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Abstract

This study proposes a joint pricing model for stocks and bonds in a no-arbitrage framework. A stock price representation is obtained in a manner consistent with the quadratic Gaussian term structure model, in which the short rate is the quadratic form of the state variables. In this study, specifying the dividend as a function using the quadratic form of the state variables leads to a stock price representation that is exponential-quadratic in the state variables. We prove that the coefficients determining the stock price have to satisfy some matrix equations, including an algebraic Riccati equation. Moreover, we specify the sufficient condition in which the matrix equations do have a unique solution. In our empirical analysis using Japanese data, we obtain estimates with a good fit to the actual data. Furthermore, we estimate the risk premiums for stocks and bonds and analyze how the BOJ's unconventional monetary policy has affected these risk premiums.

Keywords: risk premium, quadratic Gaussian term structure model, unscented Kalman filter, algebraic Riccati equation, controllability, portfolio rebalance

JEL Classification C13,E43,E44,G12

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1 Introduction

Risk premiums are basic inputs for investors' asset allocations. Needless to say, since investors make decisions on the asset allocations of several financial assets, such as stocks and bonds, they need to simultaneously estimate their risk premiums. Moreover, risk premiums are essential not only for investors but also for central bankers. In particular, it has become more difficult for central banks to ignore how risk premiums have been evolving since the financial crisis that began in August 2007. After the collapse of Lehman Brothers, some central banks in developed countries conducted unconventional monetary easing measures. For example, the U.S. Federal Reserve (Fed) began its quantitative easing in December 2008 and recently ended this policy in October 2014. The Bank of Japan (BOJ) started its comprehensive monetary easing (CME) in October 2010, and when it ended its CME, it began the qualitative and quantitative easing (QQE) in April 2013. Furthermore, the BOJ decided to expand the scale of its QQE in November 2014. These policies consisted of large-scale purchases of longer-term government bonds by the Fed and the BOJ in their attempts to lower investors' risk premiums through portfolio rebalancing. For these reasons, it is important for many investors and central bankers to simultaneously estimate the risk premiums of several financial assets.

In order to simultaneously estimate the risk premiums of several assets, we need a unified framework that jointly deals with the prices of these assets. One of the likely candidates is the no-arbitrage pricing framework. Almost all the earlier studies that price multiple assets within a no-arbitrage framework target stocks and bonds. Although there is a limited number of earlier studies on this subject, [3], [11], [5], [10], and [2] developed the joint-pricing model for stocks and bonds with the no-arbitrage framework. Except for [2], these studies are based on a Gaussian affine framework. In these studies' models, stock and bond prices are both represented using an affine function of the Gaussian state variables. They sustain theoretical consistency in the sense that there is no arbitrage opportunity in financial markets. However, these models do not ensure the positivity of nominal interest rates. A particular problem here is that this may lead to inaccurate estimation results in a low interest rate environment, such as in the current situation.

As for the previous studies focusing only on pricing bonds, some models that guarantee the interest rate's positivity have been proposed. For example, the Cox-Ingersoll-Ross (CIR) model proposed in [4] would be the most popular among positive interest rate models. In addition, the potential approach in [13] and the shadow-rate approach in [8] ensure positive interest rates. Furthermore, the quadratic Gaussian term structure model (QGTM) studied by [1] and [9] is one of these types of studies. The QGTM has an advantage over the CIR model, which is a similar and popular short-rate model. While it is possible that the interest rate in the CIR model takes a negative value in time discretization, even though it ensures positivity in a continuous time setting, the interest rate in the QGTM always takes a positive value, even in a discrete time setting. Furthermore, the QGTM with multivariate factors enables us to represent more flexible correlations among the variables. On the other hand, those of the multivariate CIR are obliged to have some restrictions in obtaining well-defined bond prices.

In this study, we want to incorporate the QGTM into the dividend-discounted-cash-flow pricing model of stocks in a manner consistent with the no-arbitrage condition[1]. We assume that the stock dividend is paid to stockholders on a continuous basis and that the dividend yield is assumed to depend on the state variables. Furthermore, setting the ex-dividend stock price as the exponential quadratic form of the state variables leads to the necessary condition for the existence of the stock price. Thus, we provide the sufficient condition of the well-defined stock price. The joint pricing model for stocks and bonds, we propose in this study, allows us to estimate market prices and their risk premiums more accurately than the affine Gaussian framework seen in the previous studies. This is because our model ensures the positivity of the nominal interest rates. In particular, our model could enable us to elaborate an empirical analysis for financial markets under a low interest rate environment.

The rest of the paper is organized as follows. In section 2, we present the theoretical basis of the study, which consists of the setup, and the bond-pricing and stock-pricing elements of our model. In section 3, we explain the estimation methodology. section 4 presents the estimation results. The conclusions are presented in section 5.

2 Theory

In this section, we explain the theoretical portion of the study. First, we prepare for the setup of our model. Here, we define the state variable processes and the short rate. Second, in order to provide the bond pricing, we review the quadratic Gaussian term structure model (QGTM) studied by [1] and [9]. The QGTM serves as a basis for the interest rate models in our study. Next, we aim to give the stock price representation. In this part of our study, after defining the finite-maturity stock, we represent the price as the general form of the conditional expectation of the discounted cash flow. Then, we specify the amount of dividend paid continuously to the stockholders, which depends on the state variables. As a result of an application of the Feynman-Kac theorem, we can derive the necessary condition for the stock price representation to be satisfied. In order for the finite-maturity stock price to become well defined, we discuss the sufficient condition for the unique existence of the stock price. Next, we define the infinite-maturity stock and give the price as the discounted cash flow representation in the conditional-expectation form. Finally, we prove that imposing two transversality conditions for the dividend and the infinite-maturity stock leads to equality between the infinite-maturity stock price and the limit of the finite-maturity stock as maturity approaches infinity. Therefore, under some proper conditions, we obtain the well-defined stock price.

2.1 Setup

In this study, we fix a probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbf{P})$ that satisfies the usual condition. Here, let $\mathbf{P}$ denote the physical measure. In addition, we assume the market to be complete, so that the risk-neutral measure, $\mathbf{Q}$, uniquely exists.

[1] [2] works on a joint pricing model for stocks and bonds. They build a bond pricing model with the potential approach and they incorporate it into the dividend discounted cash flow pricing of stocks that is consistent with the no-arbitrage condition. Our work is particularly concerned with the estimation of risk premiums; on the other hand, their work focuses only on the pricing of stocks and bonds.

Let us consider state variable X_t, following the Ornstein-Uhlenbeck process under the physical measure $\mathbf{P}$:

$$dX_t = K_X^{\mathbf{P}}(\theta^{\mathbf{P}} - X_t)dt + \Sigma_X dW_{t,1}^{\mathbf{P}}, \tag{1}$$

where $W_{t,1}^{\mathbf{P}}$ is an N dimensional Brownian motion. Moreover, we assume that Σ_X is a diagonal matrix with its diagonal elements all positive.

In addition to X_t, we define another state variable Y_t as follows:

$$dY_t = \mu^{\mathbf{P}} dt + K_Y^{\mathbf{P}} X_t dt + \Sigma_{Y,1} dW_{t,1}^{\mathbf{P}} + \Sigma_{Y,2} dW_{t,2}^{\mathbf{P}}, \tag{2}$$

where $W_{t,2}^{\mathbf{P}}$ is an M dimensional Brownian motion with $cov(W_{t,1}^{\mathbf{P}}, W_{t,2}^{\mathbf{P}}) = 0_{N\times M}$. As can be seen from equation (2), Y_t is the non-stationary process.

