria in the case of time additive utilities, in the Arrow-Debreu economy.

Keywords and Phrases: Approximately complete markets, Approximate security market equilibrium, Arrow-Debreu equilibrium, General equilibrium analysis, Infinite dimensional martingale generator, Jump-diffusion, Stochastic differential utility.

JEL Classification Numbers: C62, D51, G10.

This paper is a revision and expansion of each part of my working paper (Kusuda [31]) and Chapter 3 in my Ph.D. dissertation (Kusuda [33]) at the Department of Economics, University of Minnesota. I would like to thank my adviser Professor Jan Werner for his invaluable encouragement and advice. I am grateful for comments of participants of presentations at University of Minnesota, Japanese Economic Association Fall 2002 Conference, Institute of Economic Research, Hitotsubashi University, Institute for Advanced Studies, Vienna, and Shiga University.

1. Introduction

Jump-diffusion security market models have been intensively studied in Finance and Financial Economics, and in particular, in the context of CAPM[1], option pricing[2], and portfolio choice[3]. In most of jump-diffusion security market models, the jump magnitude is specified as a continuously distributed random variable at each jump time. In this case, the *dimensionality of martingale generator* in the markets[4], which can be interpreted as "the number of sources of uncertainty," is uncountably infinite, and no finite set of traded securities can complete the markets. While many equilibrium analyses have been conducted in security market economy with finite dimensional martingale generator,[5] in security market economy with infinite dimensional martingale generator, no equilibrium analysis had been conducted until recently. The author's companion paper (Kusuda [35]) has shown that a generalized security market equilibrium in an "approximately complete security market" (Björk *et al.* [9]) economy can be identified with an Arrow-Debreu equilibrium in the corresponding Arrow-Debreu economy.

The purpose of this paper is to present (1) a sufficient condition for the existence of the Arrow-Debreu equilibria in the case of stochastic differential utilities (SDUs, hereafter), and (2) sufficient conditions for the existence, uniqueness, and local uniqueness of ASM equilibria in the case of time additive utilities (TAUs, hereafter), in the corresponding Arrow-Debreu economy. In subsequent papers, Consumption-based CAPMs with jump risk and a broad class of jump-diffusion option pricing models are presented in the class of GE models with infinite dimensional martingale generator (Kusuda [32]), and further interest rate derivative pricing models are proposed in the class of jump-diffusion option pricing models (Kusuda [30] [34]).

A summary of this paper is as follows. A continuous-time security market economy with an infinite dimensional martingale generator, which consists of a jump process given by a *marked point process* (see Appendix A) and a Wiener process, is considered. Markets are assumed to be *approximately complete* (Björk *et al.* [9] [10]) in which every zero-coupon bond with any maturity time is traded, and any contingent claim is approximately replicated with any given precision by an admissible self-financing portfolio of the bonds. Since it is shown that a generalized security market equilibrium called *approximate security market* (ASM, hereafter) *equilibrium* in approximately complete markets can be identified with an Arrow-Debreu equilibrium (Kusuda [35]), this paper presents sufficient conditions for the existence of Arrow-Debreu equilibria in the case of SDUs, and for the existence, uniqueness, and finiteness (or local uniqueness) of Arrow-Debreu equilibria in the case of TAUs.

---

[1] Ahn and Thompson [2], Back [4], Kusuda [32], Madan [37], *etc.*

[2] Bakshi, Cao, and Chen [5], Bates [6] [7] [8], Björk *et al.* [9] [10], Duffie, Pan, and Singleton [20], Fujiwara and Miyahara [24], Merton [40], Naik and Lee [41], *etc.*

[3] Adachi [1], Daglish [12], Liu, Longstaff, and Pan [36], *etc.*

[4] Consider the case in which the information filtration in security markets is generated by a d-dimensional Wiener process and a d'-dimensional Poisson process. In this case, a martingale generator consists of the Wiener process and its compensated Poisson process, and its dimensionality is $d + d'$. In this paper, the finite dimensional Poisson process is replaced with "infinite dimensional Poisson process."

[5] In security market economy in which the filtration is generated by a finite dimensional Wiener process, Duffie [16], Duffie and Zame [22], and Huang [25] show sufficient conditions for the existence of equilibria, and Karatzas, Lakner, Lehoczky, and Shreve [27], and Karatzas, Lehoczky, and Shreve [28] present sufficient conditions for the existence and uniqueness of equilibria. Dana and Pontier [15], and Duffie [16] show sufficient conditions for the existence of equilibria in security market economy in which the filtration is more general than the one generated by finite dimensional Wiener process. However, the martingale generator in their markets is still assumed to be finite dimensional.

For the case of SDUs, a sufficient condition for the existence of Arrow-Debreu equilibria in the case of *normalized SDUs* (see Definition 7 in Section 5) is presented (Duffie, Geoffard, and Skiadas [19]). It is also shown that any SDU is normalized under diffusion information (Duffie and Epstein [18]). However, it does not hold that any SDU is normalized under jump-diffusion information. Therefore, this paper presents a necessary and sufficient condition for an SDU to be normalized under jump-diffusion information. The result of Duffie, Geoffard, and Skiadas [19] is then applied to the class of normalizable SDUs for jump-diffusion information. The class of normalizable SDUs for jump-diffusion information includes the standard TAU, the Uzawa utility (Uzawa [43]), and the Kreps-Porteus utility (Kreps and Porteus [29]). For the case of TAUs, sufficient conditions for the existence, uniqueness, and local uniqueness of Arrow-Debreu equilibria for a static economy (Dana [13] [14]). Her results, in which the Negishi approach (Negishi [42]) is exploited, are summarized in the following: (1) An Arrow-Debreu equilibrium can be identified with a representative agent equilibrium; (2) There exists a representative agent equilibrium under a regularity condition; (3) If every agent's relative risk aversion coefficient is less than or equal to one, then the representative agent equilibrium is unique; (4) If every agent's risk tolerance satisfies an integrability condition and every agent's endowment process is bounded away from zero, then the set of equilibria is generically finite. This paper extends these results to continuous-time economy.

The remainder of this paper is organized as follows. Section 2 provides a specification of security market economy with jump-diffusion information. Section 3 reviews the notions of approximately complete markets and ASM equilibrium, and the result on the equivalence of ASM and Arrow-Debreu equilibria. Section 4 presents a sufficient condition for the existence of Arrow-Debreu equilibria in the case of SDUs. Section 5 shows sufficient conditions for the existence, uniqueness, and local uniqueness of Arrow-Debreu equilibria in the case of TAUs.

2. Security Market Economy with Jump-Diffusion Information

In this section, a specification of security market economy with jump-diffusion information is provided.

A continuous-time frictionless security market economy with time span $[0, T^{\dagger}]$ (abbreviated by $\mathbf{T}$, hereafter) for a fixed horizon time $T^{\dagger} > 0$ is considered. The agents' common subjective probability and information structure is modeled by a complete filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}^{W,\nu}, P)$ where $\mathbb{F}^{W,\nu} = (\mathcal{F}_t)_{t \in \mathbf{T}}$ is the natural filtration generated by a d-dimensional Wiener process W and a marked point process $\nu(dt \times dz)$ (see Appendix A) on a Lusin space $(\mathbb{Z}, \mathcal{Z})$ (in usual applications, $\mathbb{Z} = \mathbb{R}^{d'}$, or $\mathbb{N}^{d'}$, or a finite set) with the P-intensity kernel $\lambda_t(dz)$.[6] If the mark space $\mathbb{Z}$ is infinite, then the dimensionality of martingale generator is infinite because a martingale generator in this economy is $(W, (\nu(dt \times \{z\}) - \lambda_t(\{z\}))_{z \in \mathbb{Z}})$. The author's main concern is to consider the case in which $\mathbb{Z}$ is infinite, although $\mathbb{Z}$ is unspecified.

There is a single perishable consumption commodity. The commodity space is a Banach space $\mathbf{L}^{\infty} = \mathbf{L}^{\infty}(\Omega \times \mathbf{T}, \mathcal{P}, \mu)$ where $\mathcal{P}$ is the predictable σ-algebra on $\Omega \times \mathbf{T}$, μ is the product measure of P and Lebesgue measure on $\mathbf{T}$. There are I agents. Each agent $i \in \{1, 2, \cdots, I\}$ (abbreviated by $\mathbf{I}$, hereafter) is represented by $(U^i, \bar{c}^i)$, where U^i is a strictly increasing and continuous utility on the positive cone

[6] This information structure is based on Björk, Kabanov, and Runggaldier [10]. More general information structures are considered in Björk, Di Masi, Kabanov, and Runggaldier [9] and in Jarrow and Madan [26].

$\mathbf{L}^\infty_+$ of the consumption process and $\bar{c}^i \in \mathbf{L}^\infty_+$ is an endowment process, which is assumed to be nonzero. The economy mentioned above is described by a collection

$$\mathbf{E} = ((\Omega, \mathcal{F}, \mathbb{F}^{W,\nu}, P), (U^i, \bar{c}^i)_{i \in \mathbf{I}}).$$

There are markets for the consumption commodity and securities at every date $t \in \mathbf{T}$. The traded securities are nominal-risk-free security (NOT the risk-free security) called the *money market account* and a continuum of zero-coupon bonds whose maturity times are $(0, T^\dagger]$, each of which pays one unit of cash (NOT one unit of the commodity) at its maturity time. Let p, B, and $(B^T)_{T \in (0,T^\dagger]}$ denote the consumption commodity price process, nominal money market account price process and nominal bond price processes, respectively. The collection $(B, (B^T)_{T \in (0,T^\dagger]})$ of security prices is abbreviated by $\mathbf{B}$, and called the *family of bond prices*.

Following Björk, Kabanov, and Runggaldier [10], each agent is allowed to hold a portfolio consisting of the money market account and all of bonds at one time. To do so, we define the portfolio component of bonds by a signed finite Borel measure on $[t, T^\dagger]$ for every event $\omega \in \Omega$ and time $t \in \mathbf{T}$.

Definition 1. A *portfolio* is a stochastic process $\vartheta = (\vartheta^0, \vartheta^1(\cdot))$ that satisfies:

(1) The component ϑ^0 is a real-valued $\mathcal{P}$-measurable process.
(2) The component ϑ^1 is such that:
   (i) For every $(\omega, t) \in \Omega \times \mathbf{T}$, the set function $\vartheta^1_t(\omega, \cdot)$ is a signed finite Borel measure on $[t, T^\dagger]$.
   (ii) For every Borel set A, the process $\vartheta^1(A)$ is $\mathcal{P}$-measurable.

3. Approximate Security Market Equilibrium

In this section, a review of approximately complete markets and ASM (Approximate Security Market) equilibrium, and the result on the equivalence of ASM and Arrow-Debreu equilibria is conducted following the companion paper (Kusuda [35]).

First, a class of families of bond prices is introduced such that the markets are *arbitrage-free* and *approximately complete* (for definitions, see Kusuda [35]), and that for every family of bond prices in this class, an ASM equilibrium can be identified with an Arrow-Debreu equilibrium.

Definition 2. A family of bond prices $\mathbf{B}$ is *implementable* if and only if the following three conditions hold (for definitions of *regular*, *risk-neutral measure*, and *density process*, see Kusuda [35]):

(1) $\mathbf{B}$ is *regular*.
(2) There exists a unique *risk-neutral measure* $\tilde{P}^{\mathbf{B}}$.
(3) The discounted *density process* $\frac{\Lambda^{\mathbf{B}}}{B}$ of $\tilde{P}^{\mathbf{B}}$ relative to P is bounded above and bounded away from zero μ-a.e.