In order to model the state variable process under the risk-neutral measure $\mathbf{Q}$, we define the market price of risk. This allows us to bridge between $\mathbf{P}$ and $\mathbf{Q}$. We assume that there exists the market price of risk for $W_{t,1}^{\mathbf{P}}$ and $W_{t,2}^{\mathbf{P}}$. Let $\Lambda_{t,1}$ be defined such that $dW_{t,1}^{\mathbf{Q}} = dW_{t,1}^{\mathbf{P}} + \Lambda_{t,1} dt$ where $W_{t,1}^{\mathbf{Q}}$ is an N dimensional Brownian motion under the risk-neutral measure. We also assume that $dW_{t,2}^{\mathbf{Q}} = dW_{t,2}^{\mathbf{P}} + \Lambda_{t,2} dt$ where $W_{t,2}^{\mathbf{Q}}$ is an M dimensional Brownian motion under the risk-neutral measure. Specifically, we model $\Lambda_{t,1}$ as $\Lambda_{t,1} = \lambda_0 + \Lambda_1 X_t$ and $\Lambda_{t,2}$ as $\Lambda_{t,2} = \lambda_1 + \Lambda_2 X_t$, according to [7]. This is called the essentially affine setting. Describing X_t under $\mathbf{Q}$ as follows:

$$dX_t = K_X^{\mathbf{Q}}(\theta^{\mathbf{Q}} - X_t)dt + \Sigma_X dW_{t,1}^{\mathbf{Q}}, \tag{3}$$

we find the following relationships from equations (1) and (3)

$$K_X^{\mathbf{P}}\theta^{\mathbf{P}} = K_X^{\mathbf{Q}}\theta^{\mathbf{Q}} + \Sigma_X \lambda_0, \quad K_X^{\mathbf{P}} = K_X^{\mathbf{Q}} - \Sigma_X \Lambda_1. \tag{4}$$

Under $\mathbf{Q}$, the non-stationary state process, Y_t, is given by

$$dY_t = \mu^{\mathbf{Q}} dt + K_Y^{\mathbf{Q}} X_t dt + \Sigma_{Y,1} dW_{t,1}^{\mathbf{Q}} + \Sigma_{Y,2} dW_{t,2}^{\mathbf{Q}}. \tag{5}$$

Thus, the essentially affine setting, equations (2), and (5) lead to the following relationships:

$$\mu^{\mathbf{P}} = \mu^{\mathbf{Q}} + \Sigma_{Y,1}\lambda_0 + \Sigma_{Y,2}\lambda_1, \quad K_Y^{\mathbf{P}} = K_Y^{\mathbf{Q}} + \Sigma_{Y,1}\Lambda_1 + \Sigma_{Y,2}\Lambda_2. \tag{6}$$

Next, the risk-free short rate, r_t, is defined as the quadratic form of the state variable, X_t, in this study:

$$r_t = X_t' \Psi X_t, \tag{7}$$

where the superscript of X_t represents the transposition of X_t and Ψ is assumed to be positive definite. This setting ensures the positivity of the short rate r_t. Additionally, we note that we sometimes denote r_t by $r(t, X_t)$ in order to emphasize on that r_t depends on X_t.

2.2 Bond Pricing

Under the above-indicated setting, we derive the zero-coupon bond-pricing formula. First, the zero-coupon bond price $P^{T-t}(t, X_t)$ at time t with maturity T is described as

$$P^{T-t}(t, X_t) = E_t^{\mathbf{Q}}\left[\exp\left(-\int_t^T r(u, X_u)du\right)\right], \tag{8}$$

where $E_t^{\mathbf{Q}}[\ ]$ is the conditional expectation operator under $\mathbf{Q}$, given the filtration $\mathcal{F}_t$ which $W_t^{\mathbf{Q}}$ generates until time t. We immediately find that the zero-coupon bond price always becomes less than one because the short rate takes a positive value from equation (7). This means that the zero-coupon yields also take positive values.

Applying the Feynman-Kac theorem to equation (8), we obtain the following partial differential equation for $P^\tau(t, X_t)$ $(\tau = T - t)$.

$$\begin{aligned} &\frac{\partial P^\tau(t,X_t)}{\partial t} + \kappa^{\mathbf{Q}}(t,X_t)'\frac{\partial P^\tau(t,X_t)}{\partial X_t} \\ &\qquad - r(t,X_t)P^\tau(t,X_t) + \frac{1}{2}\mathrm{Tr}\left(\Sigma_X\Sigma_X'\frac{\partial^2 P^\tau(t,X_t)}{\partial^2 X_t}\right) = 0, \\ &P^0(T, X_T) = 1, \end{aligned} \tag{9}$$

where $\kappa^{\mathbf{Q}}(t, X_t) = K_X^{\mathbf{Q}}(\theta^{\mathbf{Q}} - X_t)$.

An attempting at finding the solution to equation (9) takes the form given by

$$P^\tau(t, X_t) = \exp\left(X_t' A_\tau X_t + b_\tau' X_t + c_\tau\right). \tag{10}$$

Computing derivatives based on equation (10), we obtain

$$\begin{aligned} \frac{\partial P^\tau(t,X)}{\partial t} &= \left(X'\frac{\mathrm{d}A_\tau}{\mathrm{d}t}X + \frac{\mathrm{d}b_\tau'}{\mathrm{d}t}X + \frac{\mathrm{d}c_\tau'}{\mathrm{d}t}\right)P^\tau(t,X), \\ \frac{\partial P^\tau(t,X)}{\partial X} &= \left((A_\tau' + A_\tau)X + b_\tau\right)P^\tau(t,X), \\ \frac{\partial^2 P^\tau(t,X)}{\partial^2 X} &= \{(A_\tau' + A_\tau)XX'(A_\tau' + A_\tau) + (A_\tau' + A_\tau)Xb_\tau' \\ &+ b_\tau X'(A_\tau' + A_\tau) + (A_\tau' + A_\tau + b_\tau b_\tau')\}P^\tau(t,X). \end{aligned} \tag{11}$$

Substituting equation (11) into equation (9), we obtain the differential equations from the conditions that the coefficient corresponding to each degree of X_t must each become equal to zero, respectively:

$$\begin{aligned} &\tfrac{\mathrm{d}A_\tau}{\mathrm{d}t} = K_X^{\mathbf{Q}'}(A_\tau + A_\tau') + \Psi - \tfrac{1}{2}(A_\tau' + A_\tau)\Sigma_X\Sigma_X'(A_\tau' + A_\tau),\ \ A_0 = 0_{N\times N} \\ &\tfrac{\mathrm{d}b_\tau'}{\mathrm{d}t} = -(K_X^{\mathbf{Q}}\theta^{\mathbf{Q}})'(A_\tau' + A_\tau) + b_\tau' K_X^{\mathbf{Q}} - b_{T-t}'\Sigma_X\Sigma_X'(A_\tau' + A_\tau),\ \ b_0 = 0_{N\times 1} \\ &\tfrac{\mathrm{d}b_\tau}{\mathrm{d}t} = -(K_X^{\mathbf{Q}}\theta^{\mathbf{Q}})'b_{T-t} - \tfrac{1}{2}\mathrm{Tr}(\Sigma_X\Sigma_X'(A_\tau' + A_\tau) + b_\tau b_\tau'),\ \ c_0 = 0 \end{aligned} \tag{12}$$

In general, equation (12) does not have the closed-form solution. Therefore, to compute the solutions, we rest on the numerical method, such as the Runge-Kutta method.

2.3 Stock Price Modeling

2.3.1 Finite-Maturity Stock

In this subsection, we define the finite-maturity stock and provide a sufficient condition for the unique existence of this stock price.

First, we consider the stock with finite maturity. This stock continuously pays dividend $D(t, Z_t)$ per time to stockholders, where Z_t denotes $Z_t' = (X_t', Y_t')$. At maturity T, this stock pays the terminal dividend $\overline{D}(T, Z_T)$. For the period after T, it is assumed that this stock no longer generates cash flow. This finite-maturity stock is bought and sold for price $S^T(t)$ at time t.

The cumulative discounted gain from time 0 to time t $(t \leq T)$, $g^T(t)$ is described as

$$g^T(t) = \int_0^t \exp\left(-\int_0^s r(u, X_u)du\right) D(s, Z_s)ds + \exp\left(-\int_0^t r(s, X_s)ds\right) S^T(t). \quad (13)$$

Under the $\mathbf{Q}$ measure, $g^T(t)$ has to become a martingale. Thus, the relationship $g^T(t) = E_t^{\mathbf{Q}}[g^T(T)]$ holds. This relationship and equation (13) lead to the following equation for the stock price at t.

$$S^T(t) = E_t^{\mathbf{Q}}\left[\int_t^T \exp\left(-\int_t^s r(u, X_u)du\right) D(s, Z_s)ds + \exp\left(-\int_t^T r(s, X_s)ds\right)\overline{D}(T, Z_T)\right]. \quad (14)$$

Here, assuming the transversality condition for the terminal dividend given by

$$\lim_{T\to\infty} E_t^{\mathbf{Q}}\left[\exp\left(-\int_t^T r(s, X_s)ds\right)\overline{D}(T, Z_T)\right] = 0, \quad (15)$$

from equation (14), we can derive the limit of the finite maturity stock price as the maturity approaches infinity:

$$\lim_{T\to\infty} S^T(t) = \lim_{T\to\infty} E_t^{\mathbf{Q}}\left[\int_t^T \exp\left(-\int_t^s r(u, X_u)du\right) D(s, Z_s)ds\right]. \quad (16)$$

Here, let us specify the dividend. First, we model the dividend $D(t, Z_t)$ as follows:

$$D(t, Z_t) = (\delta_0 + \delta_1' X_t + X_t'\Phi X_t)\exp\left(kt + d'X_t + X_t'EX_t + c'Y_t\right), \quad (17)$$

where Φ and E are assumed to be symmmetric. As for the terminal dividend at maturity T, we model it by

$$\overline{D}(T, Z_T) = \exp\left(kT + d'X_T + X_T'EX_T + c'Y_T\right). \quad (18)$$

Equations (14), (17), (18), and the Feynman-Kac theorem allow us to derive the partial differential equation for $S^T(t)$. (Afterwards, we sometimes denote $S^T(t)$ by $S^T(t, Z_t)$ to emphasize that the price depends on the state variable Z_t)

$$\begin{aligned}
&\tfrac{\partial S^T(t,X)}{\partial t} + \tfrac{\partial S^T(t,Z)}{\partial X}\kappa^{\mathbf{Q}}(X) + \tfrac{\partial S^T(t,Z)}{\partial Y}\tilde{\kappa}^{\mathbf{Q}}(X) + \tfrac{1}{2}\mathrm{Tr}\left(\Sigma_X\Sigma_X'\tfrac{\partial^2 S^T(t,X)}{\partial^2 X}\right)\\
&+\tfrac{1}{2}\mathrm{Tr}\left(\Sigma_Y\Sigma_Y'\tfrac{\partial^2 S^T(t,Z)}{\partial^2 Y}\right) + \mathrm{Tr}\left(\Sigma_X\Sigma_Y'\tfrac{\partial^2 S^T(t,Z)}{\partial X\partial Y}\right)\\
&-r(t,X)S^T(t,Z) + D(t,Z) = 0, \\
&S^T(T, Z_T) = \overline{D}(T, Z_T) = \exp(kT + d'X_T + X_T'EX_T + c'Y_T),
\end{aligned} \quad (19)$$

where $\kappa^{\mathbf{Q}}(X) = K_X^{\mathbf{Q}}(\theta^{\mathbf{Q}} - X)$ and $\tilde{\kappa}^{\mathbf{Q}}(X) = \mu^{\mathbf{Q}} + K_Y^{\mathbf{Q}} X$.

An estimate of the solution to equation (19) has the form given by $S^T(t, Z) = \exp(kt + d'X_t + X_t'EX_t + c'Y_t)$. Then, we substitute this form and equation (17) into equation (19). Each coefficient of X_t with respect to degree has to be a zero matrix, zero vector, or zero; hence,

$$
\begin{aligned}
&-2EK_X^{\mathbf{Q}} + 2E\Sigma_X\Sigma_X'E + \Phi - \Psi = 0_{N\times N},\\
&(-K_X^{\mathbf{Q}\prime} + 2E\Sigma_X\Sigma_X')d + 2EK_X^{\mathbf{Q}}\theta^{\mathbf{Q}} + 2E\Sigma_X\Sigma_Y c + \delta_1 = 0_{N\times 1},\\
&k + d'K_X^{\mathbf{Q}}\theta^{\mathbf{Q}} + \tfrac{1}{2}\mathrm{Tr}(\Sigma_X\Sigma_X'(2E + dd')) + \tfrac{1}{2}\mathrm{Tr}(\Sigma_Y\Sigma_Y'cc')\\
&+c'\Sigma_Y\Sigma_X + c'\Sigma_Y\Sigma_X + \delta_0 + c'\mu^{\mathbf{Q}} = 0.
\end{aligned}
\tag{20}
$$

Here, we note that c is a free parameter and E, d, and k are variables to be solved.

Taking the transposition of the first equation of equation (20), adding it to the first equation and multiplying it by one half, we obtain the following equation:

$$
EK_X^{\mathbf{Q}} + K_X^{\mathbf{Q}\prime}E - 2E\Sigma_X\Sigma_X'E + \Psi - \Phi = 0_{N\times N}. \tag{21}
$$

Equation (21) is called an algebraic Riccati Equation. In general, there is no solution to this equation. However, a sufficient condition for the unique existence of a solution in an algebraic Riccati equation is known. The condition is that a matrix pair $(K_X^{\mathbf{Q}}, \Sigma_X)$ is controllable and $\Psi - \Phi$ is positive definite. The solution $\hat{E}$ of this equation becomes positive definite.

In terms of the existence of the solution to the second equation of equation (21), we have to examine whether or not the coefficient of d, $-K_X^{\mathbf{Q}\prime} + 2\hat{E}\Sigma_X\Sigma_X'$ has an inverse matrix. This coefficient becomes equal to $\hat{E}^{-1}(\Psi - \Phi + \hat{E}K_X^{\mathbf{Q}})$ from the first equation. Hence, we have to only examine whether $\Psi - \Phi + \hat{E}K_X^{\mathbf{Q}}$ has an inverse matrix. The matrix obtained by adding this matrix to the transposition of this matrix is positive definite. This is because the matrix becomes equal to $\Psi - \Phi + 2\hat{E}\Sigma_X\Sigma_X'\hat{E}$, which is positive, due to the assumption of the positivity of the diagonal elements of the diagonal matrix Σ_X. If the matrix is positive, then it has an inverse matrix. As a result, we find that the coefficient of d, $-K_X^{\mathbf{Q}\prime} + 2\hat{E}\Sigma_X\Sigma_X'$ has an inverse matrix, so that d in the second equation always has the solution. Once we find the solution of E and d, we easily compute k from the third equation.

Summing up the above discussion, we obtain the following proposition and lemma.

Proposition 1. State variables X_t and Y_t follow equations (3) and (5), respectively. The volatility term of X_t, Σ_X is assumed to be a diagonal matrix with positive diagonal elements. As for r_t, we define it by equation (7). In addition, the dividend $D(t, Z_t)$ of the finite-maturity stock is given by equation (17). Here, E and Φ are symmetric. Furtheremore, if we assume that $\Psi - \Phi$ is positive definite and a matrix pair $(K_X^{\mathbf{Q}}, \Sigma_X)$ is controllable, then the finite-maturity stock price $S^T(t, Z_t)$ has the following representation:

$$
S^T(t, Z_t) = \exp(kt + d'X_t + X_t'EX_t + c'Y_t),
$$

where k, d and E are obtained from equation (20) while c is a free parameter.

Lemma 1. Under the condition presented in Proposition 1, $S^T(t, X_t) = S^{T'}(t, X_t)$ $(T \neq T')$.

Proof. $S^T(t, Z_t) = \exp(kt + d'X_t + X_t'EX_t + c'Y_t)$ does not depend on the time to maturity. Hence, $S^T(t, X_t) = S^{T'}(t, X_t)$ $(T \neq T')$. $\square$

Since the dividend $D(t, Z_t) = (\delta_0 + \delta_1'X_t + X_t'\Phi X_t)\exp(kt + d'X_t + X_t'EX_t + c'Y_t)$ and $S(t, Z_t) = \exp(kt + d'X_t + X_t'EX_t + c'Y_t)$, $\delta_0 + \delta_1'X_t + X_t'\Phi X_t$ is interpreted as the dividend yield.

Next, imposing condition (15) on the finite-maturity stock price leads to the following proposition.

Proposition 2. Under the condition presented in Proposition 1, assuming the transversality condition (15), we obtain the following relationship:

$$\lim_{T\to\infty} E_t^{\mathbf{Q}}\left[\int_t^T \exp\left(-\int_t^s r(u, X_u)du\right) D(s, Z_s)ds\right] = \exp(kt + d'X_t + X_t'EX_t + c'Y_t)$$

Proof. By Lemma 1,

$$\lim_{T\to\infty} S^T(t) = S^T(t) = \exp(kt + d'X_t + X_t'EX_t + c'Y_t).$$

On the other hand, by equation(16),

$$\lim_{T\to\infty} E_t^{\mathbf{Q}}\left[\int_t^T \exp\left(-\int_t^s r(u, X_u)du\right) D(s, Z_s)ds\right] = \lim_{T\to\infty} S^T(t).$$

Therefore,

$$\lim_{T\to\infty} E_t^{\mathbf{Q}}\left[\int_t^T \exp\left(-\int_t^s r(u, X_u)du\right) D(s, Z_s)ds\right] = \exp(kt + d'X_t + X_t'EX_t + c'Y_t).$$

$\square$

2.3.2 Infinite-Maturity Stock

In this subsection, let us consider an infinite-maturity stock. As is the case with a finite-maturity stock, we postulate that an infinite-maturity stock continuously pays the dividend $D(t, Z_t)$ per time to holders. Let $S^\infty(t)$ denote the price of this stock at time t. The discounted gain process from time 0, $g^\infty(t)$ is given by

$$g^\infty(t) = \int_0^t \exp\left(-\int_0^s r_u du\right) D(s, Z_s)ds + \exp\left(-\int_0^t r_s ds\right) S^\infty(t). \tag{22}$$

We note that $g^\infty(t)$ needs to be a martingale under the $\mathbf{Q}$ measure. Hence, $g^\infty(t) = E_t^{\mathbf{Q}}[g^\infty(T)]$ holds. This relationship and equation (22) lead to the equation given by

$$S^\infty(t) = E_t^{\mathbf{Q}}\left[\int_t^T \exp\left(-\int_t^s r_u du\right) D(s, Z_s)ds + \exp\left(-\int_t^T r_s ds\right) S^\infty(T)\right]. \tag{23}$$

This holds for $T \geq t$.