Let $\bar{\mathcal{B}}$ denote the class of families of implementable bond prices. Next, we introduce a class of *admissible portfolios*.

Definition 3. Let $\mathbf{B}$ in $\bar{\mathcal{B}}$. A *feasible portfolio at* $\mathbf{B}$ (for definition, see Kusuda [35]) is *admissible at* $\mathbf{B}$ if and only if the discounted value process $\frac{V^{\mathbf{B}}(\vartheta)}{B}$ is bounded below P-a.s. where

$$V^{\mathbf{B}}_t(\vartheta) = B_t\, \vartheta^0_t + \int_t^{T^\dagger} B^T_t\, \vartheta^1_t(dT) \qquad\qquad \forall t \in \mathbf{T}.$$

Let $\underline{\Theta}(\tilde{\mathbf{B}})$ denote the class of admissible portfolios at $\mathbf{B}$. Now the notion of ASM equilibrium is introduced in which each agent is allowed to choose any consumption plan that can be approximately financed with any prescribed precision by a budgetary admissible portfolio.

Definition 4. A collection $((\hat{c}^i)_{i\in\mathbf{I}}, p, \mathbf{B}) \in \prod_{i\in\mathbf{I}} \mathbf{L}^\infty_+ \times \mathbf{L}^\infty_+ \times \bar{\mathcal{B}}$ constitutes an ASM (*Approximate Security Market) equilibrium for* $\mathbf{E}$ if and only if the following conditions hold:

(1) For every $i \in \mathbf{I}$, $\hat{c}^i$ solves the problem

$$\max_{c^i \in \bar{\mathcal{C}}^i(p,\mathbf{B})} U^i(c^i)$$

where

$$\bar{\mathcal{C}}^i(p,\mathbf{B}) = \Big\{ c^i \in \mathbf{L}^\infty_+ \;:\; \exists(\vartheta^i_n)_{n\in\mathbb{N}} \in \prod_{n\in\mathbb{N}} \underline{\Theta}(\tilde{\mathbf{B}}) \quad \text{s.t.}$$

$$V^{\mathbf{B}}_t(\vartheta^i_n) = \int_0^t \vartheta^{i0}_{ns}\, dB_s + \int_0^t \int_s^{T^\dagger} \vartheta^{i1}_{ns}(dT)\, dB^T_s + \int_0^t p_s(\bar{c}^i_s - c^i_s)\, ds \qquad \forall (n,t) \in \mathbb{N} \times \mathbf{T},$$

$$\lim_{n\to\infty} V^{\mathbf{B}}_{T^\dagger}(\vartheta^i_n) = 0 \Big\}.$$

(2) The commodity market is cleared as $\sum_{i\in\mathbf{I}} \hat{c}^i = \sum_{i\in\mathbf{I}} \bar{c}^i$.

We say that a collection $((\hat{c}^i)_{i\in\mathbf{I}}, \pi) \in \prod_{i\in\mathbf{I}} \mathbf{L}^\infty_+ \times \mathbf{L}^\infty_+$ constitutes an *Arrow-Debreu equilibrium for* $\mathbf{E}$ if and only if the following conditions hold:

(1) For every $i \in \mathbf{I}$, $\hat{c}^i$ solves the problem

$$\max_{c^i \in \mathcal{C}^i(\pi)} U^i(c^i)$$

where $\mathcal{C}^i(\pi) = \{c^i \in \mathbf{L}^\infty_+ \;:\; \int_0^{T^\dagger} c^i_s\, ds = \int_0^{T^\dagger} \bar{c}^i_s\, ds\}$.

(2) The commodity market is cleared as $\sum_{i\in\mathbf{I}} \hat{c}^i = \sum_{i\in\mathbf{I}} \bar{c}^i$.

It can be proven that for every implementable family of bond prices $\mathbf{B} \in \bar{\mathcal{B}}$, an ASM equilibrium $((\hat{c}^i)_{i\in\mathbf{I}}, p, \mathbf{B})$ for $\mathbf{E}$ is identified with an Arrow-Debreu equilibrium $((\hat{c}^i)_{i\in\mathbf{I}}, \pi)$ for $\mathbf{E}$ under the relation $\tilde{\Lambda}^{\mathbf{B}} p = \pi$. It can also be shown that if the mark space is finite, then for every implementable family of bond prices, an ASM equilibrium is reduced to be a security market equilibrium.

Theorem 1. *Let* $\mathbf{B} \in \bar{\mathcal{B}}$. *It follows that:*

(1) (i) *Let* $((\hat{c}^i)_{i\in\mathbf{I}}, \pi)$ *be an Arrow-Debreu equilibrium for* $\mathbf{E}$. *Define* $p = (\tilde{\Lambda}^{\mathbf{B}})^{-1}\pi$. *Then* $((\hat{c}^i)_{i\in\mathbf{I}}, p, \mathbf{B})$ *is an ASM equilibrium for* $\mathbf{E}$.
(ii) *Conversely, let* $((\hat{c}^i)_{i\in\mathbf{I}}, p, \mathbf{B})$ *be an ASM equilibrium for* $\mathbf{E}$. *Define* $\pi = \tilde{\Lambda}^{\mathbf{B}} p$. *Then* $((\hat{c}^i)_{i\in\mathbf{I}}, \pi)$ *is an Arrow-Debreu equilibrium for* $\mathbf{E}$.

(2) *Suppose that the mark space* $\mathbb{Z}$ *is finite. Then* $((\hat{c}^i)_{i\in\mathbf{I}}, p, \mathbf{B})$ *is an ASM equilibrium for* $\mathbf{E}$ *if and only if* $((\hat{c}^i)_{i\in\mathbf{I}}, p, \mathbf{B})$ *is a security market equilibrium for* $\mathbf{E}$.

Proof. See Kusuda [35] □

Now the task is reduced to present sufficient conditions for the existence of Arrow-Debreu equilibria in the case of SDUs, and for the existence, uniqueness, and local uniqueness of Arrow-Debreu equilibria in the case of TAUs.

4. Existence of Equilibria in Case of SDUs

In this section, a sufficient condition for the existence of Arrow-Debreu equilibria in the case of SDUs (Stochastic Differential Utilities) is presented. Duffie and Epstein [18] show that any SDU is normalized under pure diffusion information, and Duffie, Geoffard, and Skiadas [19] present sufficient conditions for the existence of Arrow-Debreu equilibria in the case of normalized SDUs. However, it does not hold that any SDU is normalized under jump-diffusion information. Therefore, this paper presents a necessary and sufficient condition for an SDU to be normalized under jump-diffusion information. The result of Duffie, Geoffard, and Skiadas [19] is then applied to the class of normalizable SDUs for jump-diffusion information. The class of normalizable SDUs for jump-diffusion information is a subclass of SDUs for diffusion information, but still includes the standard TAU (Time Additive Utility), the Uzawa utility (Uzawa [43]), the Kreps-Porteus utility (Kreps and Porteus [29]), *etc.*

The notion of SDU was first introduced by Kreps and Porteus [29], developed by Epstein and Zin [23] in discrete-time setting, and extended to continuous-time setting by Duffie and Epstein [18]. An SDU has an expected recursive utility representation and is an extension of the standard TAU. It is well known that in the standard TAU, both of risk aversion and intertemporal substitution depend on the curvature of the von Neumann-Morgenstern utility function, for instance, the relative risk aversion is reciprocal of the elasticity of intertemporal substitution in the CRRA utility. These two properties of utility can be independently given in SDUs. In this section, it is assumed that agents' common subjective probability and information structure is modeled by a complete filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$, where $\mathbb{F} = (\mathcal{F}_t)_{t\in\mathbf{T}}$ is a filtration satisfying usual conditions.

4.1. **Normalizable SDUs under Jump-Diffusion Information.** First, the notion of *SDU for diffusion information* given by Duffie and Epstein [18] is reviewed (for definitions of *aggregator*, *certainty equivalent*, and its *local gradient representation*, see Duffie and Epstein [18]).

Definition 5. Let $\mathbb{F} = \mathbb{F}^W$ where $\mathbb{F}^W$ is the natural filtration generated by d-dimensional Wiener process W. Then a utility $\bar{U} : \mathbf{L}^\infty_+ \to \mathbb{R}$ is an *SDU for diffusion information* if and only if U is characterized by an aggregator $(\bar{f}, \bar{q})$ such that $\bar{U}(c) = \bar{Y}_0$ for every $c \in \mathbf{L}^\infty_+$ where $\bar{Y}$ is a unique solution in $\mathbf{L}^\infty$ of the stochastic differential equation:

$$d\bar{Y}_t = \mu^{\bar{Y}}_t \, dt + \sigma^{\bar{Y}}_t \cdot dW_t \qquad\qquad \forall t \in \mathbf{T}$$

with $\bar{Y}_{T^\dagger} = 0$ where $\mu^{\bar{Y}} \in \mathcal{L}^1$, $\sigma^{\bar{Y}} \in \prod_{j=1}^d \mathcal{L}^2$, and satisfy

$$\mu^{\bar{Y}}_t = -\bar{f}(c_s, \bar{Y}_s) - \frac{1}{2}\, \bar{q}_{11}(\bar{Y}_s, \bar{Y}_s)\, \|\sigma^{\bar{Y}}_s\|^2$$

where $\bar{q} : \mathbb{R}^2 \to \mathbb{R}$ is the local gradient representation (LGR, hereafter) of certainty equivalent, and satisfies $\bar{q} \in \mathbf{C}^{2,0}$ and $\bar{q}_1(x, x) = 1$ for every $x \in \mathbb{R}$.

The notion of *SDU for jump-diffusion information* is introduced, which is a natural extension of the notion of SDU for diffusion information.

Definition 6. Let $\mathbb{F} = \mathbb{F}^{W,\nu}$. Then a utility $\bar{U} : \mathbf{L}^\infty \to \mathbb{R}$ is an *SDU for jump-diffusion information* if and only if $\bar{U}$ is characterized by an aggregator $(\bar{f}, \bar{q})$ such that $\bar{U}(c) = \bar{Y}_0$ for every $c \in \mathbf{L}^\infty_+$ where $\bar{Y}$ is a unique solution in $\mathbf{L}^\infty$ of the SDDE:

$$d\bar{Y}_t = \mu^{\bar{Y}}_t \, dt + \sigma^{\bar{Y}}_t \cdot dW_t + \bar{Y}_{t-} \int_{\mathbb{Z}} m^{\bar{Y}}_t(z) \{ \nu(dt \times dz) - \lambda_t(dz)\, dt \} \qquad \forall t \in \mathbf{T} \quad (4.1)$$

with $\bar{Y}_{T^\dagger} = 0$ where $\mu^{\bar{Y}} \in \mathcal{L}^1$, $\sigma^{\bar{Y}} \in \prod_{j=1}^d \mathcal{L}^2$, $m^{\bar{Y}} \in \mathcal{L}^1(\lambda_t(dz) \times dt)$, and satisfy

$$\begin{aligned}\mu^{\bar{Y}}_t &= -\bar{f}(c_s, \bar{Y}_s) - \frac{1}{2}\, \bar{q}_{11}(\bar{Y}_s, \bar{Y}_s)\, \|\sigma^{\bar{Y}}_s\|^2 \\ &\qquad - \int_{\mathbb{Z}} \big\{ \bar{q}((1 + m^{\bar{Y}}_s(z))\bar{Y}_s, \bar{Y}_s) - \bar{q}(\bar{Y}_s, \bar{Y}_s) - \bar{Y}_s m^{\bar{Y}}_s(z) \big\} \lambda_s(dz) \qquad (4.2)\end{aligned}$$

where $\bar{q}$ is the LGR of certainty equivalent $\bar{Q}$, and satisfies $\bar{q} \in \mathbf{C}^{2,0}$ and $\bar{q}_1(x, x) = 1$ for every $x \in \mathbb{R}$.

Remark 1. Equation (4.2) is derived from definitions of the aggregator $(\bar f, \bar q)$:

$$\begin{aligned}
\bar f(c_t, \bar Y_t) &= -\lim_{\Delta\downarrow 0} \frac{1}{\Delta} E_t\big[\bar q(\bar Y_t, \bar Y_{t-\Delta}) - \bar q(\bar Y_{t-\Delta}, \bar Y_{t-\Delta})\big] \\
&= -\lim_{\Delta\downarrow 0} \frac{1}{\Delta} E_t\bigg[\int_{t-\Delta}^{t} \Big\{\bar q_1(\bar Y_s, \bar Y_s)\,\mu_s^{\bar Y} + \frac{1}{2}\,\bar q_{11}(\bar Y_s, \bar Y_s)\,\|\sigma_s^{\bar Y}\|^2 \\
&\quad + \int_{\mathbb{Z}} \big\{\bar q((1+m_s^{\bar Y}(z))\bar Y_s, \bar Y_s) - \bar q(\bar Y_s, \bar Y_s) - \bar q_1(\bar Y_s, \bar Y_s)\bar Y_s m_s^{\bar Y}(z)\big\}\lambda_s(dz)\Big\}\, ds\bigg] \\
&= -\mu_t^{\bar Y} - \frac{1}{2}\,\bar q_{11}(\bar Y_t, \bar Y_t)\,\|\sigma_t^{\bar Y}\|^2 \\
&\qquad - \int_{\mathbb{Z}} \big\{\bar q((1+m_t^{\bar Y}(z))\bar Y_t, \bar Y_t) - \bar q(\bar Y_t, \bar Y_t) - \bar Y_t m_t^{\bar Y}(z)\big\}\lambda_t(dz).
\end{aligned}$$

Here the property $\bar q_1(\bar Y_s, \bar Y_s) = 1$ was used.

Remark 2. An SDU $\bar U$ for diffusion information has the following recursive expected utility representation:

$$\begin{aligned}
\bar Y_t = -E_t\bigg[\int_t^{T^\dagger} \mu_s^{\bar Y}\, ds\bigg] &= E_t\bigg[\int_t^{T^\dagger} \big\{\bar f(c_s, \bar Y_s) + \frac{1}{2}\,\bar q_{11}(\bar Y_s, \bar Y_s)\,\|\sigma_s^{\bar Y}\|^2\, ds \\
&+ \int_{\mathbb{Z}} \big\{\bar q((1+m_t^{\bar Y}(z))\bar Y_t, \bar Y_t) - \bar q(\bar Y_t, \bar Y_t) - \bar Y_t m_t^{\bar Y}(z)\big\}\lambda_t(dz)\,\big\}\bigg] \qquad \forall t \in \mathbf{T}.
\end{aligned} \tag{4.3}$$

The recursive expected utility representation (4.3) of $\bar U$ is analytically intractable. Exploiting the notion of *ordinally equivalent utility*[7] Duffie and Epstein [18], introduce the notion of a *normalizable SDU*.

Definition 7. A utility $\bar U : \mathbf{L}_+^\infty \to \mathbb{R}$ is a *normalizable SDU* if and only if there exists an ordinally equivalent utility U that is characterized by an aggregator (f, q) such that $U(c) = Y_0$ for every $c \in \mathbf{L}_+^\infty$ where Y is a unique solution in $\mathbf{L}^\infty$ of the recursive equation:

$$Y_t = E_t\left[\int_t^{T^\dagger} f(c_s, Y_s)\, ds\right] \qquad \forall t \in \mathbf{T}. \tag{4.4}$$

The functions (f, q) (or the function f) and the process Y are called the *normalized aggregator* and the *continuation utility process*, respectively. The class of normalizable SDUs depends on the filtration $\mathbb{F}$, so let it be denoted by $\mathcal{U}_{\mathrm{SD}}(\mathbb{F})$. It is shown by Duffie and Epstein [18] that any SDU for diffusion information is normalized. However, it is not true that any SDU for jump-diffusion information is normalized. The following proposition presents a necessary and sufficient condition for an SDU for jump-diffusion information to be normalized.

Proposition 1. *Let $\mathbb{F} = \mathbb{F}^{W,\nu}$. Let $\bar U$ be an SDU for jump-diffusion information characterized by an aggregator $(\bar f, \bar q)$. Then $\bar U \in \mathcal{U}_{\mathrm{SD}}(\mathbb{F}^{W,\nu})$ if and only if $\bar U$ satisfies*

$$\bar q_1(x, y) = \exp\left[\int_y^x \psi(x_1)\, dx_1\right] \qquad \forall (x, y) \in \mathbb{R}^2 \tag{4.5}$$

for some continuous function $\psi : \mathbb{R} \to \mathbb{R}$.

Let $\mathcal{U}_{\mathrm{SD}}^{W,\nu}$ denote the class of SDUs for jump-diffusion information satisfying the condition (4.5), and call it the *class of normalizable SDUs for jump-diffusion information.*

[7] We say that a utility $U : \mathbf{L}_+^\infty \to \mathbb{R}$ is an *ordinally equivalent utility* to a utility $\bar U : \mathbf{L}_+^\infty \to \mathbb{R}$ if and only if there exists a strictly increasing and continuous function $\varphi : \mathbf{R} \to \mathbb{R}$ with $\varphi(0) = 0$ such that $U = \varphi \circ \bar U$.

Remark 3. The class of normalizable SDUs for jump-diffusion information is a subclass of SDUs for jump-diffusion information, but still includes the class of SDUs, each of which is characterized by an *expected utility certainty equivalent* (for definition, see Duffie and Epstein [18]). This class of SDUs includes the Kreps-Porteus utility and the Uzawa utility as well as the standard TAU.

Proof. Suppose that a utility $\bar{U} : \mathbf{L}^\infty_+ \to \mathbb{R}$ is characterized by an unnormalized aggregator $(\bar{f}, \bar{q})$ such that $\bar{U}(c) = \bar{Y}_0$ for every $c \in \mathbf{L}^\infty_+$ where $\bar{Y}$ is a unique square-integrable process satisfying the SDDE (4.1).

Step 1 – $\bar{U}$ is normalized if and only if there exists a continuous function $\psi : \mathbb{R} \to \mathbb{R}$ satisfying (4.5): Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a strictly increasing and $\mathbf{C}^2$-function with $\varphi(0) = 0$ and let $Y_t = \varphi(\bar{Y}_t)$ for every $t \in \mathbf{T}$ and $f(x, \varphi(y)) = \varphi'(y)\bar{f}(x, y)$. Applying Ito's Formula to $Y_t = \varphi(\bar{Y})$ yields

$$dY_t = \mu^Y_t\, dt + \varphi'(\bar{Y}_t)\, \sigma^{\bar{Y}}_t \cdot dW_t + \int_{\mathbb{Z}} \big\{\varphi((1+m^{\bar{Y}}_{t-}(z))\bar{Y}_{t-}) - \varphi(\bar{Y}_{t-})\big\}\, \{\, \nu(dt\times dz) - \lambda_t(dz)\, dt \,\} \tag{4.6}$$

for every $t \in \mathbf{T}$ where

$$\mu^Y_t = \varphi'(\bar{Y}_t)\Big\{\mu^{\bar{Y}}_t - \bar{Y}_t \int_{\mathbb{Z}} m^{\bar{Y}}_t(z)\, \lambda_t(dz)\Big\} + \frac{1}{2}\, \varphi''(\bar{Y}_t)\, \|\sigma^{\bar{Y}}_t\|^2 + \int_{\mathbb{Z}} \big\{\varphi((1 + m^{\bar{Y}}_t(z))\bar{Y}_t) - \varphi(\bar{Y}_t)\big\}\lambda_t(dz). \tag{4.7}$$

Substituting (4.2) and $\varphi'(\bar{Y}_t)\bar{f}(c_t, \bar{Y}_t) = f(c_t, Y_t)$ into (4.7) yields

$$\begin{aligned}
\mu^Y_t =& -\varphi'(\bar{Y}_t)\bar{f}(c_t, \bar{Y}_t) - \frac{1}{2}\, \varphi'(\bar{Y}_t)\, \bar{q}_{11}(\bar{Y}_t, \bar{Y}_t)\, \|\sigma^{\bar{Y}}_t\|^2 \\
&- \varphi'(\bar{Y}_t) \int_{\mathbb{Z}} \big\{\bar{q}((1 + m^{\bar{Y}}_t(z))\bar{Y}_t, \bar{Y}_t) - \bar{q}(\bar{Y}_t, \bar{Y}_t) - \bar{Y}_t m^{\bar{Y}}_t(z)\big\}\lambda_t(dz) \\
&+ \frac{1}{2}\, \varphi''(\bar{Y}_t)\, \|\sigma^{\bar{Y}}_t\|^2 + \int_{\mathbb{Z}} \big\{\varphi((1 + m^{\bar{Y}}_t(z))\bar{Y}_t) - \varphi(\bar{Y}_t) - \varphi'(\bar{Y}_t)\bar{Y}_t m^{\bar{Y}}_t(z)\big\}\, \lambda_t(dz) \\
=& -f(c_t, Y_t) - \frac{1}{2}\, \big\{\varphi'(\bar{Y}_t)\, \bar{q}_{11}(\bar{Y}_t, \bar{Y}_t) - \varphi''(\bar{Y}_t)\big\}\, \|\sigma^{\bar{Y}}_t\|^2 \\
&- \int_{\mathbb{Z}} \Big[\varphi'(\bar{Y}_t)\big\{\bar{q}((1 + m^{\bar{Y}}_t(z))\bar{Y}_t, \bar{Y}_t) - \bar{q}(\bar{Y}_t, \bar{Y}_t)\big\} - \big\{\varphi((1 + m^{\bar{Y}}_t(z))\bar{Y}_t) - \varphi(\bar{Y}_t)\big\}\Big]\, \lambda_t(dz).
\end{aligned} \tag{4.8}$$

Thus, $\bar{U}$ is normalized if and only if the set of conditions hold:

$$\varphi''(y) = \bar{q}_{11}(y, y)\, \varphi'(y), \tag{4.9}$$

$$\varphi(x) - \varphi(y) = \varphi'(y)\{\bar{q}(x, y) - \bar{q}(y, y)\} \tag{4.10}$$

for every $(x, y) \in \mathbb{R}^2$. However, twice partially differentiating both sides of (4.10) with respect to x and substituting $x = y$ yields (4.9). Hence, $\bar{U}$ is normalized if and only if the condition (4.10) holds. Considering $\bar{q}_1(y, y) = 1$ for every $y \in \mathbb{R}$, the condition (4.10) is equivalent to

$$\varphi'(x) = \varphi'(y)\bar{q}_1(x, y) \qquad \forall (x, y) \in \mathbb{R}^2. \tag{4.11}$$

Taking log of both sides of (4.11) and partially differentiating with respect to x, we have

$$\frac{\varphi''(x)}{\varphi'(x)} = \frac{\bar{q}_{11}(x, y)}{\bar{q}_1(x, y)} \qquad \forall (x, y) \in \mathbb{R}^2. \tag{4.12}$$

Conversely, (4.11) follows from (4.12) and $\bar{q}_1(y, y) = 1$. Thus, the condition (4.12) is equivalent to the condition (4.11). It is straightforward to see that the condition (4.12) holds if and only if there exists a continuous function $\psi : \mathbb{R} \to \mathbb{R}$

satisfying the condition (4.5). Then a function φ satisfying (4.12) is given by

$$\varphi(x) = \int_0^x \exp\left[\int_0^{x_2} \psi(x_1)\,dx_1\right] dx_2 \qquad \forall x \in \mathbb{R}. \tag{4.13}$$

Step 2 – $\mathcal{U}_{\mathrm{SD}}(\mathbb{F}^{W,\nu}) \supset \mathcal{U}_{\mathrm{SD}}^{W,\nu}$: Let $\bar{U} \in \mathcal{U}_{\mathrm{SD}}^{W,\nu}$ be characterized by an unnormalized aggregator $(\bar{f}, \bar{q})$. It then immediately follows from Step1 that $\bar{U} \in \mathcal{U}_{\mathrm{SD}}(\mathbb{F}^{W,\nu})$.