Now, imposing the transversality condition for the infinite-maturity stock price given by

$$\lim_{T\to\infty} E_t^{\mathbf{Q}}\left[\exp\left(-\int_t^T r(s,X_s)ds\right)S^\infty(T)\right]=0, \tag{24}$$

equation (23) reduces to the following

$$S^\infty(t)=\lim_{T\to\infty} E_t^{\mathbf{Q}}\left[\int_t^T \exp\left(-\int_t^s r(u,X_u)du\right)D(s,Z_s)ds\right]. \tag{25}$$

From the above, we obtain the proposition:

Proposition 3. If two transversality conditions (15) and (24) hold under the condition presented in Proposition 1, then the infinite-maturity stock price has the following representation:

$$S^\infty(t)=\exp(kt+d'X_t+X_t'EX_t+c'Y_t).$$

Accordingly, the infinite-maturity-stock price representation is obtained when the two transversality conditions hold.

2.3.3 Theorem on Stock Price Representation

With Proposition 3, when the two transversality conditions hold, the infinite-maturity stock price has a closed-form representation. Here, let us discuss the sufficient condition for terminal transversality.

According to [11], as for the transversality condition for the dividend, we can prove the following proposition.

Proposition 4. If the dividend yield $\delta_0+\delta_1'X_t+X_t'\Phi X_t>0$, then the transversality condition for the terminal dividend (15) holds.

Proof. First, we denote the dividend yield $\delta_0+\delta_1'X_t+X_t'\Phi X_t$ by $\delta(t,X_t)$. Let us define ζ_t by

$$\zeta_t=\exp\left(\int_0^t(\delta(u,X_u)-r(u,X_u))du\right)\overline{D}(t,Z_t).$$

Applying Ito's lemma into this equation, we can write the process of ζ_t given by

$$d\zeta_t=\exp\left(\int_0^t(\delta(u,X_u)-r(u,X_u))du\right)\frac{\partial\overline{D}(t,Z_t)}{\partial Z_t}\Sigma_Z dW_t^{\mathbf{Q}}.$$

This leads to the following equation:

$$\zeta_T=\zeta_t+\int_t^T\exp\left(\int_0^s(\delta(u,X_u)-r(u,X_u))du\right)\frac{\partial\overline{D}(s,Z_s)}{\partial Z_s}\Sigma_Z dW_s^{\mathbf{Q}}.$$

The dividend yield $\delta_0+\delta_1'X_t+X_t'\Phi X_t>0$ is equivalent to $\delta_0>\frac{1}{4}\delta_1'\Phi^{-1}\delta_1$.

Let us denote the second term of the right hand side of the above equation by $I_t(T)$. $I_t(T)$ is a local martingale because it is a stochastic integral with respect to a Brownian motion. In addition, $I_t(T)$ is more than $-\zeta_t$, since ζ_t is, by definition, positive. Hence, we find that $I_t(T)$ is a super-martingale because it is a local martingale with a lower bound.

Here, we assume that $\delta(t, X_t) \geq \epsilon > 0$. From this, we have the following inequality:

$$\exp\left(\int_0^T (\delta(u, X_u) - r(u, X_u))du\right) \overline{D}(T, Z_T) > e^{\epsilon T} \exp\left(\int_0^T -r(u, X_u)du\right) \overline{D}(T, Z_T).$$

Hence,

$$e^{\epsilon T} \exp\left(\int_0^T -r(u, X_u)du\right) \overline{D}(T, Z_T) < \zeta_t + I_t(T).$$

Taking the expecataion for both sides of the above inequality,

$$e^{\epsilon T} E_t^{\mathbf{Q}}\left[\exp\left(\int_0^T -r(u, X_u)du\right) \overline{D}(T, Z_T)\right] < \zeta_t + E_t^{\mathbf{Q}}[I_t(T)] \leq \zeta_t + I_t(t) = \zeta_t,$$

where the last inequality is given by the fact that $I_t(T)$ is a super-martingale. This leads to the following inequality,

$$E_t^{\mathbf{Q}}\left[\exp\left(\int_0^T -r(u, X_u)du\right) \overline{D}(T, Z_T)\right] < e^{-\epsilon T}\zeta_t.$$

Therefore, the left hand side of the above inequality approaches zero as T approaches infinity. $\square$

Let us consider the transversality condition for the infinite-stock price.

Proposition 5. If the finite-maturity stock price and the infinite-maturity stock price uniquely exist and the transversality condition for the terminal dividend holds, then the transversality condition for the infinite-maturity stock price holds.

Proof. If $S^\infty(t, Z_t) = S(t, Z_t)$, then equation (23) is satisfied. Then, since $S^\infty(t, Z_t) = S(t, Z_t) = \overline{D}(t, Z_t)$ for any time t and the transversality condition for the dividend holds,

$$\lim_{T\to\infty} E_t^{\mathbf{Q}}\left[\exp\left(-\int_t^T r_s ds\right) \overline{D}(T, Z_T)\right] = \lim_{T\to\infty} E_t^{\mathbf{Q}}\left[\exp\left(-\int_t^T r_s ds\right) S^\infty(T, Z_T)\right] = 0.$$

Therefore, the transversality condition for the infinite-maturity stock price holds. $\square$

Summing up our discussion so far, we have the following theorem.

Theorem 1. We assume the following:

- Φ and $\Psi - \Phi$ are positive definite,
- a matrix pair $(K_X^{\mathbf{Q}}, \Sigma_X)$ is controllable,
- $\delta_0 > \frac{1}{4}\delta_1' \Phi^{-1} \delta_1$.

Then, the non-defaultable stock price is well defined and has the following representation:

$$S(t, Z_t) = exp(kt + d'X_t + X_t' E X_t + c' Y_t),$$

where the coefficients of X_t and Y_t satisfy the stock-price matrix equation given by

$$-2EK_X^{\mathbf{Q}} + 2E\Sigma_X\Sigma_X' E + \Phi - \Psi = 0_{N\times N},$$
$$(-K_X^{\mathbf{Q}} + 2E\Sigma_X\Sigma_X')d + 2EK_X^{\mathbf{Q}}\theta^{\mathbf{Q}} + \delta_1 = 0_{N\times 1},$$
$$k + d'K_X^{\mathbf{Q}}\theta^{\mathbf{Q}} + \frac{1}{2}\mathrm{Tr}(\Sigma_X\Sigma_X'(2E + dd')) + \delta_0 = 0.$$

2.4 Processes of Stock and Bond Prices, Correlation, and Risk Premiums

In this subsection, first, we write down the stochastic processes of stock and bond prices. This gives us the instantaneous excess returns and volatilities of stock and bond prices. In addition, we derive the correlation representation between stock and bond prices. These evolve, depending on the state variable X_t. We find that our model can provide more flexible structures to stock and bond dynamics and their dependencies.

Furthermore, we give the definitions of term premium and equity risk premium. In section 4, we estimate these by using Japanese data.

First of all, we write the stochastic process of the n-year zero-coupon bond price under the $\mathbf{P}$ measure. Using equation (10) and the fact that the drift term of the price return is equal to r_t under $\mathbf{Q}$, the stochastic process under $\mathbf{Q}$ is given by

$$\frac{dP^n(t)}{P^n(t)} = r_t dt + \{(A_n + A_n')X_t + b_n\}' \Sigma_X dW_{t,1}^{\mathbf{Q}}. \tag{26}$$

Here, remembering the setting of the essentially affine risk premium, or $\Lambda_t = \lambda_0 + \Lambda_1 X_t$, we obtain the bond-price process under $\mathbf{P}$ as follows:

$$\begin{aligned} \frac{dP^n(t)}{P^n(t)} &= (r_t + \{(A_n + A_n')X_t + b_n\}' \Sigma_X(\lambda_0 + \Lambda_1 X_t))dt \\ &\quad + \{(A_n + A_n')X_t + b_n\}' \Sigma_X dW_{t,1}^{\mathbf{P}}. \end{aligned} \tag{27}$$

From equation (27), we find that the instantaneous excess return and the volatility of the bond-price return depends on the state variable X_t.