Step 3 – $\mathcal{U}_{\mathrm{SD}}(\mathbb{F}^{W,\nu}) \subset \mathcal{U}_{\mathrm{SD}}^{W,\nu}$: Let $\bar{U} \in \mathcal{U}_{\mathrm{SD}}(\mathbb{F}^{W,\nu})$ be characterized by an aggregator $(\bar{f}, \bar{q})$. Assume w.l.o.g. that $(\bar{f}, \bar{q})$ is an unnormalized aggregator. Then it follows from Step 1 that there exists a continuous function $\psi : \mathbb{R} \to \mathbb{R}$ satisfying (4.5), and therefore $\bar{U} \in \mathcal{U}_{\mathrm{SD}}^{W,\nu}$. □

For the existence and uniqueness of the solution $Y \in \mathbf{L}^\infty$ of the recursive equation (4.4) for every $c \in \mathbf{L}_+^\infty$ in Definition 7, a sufficient conditions is shown by Duffie and Epstein [18].

Proposition 2. *Let $f : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ be a Borel measurable function. Suppose that f satisfies the following conditions:*

(1) *A growth condition in consumption: There exist constants k_0 and k_1 such that for every $x \in \mathbb{R}_+$, it follows that $|f(x, 0)| \le k_0 + k_1\|c\|$.*
(2) *A uniform Lipschitz condition in utility: There exists a constant k such that for every $x \in \mathbb{R}_+$ and every $(y_1, y_2) \in \mathbb{R}^2$, it follows that $|f(x, y_1) - f(x, y_2)| \le k\|y_1 - y_2\|$.*

Then, for every $c \in \mathbf{L}_+^\infty$, there exists a unique solution Y in $\mathbf{L}^\infty$ of the recursive equation (4.4).

Proof. See Appendix A in Duffie and Epstein [18]. □

4.2. **Existence of Equilibria.** It is postulated that every agent's utility is a normalizable SDU for jump-diffusion information.

Assumption 1. *For every $i \in \mathbf{I}$, $U^i \in \mathcal{U}_{\mathrm{SD}}^{W,\nu}$ is characterized by a normalized aggregator (f^i, m^i) where f^i satisfies the growth condition in consumption and the uniform Lipschitz condition in utility.*

The following assumption is introduced to ensure the existence of equilibria.

Assumption 2. (1) *For every $i \in \mathbf{I}$, it follows that:*
(i) *For every $y \in \mathbb{R}$, $f^i(\,\cdot\,, y)$ is strictly increasing.*
(ii) *The aggregator f^i is continuously differentiable on the interior of its domain.*
(iii) *The aggregator f^i is concave.*
(iv) *For every $x > 0$, $\sup_{y\in\mathbb{R}} f_c^i(x, y) < \infty$.*
(v) *The aggregator f^i satisfies $\lim_{x\downarrow 0} \inf_{y\in\mathbb{R}} f_c^i(x, y) = \infty$.*

(2) *The aggregate endowment is bounded away from zero μ-a.e.*

Remark 4. Consider a standard TAU of the form

$$U(c) = E\left[\int_0^{T^\dagger} e^{-\rho s} u(c_s)\,ds\right].$$

Then it follows from Ito's Formula that U can be interpreted as an SDU of the form

$$Y_t = E_t\left[\int_t^{T^\dagger} \big(u(c_s) - \rho Y_s\big)\,ds\right] \qquad \forall t \in \mathbf{T}.$$

It is straightforward to see that the condition 2(1)(v) is equivalent to the Inada condition in the case of TAU.

Under Assumptions 1 and 2, Duffie, Geoffard, and Skiadas [19] prove the existence of Arrow-Debreu equilibria exploiting the Negishi approach (Negishi [42]) and the results given by Duffie and Epstein [18], Duffie and Zame [22], and Mas-Collel and Zame [39].

Proposition 3. *Under Assumptions 1 and 2, there exists an Arrow-Debreu equilibrium* $((\hat{c}^i)_{i\in\mathbf{I}}, \pi)$ *for* $\mathbf{E}$. *The equilibrium state price* π *satisfies*

$$\pi_t = \hat{\alpha}_i \exp\left(\int_0^t f_y^i(\hat{c}_s^i, Y_s^i)\,ds\right) f_c^i(\hat{c}_t^i, Y_t^i) \qquad \mu\text{-a.e.}$$

for some $\hat{\alpha} \in \Delta_{++}^I$. *Moreover, the allocation* $(\hat{c}^i)_{i\in\mathbf{I}}$ *is Pareto optimal.*

Proof. See Duffie, Geoffard, and Skiadas [19]. □

The following theorem is finally obtained by combining Proposition 3 with Theorem 1.

Theorem 2. *Let* $\mathbb{F} = \mathbb{F}^{W,\nu}$. *Under Assumptions 1 and 2, for every* $\mathbf{B} \in \bar{\mathcal{B}}$, *there exists an ASM equilibrium* $((\hat{c}^i)_{i\in\mathbf{I}}, p, \mathbf{B})$ *for* $\mathbf{E}$. *In particular, if the mark set* $\mathbb{Z}$ *is finite, then* $((\hat{c}^i)_{i\in\mathbf{I}}, p, \mathbf{B})$ *is a security market equilibrium for* $\mathbf{E}$. *The equilibrium commodity price* p *satisfies*

$$p_t = \hat{\alpha}_i \frac{B_t}{\Lambda_t^{\mathbf{B}}} \exp\left(\int_0^t f_y^i(\hat{c}_s^i, Y_s^i)\,ds\right) f_c^i(\hat{c}_t^i, Y_t^i) \qquad \mu\text{-a.e.}$$

for some $\hat{\alpha} \in \Delta_{++}^I$. *Moreover, the allocation* $(\hat{c}^i)_{i\in\mathbf{I}}$ *is Pareto optimal.*

5. Existence, Uniqueness, and Finiteness of Equilibria in Case of TAUs

In this section, sufficient conditions for the existence, uniqueness, and local uniqueness (or determinacy) of Arrow-Debreu equilibria in the case of TAUs (Time Additive Utilities) are shown. For the purpose, the results of Dana [13] [14] for a static economy are extended to our continuous-time economy. Her proof, which exploits the Negishi approach, are summarized in the following: (1) An Arrow-Debreu equilibrium can be identified with a representative agent equilibrium: (2) There exists a representative agent equilibrium under a regularity condition; (3) If every agent's relative risk aversion coefficient is less than or equal to one, then the representative agent equilibrium is unique; (4) If every agent's risk tolerance coefficient satisfies an integrability condition and every agent's endowment process is bounded away from zero, then the set of equilibria is generically finite.

It is assumed that every agent's utility is a TAU with the nice properties given below.

Assumption 3. *For every agent* $i \in \mathbf{I}$, *the utility* U^i *is a TAU of the form*

$$U^i(c) = E\left[\int_0^{T^\dagger} u^i(t, c_t^i)\,dt\right]$$

where the von Neumann-Morgenstern (VNM, hereafter) utility function $u^i : \mathbf{T} \times \mathbb{R}_+ \to \mathbb{R} \cup \{-\infty\}$ *is* $\mathbf{C}^{1,2}$ *on* $\mathbb{R}_+$, *and such that* $u^i(t, \cdot)$ *is strictly increasing and strictly concave on* $\mathbb{R}_+$ *for every* $t \in \mathbf{T}$.

5.1. **Equivalence of Arrow-Debreu and Representative Agent Equilibria.** The aggregate utility is introduced to exploit the Negishi approach. Let $\alpha \in \Delta^I_+$ where $\Delta^I_+ = \{\alpha \in \mathbb{R}^I_+ \mid \sum_{i\in\mathbf{I}} \alpha_i = 1\}$, and define the *aggregate utility* $U^\alpha : \mathbf{L}^\infty_+ \to \mathbb{R}$ by

$$U^\alpha(c) = \max_{(c^1,c^2,\cdots,c^I)\in\prod_{i\in\mathbf{I}}\mathbf{L}^2_+} \sum_{i\in\mathbf{I}} \alpha_i U^i(c^i) \qquad \text{s.t.} \qquad \sum_{i\in\mathbf{I}} c^i \le c.$$

We also define a *demand function* $c^* : \mathbf{T}\times\mathbb{R}_+\times\mathbb{R}^I_+ \to \mathbb{R}^I_+$ by

$$(c^*_i(t,x,\alpha))_{i\in\mathbf{I}} = \operatorname{argmax}_{\{(x_1,x_2,\cdots,x_I)\in\mathbb{R}^I_+ : \sum_{i\in\mathbf{I}} x_i \le x\}} \sum_{i\in\mathbf{I}} \alpha_i u^i(t,x_i).$$

The following lemma shows that the aggregate utility U^α has the expected utility representation, and the properties of the VNM aggregate utility function and the demand function.

Lemma 1. *Under Assumption 3, the aggregate utility U^α is a TAU of the form*

$$U^\alpha(c) = E\left[\int_0^{T^\dagger} u(t,c_t,\alpha)\,dt\right] \quad \textit{where} \quad u(t,x,\alpha) = \sum_{i\in\mathbf{I}} \alpha_i u^i(t,c^*_i(t,x,\alpha)). \tag{5.1}$$

Moreover, u and $(c^*_i)_{i\in\mathbf{I}}$ satisfy the following conditions.