Next, let us derive the stock price process under $\mathbf{P}$. The cumulative gain associated with holding the stock from time 0, $\tilde{g}(t)$ is given by

$$\tilde{g}(t) = \int_0^t D(s)ds + S(t) = \int_0^t (\delta_0 + \delta_1' X_s + X_s'\Phi X_s)S(s)ds + S(t). \tag{28}$$

Hence, from equation (28) and Ito's lemma, the process of the return associated with holding the stock is given under $\mathbf{Q}$,

$$\begin{aligned} \frac{d\tilde{g}(t)}{S(t)} &= (\delta_0 + \delta_1' X_t + X_t'\Phi X_t)dt + \frac{dS(t)}{S(t)} \\ &= r_t dt + ((d + 2EX_t)'\Sigma_X + c'\Sigma_{Y,1})dW_{t,1}^{\mathbf{Q}} + c'\Sigma_{Y,2}dW_{t,2}^{\mathbf{Q}}. \end{aligned} \tag{29}$$

The second equality is due to the fact that the drift term must be the risk-free rate under $\mathbf{Q}$.

By equation (29) and the assumption of the essentially affine risk premium, we obtain the return process associated with holding the stock under $\mathbf{P}$,

$$\begin{aligned} \frac{d\tilde{g}(t)}{S(t)} =& r_t dt + ((d + 2EX_t)'\Sigma_X + c'\Sigma_{Y,1})(\lambda_0 + \Lambda_1 X_t)dt \\ &+ ((d + 2EX_t)'\Sigma_X + c'\Sigma_{Y,1})dW_{t,1}^{\mathbf{P}} + c'\Sigma_{Y,2}dW_{t,2}^{\mathbf{P}}. \end{aligned} \tag{30}$$

As with the bond return, excess return and the volatility of a stock holding's return depends on the state variable X_t.

Equation (27) and (30) lead to the following correlation between the bond return and the stock holding return.

$$\frac{(D'\Sigma_X + c'\Sigma_{Y,1})\Sigma_X' \left\{\tilde{A}_n X_t + b_n\right\}}{\sqrt{(D'\Sigma_X + c'\Sigma_{Y,1})(D'\Sigma_X + c'\Sigma_{Y,1})' + c'\Sigma_{Y,2}\Sigma_{Y,2}'c}\sqrt{\left\{\tilde{A}_n X_t + b_n\right\}' \Sigma_X \Sigma_X' \left\{\tilde{A}_n X_t + b_(t)\right\}}}, \tag{31}$$

where $\tilde{A}_n = A_n + A_n'$ and $D = d + 2EX_t$.

Next, let us define the bond and equity risk premium. As for the bond risk premium, the term premium defined below is estimated and analyzed in many empirical studies. The term premium TP_t^n of the n-year zero-coupon yield at time t is defined as:

$$\begin{aligned} TP_t^n &= \frac{1}{n}\log E_t^{\mathbf{P}}\left[\exp\left(-\int_t^{t+n} r(u, X_u)du\right)\right] - \frac{1}{n}\log E_t^{\mathbf{Q}}\left[\exp\left(-\int_t^{t+n} r(u, X_u)du\right)\right] \\ &= y_t^n + \frac{1}{n}\log E_t^{\mathbf{P}}\left[\exp\left(-\int_t^{t+n} r(u, X_u)du\right)\right], \end{aligned} \tag{32}$$

where y_t^n represents the n-year zero-coupon yield.

The equity risk premium EP_t^n for n years is defined as the excess return of the expected holding return of the stock for n years over the n-year zero-coupon yield. Expressing the mathematical form, we define it as follows:

$$EP_t^n = \frac{1}{n}E_t^{\mathbf{P}}\left[\int_t^{t+n} D(s)ds + S(t+n) - S(t)\right]/S(t) - y_t^n. \tag{33}$$

In section 4, we estimate and analyze the term premium and the equity risk premium based on Japanese stock and bond data.

3 Estimation Methodology

In this section, we explain the estimation methodology we use for conducting our empirical studies. Our model can be regarded as the state-space model. As the observation equation becomes a nonlinear function, we can not use the Kalman filter in order to estimate the latent state variables. Thus, we apply the unscented Kalman filter, one of the nonlinear filters, to actual financial market data. Furthermore, we estimate the model parameter by using the quasi-maximum likelihood method.

3.1 State Space Representation

Moving towards our empirical study, we approximate our continuous-time model to the discrete-time model. We set the unit of time as one month. We can write X_t's and Y_t's processes in discrete time as follows:

$$\begin{aligned} X_{t+1} &= \exp(-K_X^{\mathbf{P}})X_t + (I - \exp(-K_X^{\mathbf{P}}))\theta^{\mathbf{P}} + \sqrt{V}\epsilon_{1,t+1}^{\mathbf{P}} \\ Y_{t+1} &= Y_t + \mu^{\mathbf{P}} + K_Y^{\mathbf{P}} X_t + \Sigma_{Y,2}\epsilon_{2,t+1}^{\mathbf{P}} \end{aligned}$$

where $V = \int_{-1}^{0} e^{K_X^{\mathbf{P}} u} \Sigma_X \Sigma_X' e^{K_X^{\mathbf{P}} u'} du$ and $\sqrt{V}$ represents the Cholesky decomposition of V. In addition, $\epsilon_{1,t+1}^{\mathbf{P}}$ and $\epsilon_{2,t+1}^{\mathbf{P}}$ are random variables, each with a standard normal distribution, and each is independent from the other. For simplicity's sake here, we assume that $\Sigma_{Y,1}$ in equation (2) is the zero matrix. The above equation indicates the transition equation of the state variable Z_t. Summarizing X_t's and Y_t's processes, we write this as $Z_{t+1} = f(Z_t) + \Sigma \epsilon_{t+1}$.

We specify the setting of this transition equation. This specification imposes some restrictions on the model parameters. As a result, it becomes possible for us to estimate the parameters without ending up with over-fitting or under-fitting.

In our empirical analysis, we let N, the dimension of X_t be 3, while we do not consider Y_t; in other words, we let M be 0. This setting aims to reduce the computational burden associated with an estimation. Here, we note that X_t with 3 factors is flexible enough to represent a variety of shapes that the yield curve generally adopt.

Considering X_t's correlation structure, we find that it is determined by $K_X^{\mathbf{P}}$ and Σ_X. However, since the correlation matrix is symmetric, $K_X^{\mathbf{P}}$ and Σ_X are unidentifiable in an estimation. Thus, we assume that $K_X^{\mathbf{P}}$ is a lower triangular matrix and Σ_X is a diagonal matrix.

Furthermore, according to the invariant transformation by [6] and [1] in order to exclude an arbitrariness associated with a nonsingular linear transformation of X_t, we can assume that Σ_X is the following matrix:

$$\Sigma_X = \begin{bmatrix} 0.1 & 0 & 0 \\ 0 & 0.1 & 0 \\ 0 & 0 & 0.1 \end{bmatrix}.$$

If we deal with the case where $M > 1$, by the same argument as Σ_X, we can assume that $\Sigma_{Y,2}$ is the following matrix:

$$\Sigma_{Y,2} = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix},$$

where we let M be 2.

We impose a restriction on Λ_t, the market price of risk, in order that $K_X^{\mathbf{Q}}$ becomes a lower triangular matrix. That is, $K_X^{\mathbf{Q}}$ and $K_X^{\mathbf{P}}$ have the same matrix form. If Λ_1 is a lower triangular matrix, then $K_X^{\mathbf{Q}}$ becomes a lower triangular matrix from equation (4) because Σ_X is diagonal. Hence, we assume that Λ_1 is a lower triangular. As for λ_0, we do not have any restrictions.

The measurement equation is defined by

$$T_t = \begin{bmatrix} zyield_t^{(n_1)} \\ \vdots \\ zyield_t^{(n_l)} \\ logS(t) \\ dyield_t \end{bmatrix} = \begin{bmatrix} g_1(X_t) \\ \vdots \\ g_l(X_t) \\ g_{l+1}(X_t) \\ g_{l+2}(X_t) \end{bmatrix} + \begin{bmatrix} \eta_{t,1} \\ \vdots \\ \eta_{t,l} \\ \eta_{t,l+1} \\ \eta_{t,l+2} \end{bmatrix},$$

where $zyield_t^{(n_i)}$ is the zero-coupon yield with the n_i-month time to maturity at time t, $logS(t)$ is the log stock price, and $dyield_t$ is the dividend yield of the stock. The function $g_i(X_t)$ $(i = 1, \ldots, l)$ is given by

$$g_i(X_t) = -\frac{1}{n_i}\left(X_t' A_{n_i} X_t + b_{n_i}' X_t + c_{n_i}\right),$$

from equation (10). The coefficients A_{n_i}, b_{n_i}, and c_{n_i} are solutions to equation (12).

The function $g_{l+1}(X_t)$ is given by

$$g_{l+1}(X_t) = kt + d' X_t + X_t' E X_t,$$

where k, d, and E are solutions to equation (20), and c is a parameter to be estimated.

The function $g_{l+2}(X_t)$ is given by

$$g_{l+2}(X_t) = \delta_0 + \delta_1' X_t + X_t' \Phi X_t.$$

Furthermore, $(\eta_{t,1}, \ldots, \eta_{t,l}, \eta_{t,l+1}, \eta_{t,l+2})$ is the measurement error following a multivariate normal distribution. This variance-covariance matrix is given by

$$Cov(\eta_{t,1}, \ldots, \eta_{t,l}, \eta_{t,l+1}, \eta_{t,l+2}) = diag(h_1, \cdots, h_1, h_2, h_3) \equiv H.$$

3.2 Unscented Kalman Filter

As we can see in the previous subsection, because the form of $g_i(X_t)$ is nonliner, the measurement equation is a nonlinear function. Hence, we cannot rely on the Kalman filter to estimate the latent state variable X_t. Thus, we instead apply the unscented Kalman filter developed by [14] to the actual financial market data.

The first step of the unscented Kalman filter is an initialization:

$$\hat{X}_0 = E^{\mathbf{P}}[X_0], \quad P_0 = E^{\mathbf{P}}[(X_0 - \hat{X}_0)(X_0 - \hat{X}_0)'].$$

In this initialization, we calculate the unconditional expectation and variance-covariance matrix of the state variable X_t.

Next, let us explain the prediction step of the filter. Here, we calculate the prediction of X_t by using "sigma points." For $k = 1, \ldots, S$ where S is the number of observation dates, we calculate $2N + 1$ points called sigma points, given by

$$\chi_{k-1} = [\hat{X}_{k-1}, \hat{X}_{k-1} + \gamma\sqrt{P_{k-1}}, \hat{X}_{k-1} - \gamma\sqrt{P_{k-1}}],$$

where χ_{k-1} is an $N \times (2N + 1)$ matrix and $\sqrt{P_{k-1}}$ represents the square root matrix of P_{k-1}. $\hat{X}_{k-1} \pm \gamma\sqrt{P_{k-1}}$ in the above equation is defined as follows:

$$\hat{X}_{k-1} \pm \gamma\sqrt{P_{k-1}} = (\hat{X}_{k-1}, \ldots, \hat{X}_{k-1}) \pm \gamma\sqrt{P_{k-1}},$$

where $\gamma = \sqrt{\alpha^2(N + \kappa)}$ is called the scaling parameter. In this work, we assume that $\alpha = 1$ and $\kappa = 0$.

The next step is the time updating of the state variable X_t. Here, we transform the above-defined sigma points by function f as follows:

$$\chi^*_{k|k-1} = [f(\hat{X}_{k-1}), f(\hat{X}_{k-1} + \gamma\sqrt{P_{k-1}}), f(\hat{X}_{k-1} - \gamma\sqrt{P_{k-1}})],$$

where $\chi^*_{k|k-1}$ has an $N \times (2N + 1)$ matrix.

Using each column of $\chi^*_{k|k-1}$, $\chi^*_{i,k|k-1}$, we calculate the“ mean” and “ covariance” of the state variable given by

$$\hat{X}_k^- = \sum_{i=0}^{2N} W_i^m \chi^*_{i,k|k-1},$$
$$P_k^- = \sum_{i=0}^{2N} W_i^c (\chi^*_{i,k|k-1} - \hat{X}_k^-)(\chi^*_{i,k|k-1} - \hat{X}_k^-)' + \Sigma_X \Sigma_X',$$

where weights are assigned as follows:

$$W_0^m = \frac{\alpha^2(N+\kappa) - N}{\alpha^2(N+\kappa)}, \quad W_0^c = W_0^m + 1 - \alpha^2 + \beta,$$
$$W_i^m = W_i^c = \frac{1}{2\alpha^2(N+\kappa)}, \quad i = 1, \dots, 2N,$$

where β, above, is a non-negative weighting parameter used to incorporate knowledge of higher order distribution moments. In this work, we set $\beta = 0$. We define the sigma points of the predicted state variables as follows:

$$\chi_{k|k-1} = [\hat{X}_k^-, \hat{X}_k^- + \gamma\sqrt{P_k^-}, \hat{X}_k^- - \gamma\sqrt{P_k^-}].$$

Next, we transform these sigma points $\chi_{k|k-1}$ with a g function that gives the yields and the log stock price.

$$T_{k|k-1} = [g(\hat{X}_k^-), g(\hat{X}_k^-) + \gamma\sqrt{P_k^-}, g(\hat{Z}_k^- - \gamma\sqrt{P_k^-})],$$
$$T_k^- = \sum_{i=0}^{2N} W_i^m T_{i,k|k-1}.$$

Finally, we obtain the following measurement-update equations,

$$P_{T_k,T_k} = \sum_{i=0}^{2N} W_i^c (T_{i,k|k-1} - \hat{T}_k^-)(T^*_{i,k|k-1} - \hat{T}_k^-)',$$
$$P_{X_k,T_k} = \sum_{i=0}^{2N} W_i^c (\chi_{i,k|k-1} - \hat{X}_k^-)(T_{i,k|k-1} - \hat{T}_k^-)' + H.$$

Using these variance-covariance matrices, we compute the Kalman gain, K_k, given by

$$K_k = P_{X_k,T_k} P_{T_k,T_k}^{-1}.$$

With the Kalman gain, the time update of the conditional expectation and variance of X_t is given by

$$X_k = \hat{X}_k^- + K_k(T_k - \hat{T}_k^-), \quad P_k = P_k^- - K_k P_{T_k,T_k} K_k'.$$

Given the model parameters, we can estimate the latent state variables through the above steps of the unscented Kalman filter.

3.3 Quasi-Maximum Likelihood Method

As for the model parameters, we estimate them by the quasi-maximum likelihood method. The log likelihood function in this model is given by

$$\log L(\Theta) \;=\; -\frac{NS}{2}\log 2\pi - \frac{1}{2}\sum_{k=1}^{S}\left(\log|P_{T_k,T_k}| + (T_k - T_k^-)' P_{T_k,T_k}^{-1}(T_k - T_k^-)\right),$$

where Θ is a set of model parameters and S is the number of our observation dates.

The optimal model parameter Θ is estimated as the solution of the maximization of $\log L(\Theta)$.

3.4 Data

We use monthly data from January 1996 to September 2013 for the JGB zero-coupon yields, the Topix, and the dividend yield of the Topix. Zero-coupon yields are computed from the Broker's JGB prices by the method presented in [12]. The maturities included are six months and 2, 5, 10, and 20 years. Data for the JGB prices, the Topix, and its monthly dividend yield are downloaded from the Nikkei NEEDS Financial Quest.

4 Estimation Result

4.1 Model Fit

In this subsection, we illustrate the fit of our model to the actual financial market data. The estimation of the state variable X_t and the model parameters is conducted based on the methodology described in section 3. The Appendix presents estimates of the model parameters. Estimates for each factor of the state variable X_t are shown in Figure 1.