(1) (i) *The function u is a real-valued $\mathbf{C}^{1,1,0}$-function on $\mathbf{T}\times\mathbb{R}_+\times\mathbb{R}^I_+$ such that $u(t,\cdot,\alpha)$ is strictly increasing and strictly concave on $\mathbb{R}_+$ for every $(t,\alpha)\in\mathbf{T}\times\mathbb{R}^I_+$.*

(ii) *Let $i\in\mathbf{I}$. For every $(t,x,\alpha)\in\mathbf{T}\times\mathbb{R}_+\times\mathbb{R}^I_+$ satisfying $c^*_i(t,x,\alpha)>0$, the first partial derivative of u with respect to x denoted by $u_c(t,x,\alpha)$ satisfies*

$$u_c(t,x,\alpha) = \alpha_i u^i_c(t,c^*_i(t,x,\alpha)). \tag{5.2}$$

(2) *Let $i\in\mathbf{I}$.*

(i) *The function c^*_i is continuous on $\mathbf{T}\times\mathbb{R}_+\times\mathbb{R}^I_+$.*

(ii) *For every $(t,x)\in\mathbf{T}\times\mathbb{R}_{++}$, the function $c^*_i(t,x,\cdot)$ is homogeneous of degree zero.*

(iii) *For every $(t,\alpha)\in\mathbf{T}\times\mathbb{R}^I_+$, $c^*_i(t,0,\alpha)=0$.*

(3) (i) *The functions u_c and c^*_i for every $i\in\mathbf{I}$ are differentiable off the set $\mathcal{D}$ of Lebesgue measure zero:*

$$\mathcal{D} = \{(t,x,\alpha)\in\mathbf{T}\times\mathbb{R}_{++}\times\mathbb{R}^I_{++} : u_c(t,x,\alpha) = \alpha_i u^i_c(t,c^*_i(t,0,\alpha)) \textit{ for some } i\in\mathbf{I}\}.$$

(ii) *For every $t\in\mathbf{T}$, the functions $u_c(t,\cdot,\cdot)$ and $c^*_i(t,\cdot,\cdot)$ for every $i\in\mathbf{I}$ are Lipschitz continuous on compact subsets of $\mathbb{R}_+\times\mathbb{R}^I_+$..*

(iii) *Let $(t,x,\alpha)\in\mathcal{D}^c$. Assume that $c^*_i(t,x,\alpha)>0$ for every $i\in\mathbf{I}$. Then it follows that for every $i,j\in\mathbf{I}$,*

$$\frac{\partial c^*_i}{\partial \alpha_j}(t,x,\alpha) = \frac{u^j_c(t,c^*_j(t,x,\alpha))}{\alpha_i\alpha_j u^i_{cc}(t,c^*_i(t,x,\alpha))u^j_{cc}(t,c^*_j(t,x,\alpha))\eta(t,x,\alpha)} \tag{5.3}$$

where

$$\eta(t,x,\alpha) = \sum_{i\in\mathbf{I}} \frac{1}{\alpha_i u^i_{cc}(t,c^*_i(t,x,\alpha))}.$$

Proof. Proofs of (1) and (2) are easy. For proofs of (3)(i)(ii), see the proof of Proposition 2.3 in Dana [13]. (3)(iii) is obtained by differentiating the first order condition

$$\alpha_1 u^i_c(t,c^*_i(t,x,\alpha)) = \alpha_2 u^2_c(t,c^*_2(t,x,\alpha)) = \cdots = \alpha_I u^I_c(t,c^*_I(t,x,\alpha))$$

and the relation $\sum_{i\in\mathbf{I}} c^*_i(t,x,\alpha) = x$ with respect to α_j. □

The notion of a *representative agent equilibrium* $\hat{\alpha} \in \Delta^I_+$ *for* $\mathbf{E}$ is introduced, which is characterized by the Pareto optimal allocation $(c^*_i(t, \bar{c}_t, \hat{\alpha}))$ without transfer payments under the supporting state price $u_c(s, \bar{c}_s, \hat{\alpha})$.

Definition 8. A utility weight $\hat{\alpha} \in \Delta^I_{++}$ constitutes a *representative agent equilibrium for* $\mathbf{E}$ if and only if $\hat{\alpha}$ is a solution of the equation $\xi(\hat{\alpha}) = 0$ where $\xi : \mathbb{R}^I_{++} \to \mathbb{R}^I$ is the *excess utility function* defined by

$$\xi_i(\alpha) = \frac{1}{\alpha_i} E\left[\int_0^{T^\dagger} u_c(s, \bar{c}_s, \alpha)(c^*_i(s, \bar{c}_s, \alpha) - \bar{c}^i_s)\, ds\right] \qquad \forall i \in \mathbf{I}.$$

One can show that a representative agent equilibrium for $\mathbf{E}$ can be identified with an Arrow-Debreu equilibrium for $\mathbf{E}$. To do so, the following lemma is used.

Lemma 2. *Under Assumption 3, for any Pareto optimal allocation* $(c^i)_{i\in\mathbf{I}}$ *for* $\mathbf{E}$*, there exists a solution* $\hat{\alpha} \in \Delta^I_{++}$ *of the equation:*

$$c^*(t, \bar{c}_t(\omega), \hat{\alpha}) = (c^i_t(\omega))_{i\in\mathbf{I}} \qquad \mu\text{-a.e.}$$

Proof. See Huang [25]. □

Proposition 4. *Under Assumption 3, it follows that:*

(1) *Let* $\hat{\alpha}$ *be a representative agent equilibrium for* $\mathbf{E}$*. Define* $((\hat{c}^i)_{i\in\mathbf{I}}, \pi)$ *by* $(\hat{c}^i_t(\omega))_{i\in\mathbf{I}} = c^*(t, \bar{c}_t(\omega), \hat{\alpha})$ *and* $\pi_t = u_c(t, \bar{c}_t(\omega), \hat{\alpha})$ *for every* $(\omega, t) \in \Omega \times \mathbf{T}$*. Then* $((\hat{c}^i)_{i\in\mathbf{I}}, \pi)$ *is an Arrow-Debreu equilibrium for* $\mathbf{E}$*.*
(2) *Conversely, let* $((\hat{c}^i)_{i\in\mathbf{I}}, \pi)$ *be an Arrow-Debreu equilibrium for* $\mathbf{E}$*. Define* $\hat{\alpha} \in \Delta^I_{++}$ *by a solution of the equation* $c^*(t, \bar{c}_t(\omega), \hat{\alpha}) = \hat{c}_t(\omega)$ μ*-a.e. Then* $\hat{\alpha}$ *is a representative agent equilibrium for* $\mathbf{E}$*.*

Proof. See Appendix D. □

Now our task is reduced to show sufficient conditions for the existence, uniqueness, and local uniqueness of representative agent equilibria.

5.2. **Existence of Representative Agent Equilibria.** To prove the existence of representative agent equilibria, the following assumption is imposed on the aggregate endowment.

Assumption 4. *The aggregate endowment is bounded away from zero, i.e. there exists a positive constant* $\underline{\delta}$ *such that* $\bar{c}_t(\omega) \geq \underline{\delta}$ μ*-a.e.*

Note that this assumption implies that $u_c(t, \bar{c}_t(\omega), \alpha) \in \mathbf{L}^\infty_+$ because

$$u_c(t, \bar{c}_t(\omega), \alpha) \leq \max_{(t,\alpha)\in\mathbf{T}\times\Delta^I_+} u_c(t, \underline{\delta}, \alpha) \qquad \mu\text{-a.e.},$$

and $u_c(\,\cdot\,, \underline{\delta}, \,\cdot\,)$ is continuous on $\mathbf{T} \times \Delta^I_+$. Then the excess utility function has the following desired properties for proving the existence of representative agent equilibria.

Lemma 3. *Under Assumptions 3 and 4, it follows that:*

(1) *The excess utility function* ξ *is homogeneous of degree zero, and satisfies* $\alpha \cdot \xi(\alpha) = 0$ *for every* $\alpha \in \mathbb{R}^I_+$*, and bounded above on* $\mathbb{R}^I_+$*.*
(2) *The excess utility function* ξ *is continuous on* $\mathbb{R}^I_{++}$*, and* $\xi_i(\alpha) \to -\infty$ *whenever* $\alpha_i \to 0$ *for some* $i \in \mathbf{I}$*.*

Proof. Note that there exists a positive constant $\bar{\delta}$ such that $\bar{c}_t(\omega) \leq \bar{\delta}$ μ-a.e. because $\bar{c} \in \mathbf{L}^\infty_+$

Step 1 – (1): It is obvious that ξ is homogeneous of degree zero, and satisfies $\alpha \cdot \xi(\alpha) = 0$ for every $\alpha \in \mathbb{R}^I_+$. Therefore, it is proven that ξ is bounded above on $\mathbb{R}^I_+$. Let $i \in \mathbf{I}$ and $\alpha^0 \in \Delta^I_+$ be such that $\alpha^0_i = 0$. It is sufficient to show that

$\xi_i(\alpha)$ is bounded above as $\alpha \in \Delta_+^I$ tends to α^0. First, it follows from the Lipschitz continuity of $c_i^*(t,\cdot,\cdot)$ and $c_i^*(t,\bar{c}_i(\omega),\alpha^0)=0$ that there exists a K such that

$$c_i^*(t,\bar{c}_t(\omega),\alpha) \le \max_{t\in\mathbf{T}} c_i^*(t,\bar{\delta},\alpha) \le K\|\alpha-\alpha^0\| \qquad \mu\text{-a.e.}$$

Thus, it follows that

$$\frac{1}{\alpha_i}u_c\left(t,\bar{c}_t(\omega),\alpha\right)\{c_i^*(t,\bar{c}_t(\omega),\alpha)-\bar{c}_t^i(\omega)\} \le \frac{\|\alpha-\alpha^0\|}{\alpha_i}K \max_{(t,\alpha')\in\mathbf{T}\times\Delta_+^I}\{u_c(t,\underline{\delta},\alpha')\} \qquad \mu\text{-a.e.}$$

The right-hand side of the above equation converges to $K\max_{(t,\alpha')\in\mathbf{T}\times\Delta_+^I}\{u_c(t,\underline{\delta},\alpha')\}$ as α tends to α^0. Therefore, it follows from Lebesgue Dominated Convergence Theorem that $\xi(\alpha)$ is bounded above as α tends to α^0.

Step 2 – (2) Continuity on $\mathbb{R}_{++}^I$: It is enough to present the continuity of ξ on a compact subset S of $\mathbb{R}_{++}^I$ bounded away from the boundary. Since ξ is homogeneous of degree zero on α, it follows that for every $i\in\mathbf{I}$,

$$\begin{aligned}&\left|\frac{1}{\alpha_i}u_c\left(t,\bar{c}_t(\omega),\alpha\right)\{c_i^*(t,\bar{c}_t(\omega),\alpha)-\bar{c}_t^i(\omega)\}\right|\\ &\quad=\frac{\sum_{j\in\mathbf{I}}\alpha_j}{\alpha_i}\left|u_c\left(t,\bar{c}_t(\omega),\frac{\alpha}{\sum_{j\in\mathbf{I}}\alpha_j}\right)\left\{c_i^*\left(t,\bar{c}_t(\omega),\frac{\alpha}{\sum_{j\in\mathbf{I}}\alpha_j}\right)-\bar{c}_t^i(\omega)\right\}\right|\\ &\quad\le\frac{\sqrt{I}\,\|\alpha\|}{\alpha_i}\max_{(t,\alpha')\in\mathbf{T}\times\Delta_+^I}\{u_c(t,\underline{\delta},\alpha')\}\,\bar{\delta} \qquad \mu\text{-a.e.}\end{aligned}$$

Thus, the continuity of ξ on S follows from Lebesgue Dominated Convergence Theorem.

Step 3 – (2) Boundary condition: Let $i\in\mathbf{I}$ and $\alpha^0\in\Delta_+^I$ be such that $\alpha_i^0=0$. It suffices to show that $\xi_i(\alpha)$ tends to $-\infty$ as $\alpha\in\Delta_+^I$ tends to α^0. Note that there exists $\mathbf{A}\in\mathcal{P}$ such that $\mu(\mathbf{A})>0$ and $\bar{c}_t^i(\omega)>0$ for every $(\omega,t)\in\mathbf{A}$ since every agent's endowment process is assumed to be nonzero. Then it follows that

$$\begin{aligned}\xi_i(\alpha)&\le\frac{1}{\alpha_i}E\left[\int_0^{T^\dagger}u_c(s,\bar{c}_s(\omega),\alpha)\,\bar{c}_i^*(s,\bar{c}_s,\alpha)\,ds\right]-\frac{1}{\alpha_i}\int_{\mathbf{A}}u_c(s,\bar{c}_s(\omega),\alpha)\,\bar{c}_s^i(\omega)\,\mu(d\omega\times ds)\\ &\le\frac{\|\alpha-\alpha^0\|}{\alpha_i}KT^\dagger\max_{(t,\alpha')\in\mathbf{T}\times\Delta_+^I}\{u_c(t,\underline{\delta},\alpha')\}-\frac{1}{\alpha_i}\min_{(t,\alpha')\in\mathbf{T}\times\Delta_+^I}\{u_c(t,\bar{\delta},\alpha')\}\int_{\mathbf{A}}\bar{c}_s^i(\omega)\,\mu(d\omega\times ds),\end{aligned}$$

which tends to $-\infty$ as α tends to α^0. □

5.3. **Uniqueness of Representative Agent Equilibria.** To prove uniqueness of equilibria, the following two assumptions are imposed.