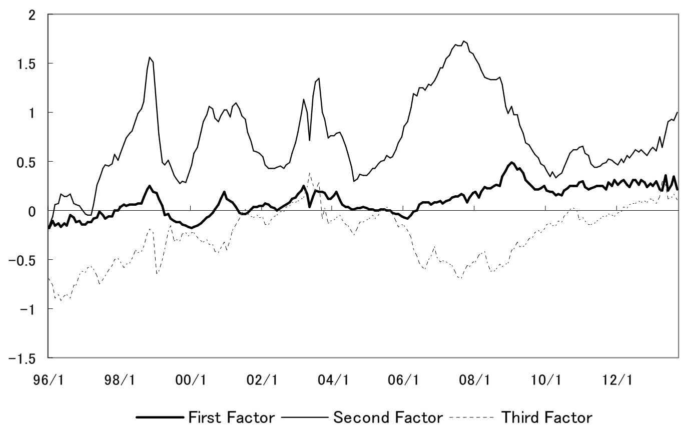


Figure 1: Estimates of the state variable X_t

Based on the estimated parameters and state variable, we can examine the performance of our model. Figure 2 displays the comparison between the observation data

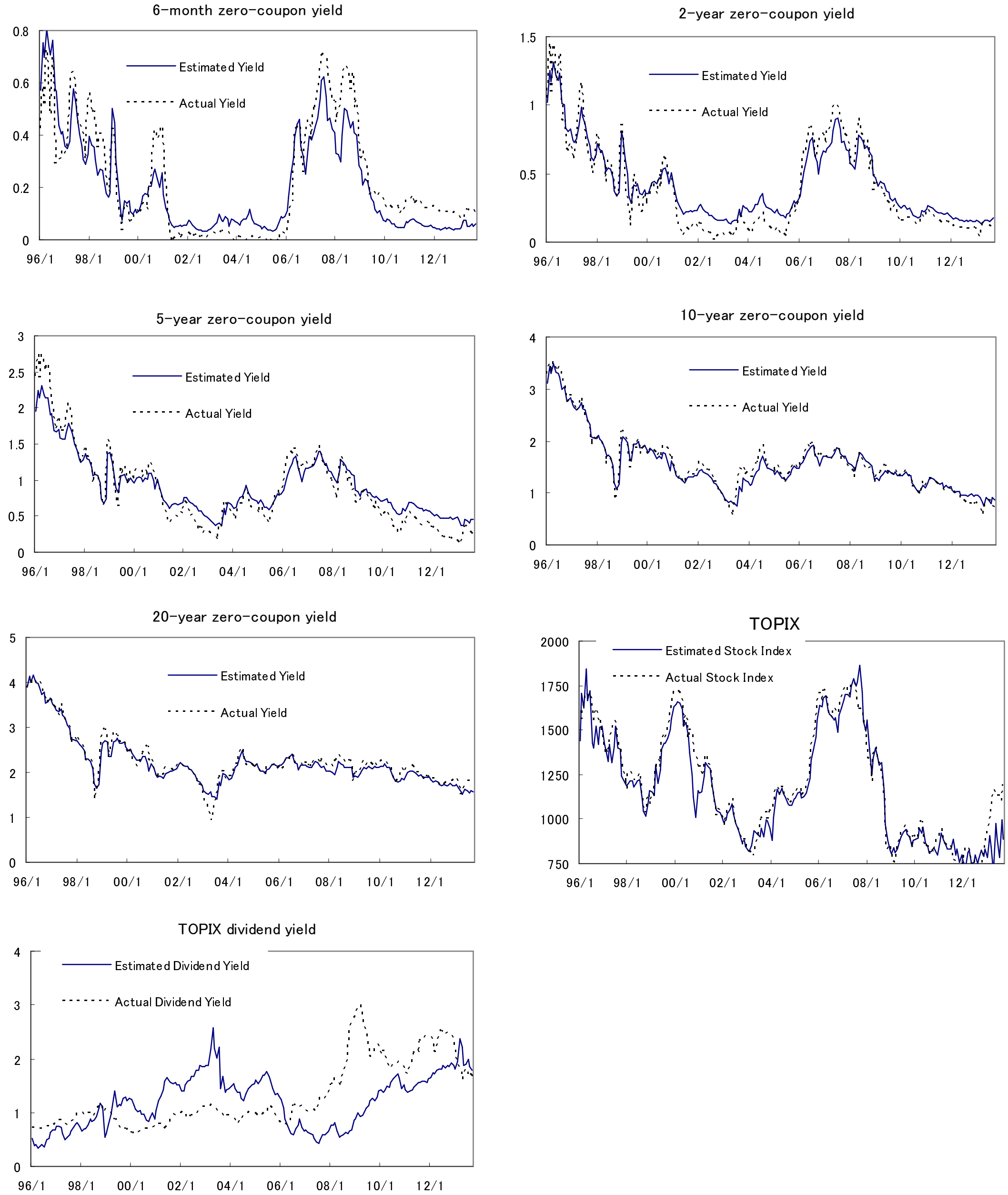


Figure 2: Model Fit: The graphs show a comparison between estimated values and actual data. Yields, including zero coupon and dividend yields, are shown in percentage terms. Two values for Topix are shown in points.

and the estimates of zero-coupon yields, stock index, and dividend yield as calculated from the filtered variable $X_{t|t}$. For the zero-coupon yields and stock price, the estimation results show a good fit to the market data. Table 1 reports the summary statistics of the estimation errors. The table indicates that the mean absolute errors for almost all zero-coupon yields are below ten basis points. For the Topix, the mean absolute relative error is about 5 %. Measuring the explained percentage variation for the Topix, defined as one minus the ratio of error variance to the observation variance, we obtain 91.8 %. Accordingly, our model displays a good fit to the zero-coupon yields and stock index.

The model's performance for the dividend yield is worse than that for the zero-coupon yields and stock index. This can most likely be attributed to the misspecification of our model. In reality, investors value a company's stock not only by dividend amount but also internal reserve amount. With this in mind, the estimated dividend yield of our model can be regarded as the adjustment of the realized dividend yield by internal reserves. Hence, the estimated dividend yield could take a value different from the observed dividend yield. Conducting an estimation with the exclusion of the dividend yield from the measurement data is possible; however, this approach can lead to an unrealistic estimate of the dividend yield. Thus, to obtain a realistic estimate, we incorporate the actual dividend yield into the observation data, although this might introduce measurement errors for the dividend yield to some extent.

	0.5-year	2-year	5-year	10-year
Mean Estimated Error	-1.713	4.186	5.044	-2.400
Mean Absolute Error	7.677	8.320	14.48	9.593

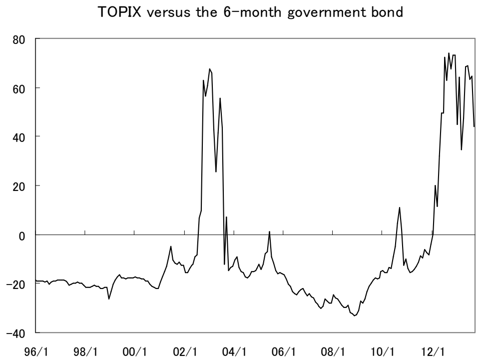
	20-year	Stock	dividend yield
Mean Estimated Error	-1.852	-1.949	-12.60
Mean Absolute Error	8.887	4.946	56.33

Table 1: Summary statistics of the estimation errors on the zero-coupon yields, the Topix, and its monthly dividend yield: The estimation errors on the zero-coupon yields and dividend yield are defned as the difference between the observed data and estimates. These are indicated in basis points. As for the Topix, the estimation error is defined as the relative error, indicated in percentage terms.

4.2 Correlation between Stocks and Bonds

When investors make decisions regarding their asset allocations, they need correlations among financial assets as the input data. However, the use of correlations computed directly from historical returns is at a risk of deterioration in the portfolio diversification in case where correlations of the future largely vary from those of the past. Hence, the correlations should contain forward-looking information on asset prices. As shown in equation (31), our model enables us to compute the implied correlation between stocks and bonds once we estimate the filtered state variable.

Figure 3 displays the implied correlations between the Topix and the Japanese government bond prices. One is the correlation between the Topix and bonds with a six-month time period to maturity, and the other is the Topix's correlation with bonds having a 10-year time to maturity.



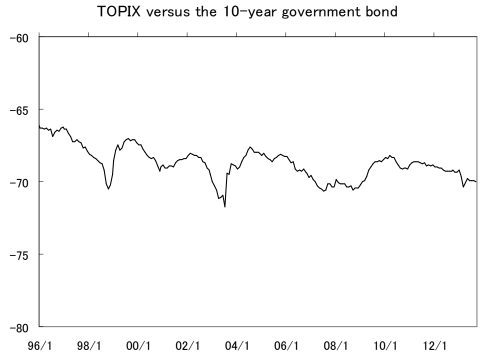


Figure 3: Correlations between the Topix and the Japanese government bonds: the graphs are shown in percentage terms.

As for the correlation between the Topix and the longer-term to maturity bonds, no large fluctuation continued to have negative values over the sample period, as the right-hand graph indicates. On the other hand, the correlation between the Topix and the shorter-maturity term bonds takes mild negative values for most of the sample period; however, it takes positive values during a certain interval of the sample period. Since the spring of 2012 when the BOJ decided to further enhance the CME introduced in October 2010, the correlation has taken significant positive values. This decision by the BOJ is likely to cause widespread expectations that prolonged quantitative easing may increase interest in the liquidity-driven market among investors. Accordingly, from the above discussion, it can be said that the sign of the implied correlation changed from negative to positive because of the BOJ's action.

4.3 Risk Premium

In this subsection, we analyze the estimates of bond and stock risk premiums. In particular, we are interested in how the unconventional monetary policies that the BOJ introduced after the financial crisis beginning in the fall of 2008 have affected risk premiums.

Figure 4 illustrates estimates of the term premiums for the shorter- and the longer-term maturity bonds, defined as equation (32). Focusing on developments after the beginning of 2009, we can observe the contrast between two developments. For the shorter term maturity bonds, premiums have remained approximately constant. On the other hand, the term premium for longer-term maturity bonds has largely continued to decline since the beginning of 2011. This might reflect the fact that the BOJ began the CME in October 2010. Before introducing this policy, the BOJ had primarily bought the Japanese government bonds with relatively shorter durations. However, the asset purchase program established via the CME attempted to buy longer-term maturity bonds on a large scale and lower both the longer-term interest rates and their term premiums. Notably, the BOJ's purchase of longer-term bonds could reduce the investors' concerns about a deteriorated balance of supply and demand and thus lead to a decline in the longer-term maturity bonds' risk premium.

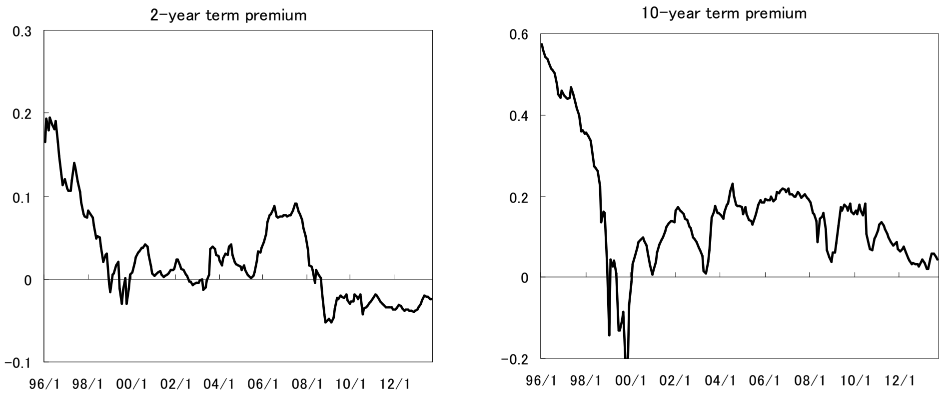


Figure 4: Term Premiums (shown in percentage terms)

Figure 5 displays the equity risk premiums for two, five, and 10 years, defined as equation (33). From this figure, we can observe that the equity risk premiums rose sharply when Lehman Brothers collapsed. Looking back at past stress events in Japan, for example, in November 1997 when Hokkaido Takushoku Bank, one of the city banks, and Yamaichi Securities Company went bankrupt and in early 2003 when the Resona Holdings capital adequacy ratio fell drastically low, equity risk premiums showed large increases. In this manner, the equity risk premiums estimated based on our model appear to capture the developments following the stress events.

Turning focus to developments in the equity risk premiums after the Lehman crash, we can observe that the risk premiums have remained high. Until at least the end of our sample period, September 2013, derailing the course of the BOJ, the decline in the term premium of the longer-term maturity bonds has not impacted equity risk premiums.

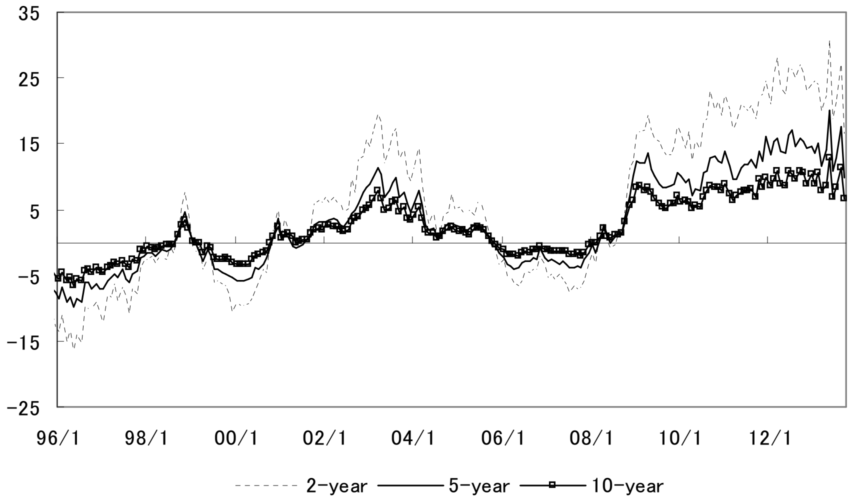


Figure 5: Equity Risk Premiums (shown in percentage terms)

5 Conclusion

This paper proposed a joint pricing model for stocks and bonds in a no-arbitrage framework. Specifically, our bond-pricing model is based on the quadratic Gaussian term structure model studied in [1] and [9]. This setting ensures a positive nominal interest rate. On the other hand, our stock price is defined as the dividend discount cash flow model incorporating the quadratic Gaussian term structure model in a no-arbitrage condition. Specifying the dividend as a function using the quadratic form of the state variables leads to a stock price representation that is exponential-quadratic in the state variables. We proved that the coefficients determining the stock price have to satisfy some matrix equations, including an algebraic Riccati equation. Moreover, in general, these matrix equations do not have any solutions; however, we specified the sufficient condition in which the matrix equations do have a unique solution.

In an empirical analysis using Japanese data, we estimated the latent state variables and the model parameters based on the quasi-maximum likelihood method with an unscented Kalman filter. As a result, we obtained a good fit to the actual financial market data. This could be because our model, which ensures a positive nominal interest rate, works well with Japanese data, which contain a lengthly low interest environment. Using estimated filtered state variables, we computed the implied correlation between stocks and bonds. While the correlation between the Topix and longer-term maturity government bond price evolves in a relatively stable manner with negative values, the correlation between the shorter-term bonds and the stock index takes positive values since the beginning of the BOJ's CME, although it takes negative values for most of the sample period. As for the risk premiums, the term premium of bonds with the longer-term maturities has continued to decrease since the introduction of CME. On the other hand, equity risk premiums have not decreased since the collapse of Lehman Brothers. At least until September 2013, the end of our sample period, we did not observe spillover effects of prompting to lower the risk premiums of more risky assets.

Although our study focused on Japanese data, our model can be applied to other countries. In particular, our model should be effective in analyzing countries with low interest rates, such as the U.S. and European countries. In the U.S, the quantitative easing policy ended in October 2014 and investors have now developed a strong interest in the period of the next policy rate rise. An analysis using our model would clarify the development of the risk premiums for the stocks and bonds during the exit from an accommodative monetary policy.

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Appendix: Estimated Model Parameters

$\Phi_{1,1}$	$\Phi_{2,1}$	$\Phi_{2,2}$	$\Phi_{3,1}$	$\Phi_{3,2}$	$\Phi_{3,3}$
1.130×10^{-5}	4.471×10^{-6}	8.449×10^{-6}	-1.571×10^{-5}	-1.818×10^{-5}	5.534×10^{-4}
$K^{\mathbf{P}}_{X,1,1}$	$K^{\mathbf{P}}_{X,2,1}$	$K^{\mathbf{P}}_{X,2,2}$	$K^{\mathbf{P}}_{X,3,1}$	$K^{\mathbf{P}}_{X,3,2}$	$K^{\mathbf{P}}_{X,3,3}$
0.05021	0.01031	0.00541	-0.01226	-0.01643	0.06484
$\Psi_{1,1}$	$\Psi_{2,1}$	$\Psi_{2,2}$	$\Psi_{3,1}$	$\Psi_{3,2}$	$\Psi_{3,3}$
1.464×10^{-5}	5.319×10^{-6}	1.250×10^{-5}	-1.489×10^{-5}	-1.796×10^{-5}	5.895×10^{-4}
$\Lambda_{1,1,1}$	$\Lambda_{1,2,1}$	$\Lambda_{1,2,2}$	$\Lambda_{1,3,1}$	$\Lambda_{1,3,2}$	$\Lambda_{1,3,3}$
-0.4928	0.1552	-0.02693	-0.02056	0.1191	-0.6327
$\theta^{\mathbf{P}}_1$	$\theta^{\mathbf{P}}_2$	$\theta^{\mathbf{P}}_3$	$\lambda_{0,1}$	$\lambda_{0,2}$	$\lambda_{0,3}$
0.2055	0.2522	-0.1511	-0.1199	-0.0915	0.1486
$\delta_{1,1}$	$\delta_{1,2}$	$\delta_{1,3}$			
-4.262×10^{-5}	-1.074×10^{-4}	0.001731			