Assumption 5. (1) *For every* $i\in\mathbf{I}$*, the agent* i*'s relative risk aversion coefficient satisfies*

$$\gamma^i(t,x)\stackrel{\text{def}}{=}-\frac{x\,u_{cc}^i(t,x)}{u_c^i(t,x)}\le 1 \qquad \forall(t,x)\in\mathbf{T}\times\mathbb{R}_+.$$

(2) *Either of the following two conditions is satisfied:*
   (i) *Every agent's endowment is positive* μ*-a.e., i.e.* $\bar{c}^i>0$ μ*-a.e. for every* $i\in\mathbf{I}$.
   (ii) *Every agent's utility satisfies the Inada condition, i.e.* $\lim_{x\downarrow 0}u_c^i(t,x)=\infty$ *for every* $i\in\mathbf{I}$.

Then it is shown that the excess utility function is *strongly gross substitute.*

Lemma 4. *Under Assumptions 3-5,* ξ *is strongly gross substitute, i.e.:*

(1) *For every* (i,j) *such that* $i\neq j$, $\xi_i(\alpha_1,\cdots,\alpha_{j-1},\cdot,\alpha_{j+1},\cdots,\alpha_I)$ *is non-increasing and for every* i, $\xi_i(\alpha_1,\cdots,\alpha_{i-1},\cdot,\alpha_{i+1},\cdots,\alpha_I)$ *is non-decreasing.*

(2) *If* $c_i^*(t, \bar{c}_t(\omega), \alpha) > 0$ *on some* $\mathbf{A} \in \mathcal{P}$ *with* $\mu(\mathbf{A}) > 0$*, then for every* $j \neq i$*,* $\xi_i(\alpha_1, \cdots, \alpha_{j-1}, \cdot, \alpha_{j+1}, \cdots, \alpha_I)$ *is strictly decreasing on a neighborhood of* α*.*

Proof. See the proof of Theorem 3.1 in Dana [13]. □

5.4. **Local Uniqueness of Representative Agent Equilibria.** Unfortunately, there is no strong evidence which supports Assumption 5. Therefore, it is shown that under more reasonable assumptions, the local uniqueness of equilibria, or equivalently the finiteness of the set of equilibria, is a generic property of our economies using the Negishi approach given by Dana [13] for static economies.

The space of economies is parameterized by keeping agents' common subjective probability and information structure $(\Omega, \mathcal{F}, \mathbb{F}^{W,\nu}, P)$, utilities $(U^i)_{i\in\mathbf{I}}$, and the aggregate endowment $\bar{c}$ fixed, and varying the distribution of individual endowments. The following assumptions are imposed on utilities and endowments.

Assumption 6. *For every* $i \in \mathbf{I}$*, the von Neumann-Morgenstern utility function satisfies*

$$-\frac{u_c^i(t,x)}{u_{cc}^i(t,x)} \leq \beta_1^i x + \beta_2^i \qquad \forall (t,x) \in \mathbf{T} \times \mathbb{R}_+$$

for some $\beta^i \in \mathbb{R}_+^2$*.*

Assumption 7. *There exists* $\delta \in \mathbb{R}_{++}^{\mathbf{I}}$ *such that* $\bar{c}_t^i > \delta_i$ μ*-a.e. on* $\mathbf{T} \times \Omega$ *for every* $i \in \mathbf{I}$*.*

The following space of economies is introduced in which each economy is characterized by the distribution of individual endowments.

$$\mathcal{E}_\delta = \Big\{ \mathbf{E} = ((\Omega, \mathcal{F}, \mathbb{F}^{W,\nu}, P), (U^i, \bar{c}^i)_{i\in\mathbf{I}}) \,\Big|\, (\bar{c}^i)_{i\in\mathbf{I}} \in \prod_{i\in\mathbf{I}} \mathbf{L}_+^\infty, \quad \sum_{i\in\mathbf{I}} \bar{c}^i = \bar{c}, \text{ and } (\bar{c}^i)_{i\in\mathbf{I}} \text{ satisfies Assumption 7 for } \delta \Big\}$$

A function $\hat{\xi} : \Delta_+^I \times \mathcal{E}_\delta \to \mathbb{R}^I$ is defined by

$$\hat{\xi}_i(\alpha, \mathbf{E}) = \frac{1}{\alpha_i} E\left[\int_0^{T^\dagger} u_c(s, \bar{c}_s, \alpha)(c_i^*(s, \bar{c}_s, \alpha) - \bar{c}_s^i)\, ds\right] \qquad \forall i \in \mathbf{I}.$$

The continuity of $\hat{\xi}$ follows from Dominated Convergence Theorem. The differentiability of $\hat{\xi}$ with respect to α and the continuity of the derivative can also be shown.

Lemma 5. *Under Assumptions 3, 4, 6, and 7,* $\hat{\xi}$ *is differentiable with respect to* α *on* $\Delta_{++}^{\mathbf{I}}$ *and its derivative is continuous on* $\Delta_{++}^I \times \mathcal{E}_\delta$*.*

Proof. See Appendix E □

Since $\sum_{i\in\mathbf{I}} \hat{\xi}_i(\alpha, \mathbf{E}) = 0$ for every $\alpha \in \Delta_+^I$, it follows that $\operatorname{rank} D_\alpha \hat{\xi}(\alpha, \mathbf{E}) \leq I - 1$. It is said that the economy $\mathbf{E}$ is *regular* if and only if $\hat{\xi}(\hat{\alpha}, \mathbf{E}) = 0$ implies $\operatorname{rank} D_\alpha \hat{\xi}(\alpha, \mathbf{E}) = I - 1$. Let $\mathcal{R}_\delta$ denote the set of all regular economies in $\mathcal{R}_\delta$. It is well known that that any regular economy can only have a finite number of equilibria (see Proposition 17.D.1 in Mas-Collel, Whinston, and Green [38]). Thus, to see that the number of equilibria is generically finite, it is enough to show that the set of regular economies $\mathcal{R}_\delta$ is open and dense in $\mathcal{E}_\delta$. In order to do so, a correspondence $\{\hat{\alpha}\}(\mathbf{E}) : \mathcal{E}_\delta \to \Delta_+^I$ is defined by

$$\{\hat{\alpha}\}(\mathbf{E}) = \{\, \alpha \in \Delta_+^I \,:\, \hat{\xi}(\alpha, \mathbf{E}) = 0 \,\},$$

Then the following lemma is shown.

Lemma 6. *Under Assumptions 3, 4, 6, and 7, it follows that:*

(1) *The correspondence $\{\hat{\alpha}\}$ is u.h.c., and for every $\mathbf{E} \in \mathcal{E}_\delta$, $\{\hat{\alpha}\}(\mathbf{E})$ is compact.*
(2) *If $\mathbf{E}$ is regular then $\{\hat{\alpha}\}(\mathbf{E})$ is finite.*

Proof. Proof of (1) immediately follows from the continuity of $\hat{\xi}$. Let $\mathbf{E}$ be a regular economy. Suppose $\{\hat{\alpha}\}(\mathbf{E})$ is infinite. Then since $\{\hat{\alpha}\}(\mathbf{E})$ is compact, it has an accumulation point $\hat{\alpha} \in \{\hat{\alpha}\}(\mathbf{E})$. This implies that $\hat{\alpha}$ is not locally unique. This is a contradiction. □

5.5. **Existence, Uniqueness, and Local Uniqueness of Equilibria.** Now the existence, uniqueness, and local uniqueness of ASM equilibria can be proven.

Theorem 3. *Under Assumptions 3 and 4, it follows that for every $\mathbf{B} \in \bar{\mathcal{B}}$:*

(1) *There exists an ASM equilibrium $((\hat{c}^i)_{i\in\mathbf{I}}, p, \mathbf{B})$ for $\mathbf{E}$. In particular, if the mark set $\mathbb{Z}$ is finite, then $((\hat{c}^i)_{i\in\mathbf{I}}, p, \mathbf{B})$ is a security market equilibrium for $\mathbf{E}$. The equilibrium $((\hat{c}^i)_{i\in\mathbf{I}}, p, \mathbf{B})$ is characterized by the corresponding representative agent equilibrium $\hat{\alpha}$ for $\mathbf{E}$, i.e. $((\hat{c}^i)_{i\in\mathbf{I}}, p)$ satisfies*

$$(\hat{c}^i_t(\omega))_{i\in\mathbf{I}} = c^*(t, \bar{c}_t(\omega), \hat{\alpha}),$$
$$p_t(\omega) = \frac{B_t(\omega)}{\Lambda^{\mathbf{B}}_t(\omega)} u_c(t, \bar{c}_t(\omega), \hat{\alpha})$$

for almost every $(\omega, t) \in \Omega \times \mathbf{T}$. Moreover, the allocation $(\hat{c}^i)_{i\in\mathbf{I}}$ is Pareto optimal.

(2) *If Assumption 5 is satisfied, then the ASM equilibrium is unique.*
(3) *If Assumptions 6 and 7 are satisfied, then the set of regular economies $\mathcal{R}_\delta$ is open and dense in $\mathcal{E}_\delta$.*

Proof. Step 1 – Existence: It follows from Lemma 3 and Kakutani's Fixed Point Theorem that there exists an $\hat{\alpha} \in \Delta^I_+$ such that $\xi(\hat{\alpha}) = 0$, *i.e.* there exists a representative agent equilibrium $\hat{\alpha}$ for $\mathbf{E}$ (for the proof, see pp. 585-587 in Mas-Collel, Whinston, and Green [38]). Define $(\hat{c}^i)_{i\in\mathbf{I}}$ and p by $(\hat{c}^i_t(\omega))_{i\in\mathbf{I}} = c^*(t, \bar{c}_t(\omega), \hat{\alpha})$ and $p_t(\omega) = (\tilde{\Lambda}^{\mathbf{B}}_t(\omega))^{-1} u_c(t, \bar{c}_t(\omega), \hat{\alpha})$ for every $(\omega, t) \in \Omega \times \mathbf{T}$, respectively. Then by Proposition 4(1) and Theorem 1(1), $((\hat{c}^i)_{i\in\mathbf{I}}, p, \mathbf{B})$ is an ASM equilibrium for $\mathbf{E}$, and $(\hat{c}^i)_{i\in\mathbf{I}}$ is a Pareto optimal allocation. Suppose that the mark set $\mathbb{Z}$ is finite. It then follows from Theorem 1(2) that $((\hat{c}^i)_{i\in\mathbf{I}}, p, \mathbf{B})$ constitutes a security market equilibrium for $\mathbf{E}$.

Step 2 – Uniqueness: By Theorem 1 and Proposition 4, it is sufficient to show that the representative agent equilibrium is unique. The proof of Dana [13] is used. Assume that there exist two non-collinear solutions for $\xi(\alpha) = 0$ and let them be $\hat{\alpha}$ and $\check{\alpha}$. Since E is homogeneous of degree zero by Lemma 3, let w.l.o.g. $\hat{\alpha} < \check{\alpha}$ with $\hat{\alpha}_i = \check{\alpha}_i$ for some $i \in \mathbf{I}$. As $\check{\alpha}$ is a solution for $\xi(\alpha) = 0$, $c^*_i(t, \bar{c}_t(\omega), \check{\alpha}) \neq 0$ for every j. Therefore, ξ_i is strictly increasing at $\check{\alpha}$. Let $\hat{\alpha} < \alpha < \check{\alpha}$. Then $0 = \xi_i(\hat{\alpha}) < \xi_i(\alpha) < \xi_i(\check{\alpha}) = 0$, which is a contradiction.

Step 3 – Local Uniqueness: See Appendix F.

□

Appendix A. Marked Point Process

A double sequence $(s_n, Z_n)_{n\in\mathbb{N}}$ is considered, where s_n is the occurrence time of an nth jump and Z_n is a random variable taking its values on a measurable space $(\mathbb{Z}, \mathcal{Z})$ at time s_n. Define a random counting measure $\nu(dt \times dz)$ by

$$\nu([0,t] \times A) = \sum_{n\in\mathbb{N}} 1_{\{s_n \le t,\, Z_n \in A\}} \qquad \forall (t, A) \in [0, T^\dagger] \times \mathcal{Z}.$$

This counting measure $\nu(dt \times dz)$ is called the $\mathbb{Z}$-*marked point process.*

Let λ be such that

(1) For every $(\omega, t) \in \Omega \times (0, T^\dagger]$, the set function $\lambda_t(\omega, \cdot)$ is a finite Borel measure on $\mathbb{Z}$.

(2) For every $A \in \mathcal{Z}$, the process $\lambda(A)$ is $\mathcal{P}$-measurable and satisfies $\lambda(A) \in \mathcal{L}^1$.

The marked point process $\nu(dt \times dz)$ is said to have the *P-intensity kernel* $\lambda_t(dz)$ if and only if the following equation

$$E\left[\int_0^{T^\dagger} Y_s \, \nu(ds \times A)\right] = E\left[\int_0^{T^\dagger} Y_s \lambda_s(A)\, ds\right] \qquad \forall A \in \mathcal{Z}$$

holds for any nonnegative $\mathcal{P}$-measurable process Y, then it is said that the marked point process $\nu(dt \times dz)$ has the *P-intensity kernel* $\lambda_t(dz)$.

Let $\nu(dt \times dz)$ be a $\mathbb{Z}$-marked point process with the P-intensity kernel $\lambda_t(dz)$. Let H be a $\mathcal{P} \otimes \mathcal{Z}$-measurable function. It follows that:

(1) If the following integrability condition

$$E\left[\int_0^{T^\dagger} \int_{\mathbb{Z}} |H_s(z)| \lambda_s(z)\, ds\right] < \infty$$

holds, then the process $\int_0^t \int_{\mathbb{Z}} H_s(z) \{\, \nu(ds \times dz) - \lambda_s(dz)\, ds \,\}$ is a P-martingale.

(2) If $H \in \mathcal{L}^1(\lambda_t(dz) \times dt)$, then the process $\int_0^t \int_{\mathbb{Z}} H_s(z) \{\, \nu(ds \times dz) - \lambda_s(dz)\, ds \,\}$ is a local P-martingale.

Proof. See p. 235 in Brémaud [11]. □

Appendix B. Ito's Formula

Let $X = (X^1, ..., X^d)'$ be a d-dimensional semimartingale, and g be a real-valued $\mathbf{C}^2$ function on $\mathbb{R}^d$. Then $g(X)$ is a semimartingale of the form

$$g(X_t) = g(X_0) + \sum_{i=1}^{d} \int_0^t \frac{\partial}{\partial x_i} g(X_{s-})\, dX_s^i + \frac{1}{2} \sum_{i=1}^{d} \sum_{j=1}^{d} \int_0^t \frac{\partial^2}{\partial x_i \partial x_j} g(X_{s-})\, d\langle X^{ic}, X^{jc} \rangle$$
$$+ \sum_{0 \le s \le t} \left\{ g(X_s) - g(X_{s-}) + \sum_{i=1}^{d} \frac{\partial}{\partial x_i} g(X_{s-})\, \Delta X_s^i \right\}$$

where X^{ic} is the continuous part of X^{ic} and $\langle X^{ic}, X^{jc} \rangle$ is the quadratic covariation of X^{ic} and X^{jc}.

Appendix C. Girsanov's Theorem

(1) Let $v \in \prod_{j=1}^d \mathcal{L}^2$ and $H \in \mathcal{L}^1(\lambda_t(dz) \times dt)$. Define a process Λ by

$$\frac{d\Lambda_t}{\Lambda_{t-}} = -v_t \cdot dW_t - \int_{\mathbb{Z}} H_t(z) \{\, \nu(dt \times dz) - \lambda_t(dz)\, dt \} \qquad \forall t \in [0, T^\dagger)$$

with $\Lambda_0 = 1$, and suppose $E[\Lambda_{T^\dagger}] = 1$. Then there exists a probability measure $\tilde{P}$ on $(\Omega, \mathcal{F}, \mathbb{F})$ given by the Radon-Nikodym derivative

$$d\tilde{P} = \Lambda_{T^\dagger}\, dP$$

such that:

(i) The measure $\tilde{P}$ is equivalent to P.

(ii) The process given by

$$\tilde{W}_t = W_t + \int_0^t v_s\, ds \qquad \forall t \in \mathbf{T}$$

is a $\tilde{P}$-Wiener process.

(iii) The marked point process $\nu(dt \times dz)$ has the $\tilde{P}$-intensity kernel such that

$$\tilde{\lambda}_t(dz) = (1 - H_t(z))\lambda_t(dz) \qquad \forall (t, z) \in \mathbf{T} \times \mathbb{Z}.$$

(2) Every probability measure equivalent to P has the structure above.

Appendix D. Proof of Proposition 4

Proof of (1). Let $\hat{\alpha}$ be a representative agent equilibrium for $\mathbf{E}$. Define $((\hat{c}^i)_{i\in\mathbf{I}}, \pi)$ by $(\hat{c}^i_t(\omega))_{i\in\mathbf{I}} = c^*(t, \bar{c}_t(\omega), \hat{\alpha})$ and $\pi_t = u_c(t, \bar{c}_t(\omega), \hat{\alpha})$ for every $(\omega, t) \in \Omega \times \mathbf{T}$. Then $\hat{c}^i \in \mathbf{L}^\infty_+$ for every $i \in \mathbf{I}$ and $\pi \in \mathbf{L}^\infty_+$. It also follows that $\sum_{i\in\mathbf{I}} \hat{c}^i = \bar{c}$ by definition of c^* and that $\hat{c}^i_t$ satisfies the necessary and sufficient condition for every agent's optimality $u^i_c(t, \hat{c}^i_t) = \frac{1}{\hat{\alpha}_i}\pi_t$ for every $i \in \mathbf{I}$.

Proof of (2). Let $((\hat{c}^i)_{i\in\mathbf{I}}, \pi)$ be an Arrow-Debreu equilibrium for $\mathbf{E}$. Since $(u^i)_{i\in\mathbf{I}}$ are strictly increasing by Assumption 3, the allocation $(\hat{c}^i)_{i\in\mathbf{I}}$ is Pareto optimal by First Welfare Theorem (see Mas-Collel and Zame [39]). Then by Lemma 2, there exists $\hat{\alpha} \in \Delta^I_+$ such that

$$c^*(t, \bar{c}_t(\omega), \hat{\alpha}) = (\hat{c}^i_t(\omega))_{i\in\mathbf{I}} \qquad \mu\text{-a.e.} \tag{D.1}$$

Combining (5.2) with (D.1) yields

$$u_c(t, \bar{c}_t(\omega), \hat{\alpha}) = \hat{\alpha}_i u^i_c(t, \hat{c}^i_t(\omega)) \qquad \mu\text{-a.e.} \tag{D.2}$$

for every $i \in \mathbf{I}$. In the meantime, the optimality of consumption plans implies that there exists a rescaled Lagrange multiplier $\hat{\alpha}^- \in \{\alpha \in \mathbb{R}_{++} \mid \sum_{i\in\mathbf{I}} \frac{1}{\hat{\alpha}^-_i} = 1\}$ such that for every $i \in \mathbf{I}$ and

$$u^i_c(t, \hat{c}^i_t) = \hat{\alpha}^-_i \pi_t \qquad \mu\text{-a.e.} \tag{D.3}$$

Comparing (D.2) with (D.3) yields $u_c(t, \bar{c}_t(\omega), \hat{\alpha}) = \pi_t(\omega)$, which implies $\xi(\hat{\alpha}) = 0$.

Appendix E. Proof of Lemma 5

The proof of Dana [13] is exploited. Let S be a compact subset of Δ^I_+ bounded away from the boundary. It suffices to prove the differentiability of $\hat{\xi}$ with respect to α on S. Define a function $\zeta : \mathbf{T} \times \mathbb{R}_+ \times S \to \mathbb{R}^I$ by

$$\zeta_i(t, \bar{c}_t, \alpha) = \frac{1}{\alpha_i} u_c(t, \bar{c}_t, \alpha)(c^*_i(t, \bar{c}_t, \alpha) - \bar{c}^i_t).$$

Partially differentiating ζ with respect to α_j yields

$$\frac{\partial \zeta_i}{\partial \alpha_j}(t, \bar{c}_t, \alpha) = \frac{\partial c^*_i}{\partial \alpha_j}(t, \bar{c}_t, \alpha)\{u^i_{cc}(t, c^*_i)(c^*_i(t, \bar{c}_t, \alpha) - \bar{c}^i_t) + u^i_c(t, c^*_i)\}. \tag{E.1}$$

In the meantime, it follows from (5.3) that

$$\frac{\partial c^*_i}{\partial \alpha_j}(t, \bar{c}_t, \alpha)\, u^i_{cc}(t, c^*_i) \le \frac{1}{\alpha_i} u^j_c(t, c^*_j) = \frac{1}{\alpha_i \alpha_j} u_c(t, \bar{c}_t, \alpha). \tag{E.2}$$

It follows from (E.1),(E.2), and Assumptions 6 and 7 that

$$\left| \frac{\partial \zeta_i}{\partial \alpha_j}(t, \bar{c}_t, \alpha) \right| \le \frac{1}{\alpha_i \alpha_j} \max_{(\alpha') \in \Delta^I} [u_c(t, \bar{c}_t(\omega), \alpha')]\{(\beta^i_1 + 2)\bar{c}_t + \beta^i_2\}. \tag{E.3}$$

Thus, by Lebesgue Dominated Convergence Theorem, $\hat{\xi}$ is differentiable with respect to α on S, and its derivative is

$$\frac{\partial \hat{\xi}_i}{\partial \alpha_j}(\alpha, \mathbf{E}) = E\left[\int_0^{T^\dagger} \frac{\partial c^*_i}{\partial \alpha_j}(s, \bar{c}_s, \alpha) \left\{ u^i_{cc}(s, c^*_i(s, \bar{c}_s, \alpha)(c^*_i(s, \bar{c}_s, \alpha) - \bar{c}^i_s) + u^i_c(t, c^*_i(s, \bar{c}_s, \alpha) \right\} ds \right].$$

Since $\bar{c}$ is fixed, $\left|\frac{\partial F_i}{\partial \alpha_j}\right|$ are bounded independently of $(\alpha, \mathbf{E})$ on S. Therefore, $\frac{\partial \hat{\xi}_i}{\partial \alpha_j}$ is continuous on $\Delta^I_{++} \times \mathcal{E}_\delta$.

Appendix F. Proof of Theorem 3(3)

The proof of Dana [13] is extended. First, the openness of $\mathcal{R}_\delta$ is shown. Let $\mathbf{E}_0 \in \mathcal{R}_\delta$. Then for any $\alpha_0 \in \delta^I$ such that $\hat{\xi}(\alpha_0, \mathbf{E}_0) = 0$, and that rank $D_\alpha \hat{\xi}(\alpha_0, \mathbf{E}_0) = I - 1$. Since $\{\hat{\alpha}\}(\mathbf{E})$ is compact and $D_\alpha \hat{\xi}$ is continuous, there exists neighborhoods $\mathcal{V} \subset \mathcal{E}_\delta$ of $\mathbf{E}_0$ and $V \subset \Delta^I_+$ of α_0 such that $D_\alpha \hat{\xi}(\alpha, \mathbf{E}) = I - 1$ for every $(\alpha, \mathbf{E}) \in V \times \mathcal{V}$. Since $\{\hat{\alpha}\}$ is u.h.c., there exists $\mathcal{V}' \subset \mathcal{V}$ such that $\{\hat{\alpha}\}(\mathcal{V}') \subset \mathcal{V}$. Thus, if $\mathbf{E} \in \mathcal{V}'$,, then the rank $D_\alpha \hat{\xi}(\alpha_0, \mathbf{E}_0) = I - 1$ for every $\alpha \in \{\hat{\alpha}\}(\mathbf{E})$. Therefore, $\mathcal{V}' \subset \mathcal{R}_\delta$ and $\mathcal{R}_\delta$ is open in $\mathcal{E}_\delta$. Next, the denseness of $\mathcal{R}_\delta$ is proven. Let $\mathbf{E} \in \mathcal{E}_\delta$. Let $\varepsilon > 0$ such that $\bar{c}^i - \varepsilon > \delta_i$ μ-a.e. for every $i \in \{1, 2, \cdots, I-1\}$. Let $(X^\varepsilon_i)_{i \in \{1,2,\cdots,I-1\}}$ such that $\max\{\|X_i\|_{\mathbf{L}^2}, \|X_i\|_{\mathbf{L}^\infty}\} \le \varepsilon \quad \forall i \in \{1, 2, \cdots, I-1\}$. Let $A = \{(a_i)_{i \in \{1,2,\cdots,I-1\}} \in \mathbb{R}^{I-1} : 0 \le a_i \le 1 \quad \forall i \in \{1, 2, \cdots, I-1\}\}$. Define a function $h : \Delta^I \times A \to \mathbb{R}^I$ by

$$h_i(\alpha, a) = E\left[\int_0^{T^\dagger} u_c(s, \bar{c}_s, \alpha)(c^*_i(s, \bar{c}_s, \alpha) - \bar{c}^i_s - a_i X^\varepsilon_{is})\, ds\right] \qquad \forall i \in \{1, 2, \cdots, I-1\},$$

and

$$h_I(\alpha, a) = E\left[\int_0^{T^\dagger} u_c(s, \bar{c}_s, \alpha)(c^*_I(s, \bar{c}_s, \alpha) - \bar{c}^I_s + \sum_{i=1}^{I-1} a_i X^\varepsilon_{is})\, ds\right].$$

One can easily check that rank $D_a g(\alpha, a) = I - 1$. By Transversality Theorem, there exists $a \in A$ such that 0 is a regular value of $h(\cdot a)$ that is 0 is a regular value of the economy in $\mathcal{E}$, $(\bar{c}^1 + a_1 X^\varepsilon_1, \bar{c}^2 + a_2 X^\varepsilon_2, \cdots, \bar{c}^{I-1} + a_{I-1} X^\varepsilon_{I-1}, \bar{c}^I - \sum_{i=1}^{I-1} a_i X^\varepsilon_i)$, arbitrarily close to $\mathbf{E}$, since ε can be chosen arbitrarily close to zero.

References

[1] Adachi, T. (2003): *Estimating Consumption and Portfolio Decisions in an Economy with Jump and Volatility Risk*, Ph.D. Dissertation, Department of Economics, University of Minnesota.

[2] Ahn, C.-M., and H. E. Thompson (1988): "Jump-Diffusion Processes and the Term-Structure of Interest Rates," *Journal of Finance*, 43, 155-174.

[3] Araujo, A., and P. K. Monteiro (1989): "Equilibrium without Uniform Conditions," *Journal of Economic Theory*, 48, 416-427.

[4] Back, K. (1991): "Asset Pricing for General Processes," *Journal of Mathematical Economics*, 20, 371-395.

[5] Bakshi, G., C. Cao, and Z. Chen (1997): "Empirical Performance of Alternative Option Pricing Models," *Journal of Finance*, 52, 2003-2049.

[6] Bates, D. S. (1991): "The Crash of 87: Was It Expected? The Evidence from Options Markets," *Journal of Finance*, 46, 1009-1044.

[7] Bates, D. S. (1996): "Jump and Stochastic Volatility: Exchange Rate Processes Implicit in Deutsche Mark Options," *Review of Financial Studies*, 9, 69-107.

[8] Bates, D. S. (2000): "Post '87 Crash Fears in the S&P 500 Futures Option Market," *Journal of Econometrics*, 94, 181-238.

[9] Björk, T., G. Di Masi, Y. Kabanov, and W. Runggaldier (1997a): "Towards a General Theory of Bond Markets," *Finance and Stochastics*, 1, 141-174.

[10] Björk, T., Y. Kabanov, and W. Runggaldier (1997b): "Bond Market Structure in the Presence of Marked Point Processes," *Mathematical Finance*, 7, 211-239.

[11] Brémaud, T. (1981): *Point Processes and Queues: Martingale Dynamics*. New York: Springer-Verlag.

[12] Daglish, T. (2003): *The Economic Significance of Jump Diffusions: Portfolio Allocations*, Ph.D. Dissertation, Department of Finance, Rotman School of Management, University of Tronto.

[13] Dana, R.-A. (1993a): "Existence, Uniqueness, and Determinacy of Arrow-Debreu Equilibria in Finance Models," *Journal of Mathematical Economics*, 22, 563-579.

[14] Dana, R.-A. (1993b): "Notes and Comments: Existence, Uniqueness, and Determinacy of Arrow-Debreu Equilibria when Preferences are Additively Separable," *Econometrica*, 61, 953-957.

[15] Dana, R.-A., and M. Pontier (1992): "On Existence of Arrow-Radner Equilibrium in the Case of Complete Markets. A Remark," *Mathematics of Operations research*, 17, 148-163.

[16] Duffie, D. (1986): "Stochastic Equilibria: Existence, Spanning, and the 'No Expected Financial Gain from Trade' Hypothesis," *Econometrica*, 54, 1161-1183.

[17] Duffie, D. (1992): *Dynamic Asset Pricing Theory*, 2nd Ed. Princeton, NJ: Princeton University Press.

[18] Duffie, D., and L. Epstein (1992): "Stochastic Differential Utility," *Econometrica*, 60, 353-394.

[19] Duffie, D., P.-Y. Geoffard, and C. Skiadas (1994): "Efficient and Equilibrium Allocations with Stochastic Differential Utility," *Journal of Mathematical Economics*, 23, 133-146.

[20] Duffie, D., J. Pan, and K. Singleton (2000): "Transform Analysis and Asset Pricing for Affine Jump-Diffusions," *Econometrica*, 68, 1343-1376.

[21] Duffie, D., and C. Skiadas (1994): "Continuous-Time Security Pricing: A Utility Gradient Approach," *Journal of Mathematical Economics*, 23, 107-131.

[22] Duffie, D., and W. Zame (1989): "The Consumption-Based Capital Asset Pricing Model," *Econometrica*, 57, 1279-1297.

[23] Epstein, L. G., and S. E. Zin (1989): "Substitution, Risk Aversion, and the Temporal Behavior of Consumption and Asset Returns: A Theoretical Framework," *Econometrica*, 57, 937-969.

[24] Fujiwara, T., and Y. Miyahara (2003): "The Minimal Entropy Martingale Measures for Geometric Lévy Processes," *Finance and Stochastics*, 7, 509-531.

[25] Huang, C. (1987): "An Intertemporal General Equilibrium Asset Pricing Model: The Case of Diffusion Information," *Econometrica*, 55, 117-142.

[26] Jarrow, R. A., and D. Madan (1999): "Hedging Contingent Claims on Semimartingales," *Finance and Stochastics*, 3, 111-134.

[27] Karatzas, I., P. Lakner, J. P. Lehoczky, and S. E. Shreve (1991): "Dynamic Equilibrium in a Multi-Agent Economy: Construction and Uniqueness," in *Stochastic Analysis: Liber Amicorum for Moshe Zakai*, ed. by E. Meyer-Wolf, A. Schwartz, and O. Zeitouni. San Diego, CA: Academic Press.

[28] Karatzas, I., J. P. Lehoczky, and S. E. Shreve (1990): "Existence and Uniqueness of Multi-Agent Equilibrium in a Stochastic, Dynamic Consumption/Investment Model," *Mathematics of Operations Research*, 15, 80-128.

[29] Kreps, D., and E. Porteus (1978): "Temporal Resolution of Uncertainty and Dynamic Choice Theory," *Econometrica*, 46, 185-200.

[30] Kusuda, K. (2002a): "Jump-Diffusion LIBOR Rate Model and Equilibrium Pricing of Interest Derivatives," Proceedings of Japanese Association of Financial Econometrics and Engineering Summer 2002 Conference, 340-361.

[31] Kusuda, K. (2002b): "Existence, Uniqueness, and Determinacy of Equilibria in Complete Security Markets with Infinite Dimensional Martingale Generator," Discussion Paper 316, Center for Economic Research, Department of Economics, University of Minnesota.

[32] Kusuda, K. (2003a): "Consumption-Based CAPM and Option Pricing under Jump-Diffusion Uncertainty," Discussion Paper 317, Center for Economic Research, Department of Economics, University of Minnesota.

[33] Kusuda, K. (2003b): *Term Structure Models of Interest Rates with Jump-Diffusion Information: Equilibrium, CAPM, and Derivative Asset Pricing*, Ph.D. Dissertation, Department of Economics, University of Minnesota.

[34] Kusuda, K. (2003c): "Jump-Diffusion LIBOR Rate Model with Stochastic Volatility, and Equilibrium Pricing of Interest Derivatives," Proceedings of Japanese Association of Financial Econometrics and Engineering Winter 2003 Conference, 200-219.

[35] Kusuda, K. (2004): "Implementing Arrow-Debreu Equilibria in Security Markets with Infinite Dimensional Martingale Generator," Working Paper B-1, Center for Risk Research, Faculty of Economics, Shiga University.

[36] Liu, J., F. Longstaff, and J. Pan (2003): "Dynamic Asset Allocation with Event Risk," *Journal of Finance*, 58, 231-259.

[37] Madan D. B. (1988): "Risk Measurement in Semimartingale Models with Multiple Consumption Goods," *Journal of Economic Theory*, 44, 398-412.

[38] Mas-Collel, A., M. D. Whinston, and J. R. Green (1995): *Microeconomic Theory*. Oxford: Oxford University Press.

[39] Mas-Collel, A., and W. Zame (1991): "Equilibrium Theory in Infinite Dimensional Spaces," in *Handbook of Mathematical Economics, Volume IV*, ed. by W. Hildenbrand, and H. Sonnenschein. Amsterdam: North-Holland.

[40] Merton, V. (1976): "Option Pricing when Underlying Stock Returns are Discontinuous," *Journal of Financial Economics*, 3, 125-144.

[41] Naik, V., and M. Lee (1990): "General Equilibrium Pricing of Options on the Market Portfolio with Discontinuous Returns," *Review of Financial Studies*, 3, 493-521.

[42] Negishi, T. (1960): "Welfare Economics and Existence of an Equilibrium for a Competitive Economy," *Metroeconomica*, 12, 92-97.

[43] Uzawa, H. (1968): "Time Preference, the Consumption Function and Optimum Asset Holdings," in *Value, Capital and Growth: Papers in Honour of Sir John Hicks*, ed. by J. Wolfe. Chicago IL: Aldine.