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Productivity and Asset Distribution in a Dynamic Economy with Heterogeneous Agents

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Abstract

This paper examines the dynamic behavior of real assets within a dynamic equilibrium model with heterogeneous agents. The heterogeneity is characterized by productivity growth rates and discount factors of consumers for future utilities. A steady state to which an equilibrium path converges is highlighted. When productivity growth rates of impatient agents are low, patient consumers accumulate more and more assets. However, if the impatient agents strive to increase their productivity growth rates, the dynamic properties of assets change, and the patient consumers continue to accumulate their assets until a certain time point, after which they begin to exhaust their assets.

Key Words: inequality, asset dynamics, heterogeneous agents, productivity growth rate

JEL Classification Numbers: D24, D31, O41

1 Introduction

Recently, rising inequality—among countries, regions, and social strata—has become a more and more serious problem. To explain this economic phenomenon, on the one hand, Krugman (2007) and Stiglitz (2012) commonly emphasized that great divergence of wealth distribution in the United States can primarily be attributed to political aspects rather than an economic trend. On the other hand, sound economic foundations for explaining the rising inequality have been proposed.

One strand for explaining superfluous inequality from a standpoint of economic theory can be found in Ramsey (1928) and Becker (1980). To refine Ramsey's conjecture, Becker exploited a one-sector dynamic general equilibrium framework with many heterogeneous consumers who discount future utilities at various rates, and proved that impatient consumers will own almost no capital asset and consume very little in future periods. These results can be interpreted as the emergence of inequality as a dynamic general equilibrium path.[1]

The results of Ramsey and Becker have been extended in various directions. Becker and Foias (1987) illustrated the existence of a period-two equilibrium cycle, and proved the turnpike property of that model; i.e., total capital will approach a steady state that is independent of the initial conditions. Becker and Tsyganov (2002) extended similar results to a two-sector setting. Sorger (2002) and Bosi and Seegmuller (2010a) introduced progressive taxation by governments into the Ramsey model with heterogeneous agents. Bosi and Seegmuller (2010b) extended the model by taking endogenous labor supply by consumers into account. Becker and Mitra (2012) studied the efficiency of the aggregated path of capital accumulation in the Ramsey model with heterogeneous agents by applying Malinvaud's criterion (1953). Assuming the logarithmic utility functions of consumers, Becker (2012) demonstrated the "twisted turnpike property" of the model, which means optimal paths starting from different initial capital stocks will closely run together in future periods. However, the above authors did not investigate effects of heterogeneity of productivity growth rates of many agents on equilibrium allocations.

The purpose of this paper is to analyze dynamics of real assets in an economy with heterogeneous agents. The heterogeneity is characterized by productivity growth rates, which are assumed to be different across the

[1] Another economic thought for explaining the rising inequality is, for example, the super-star theory developed by Rosen (1981) and Adler (1985).

agents, in addition to discount factors for their future utilities. Given various types of inequalities observed in the real world, it is an important research topic to examine how strenuous and meaningful efforts made by a class of people to raise their productivities affect others' economic positions. This paper demonstrates that, assuming logarithmic utility, consumption levels of impatient consumers converge to zero as described by Becker (2012). In the next step, a steady state to which the equilibrium path will converge is highlighted. In the steady state, the impatient consumers consume nothing, and their decision making concerning consumption does not affect the real interest rate because their presences in relation to the aggregate consumption in the markets are ignorable. In contrast, the productivity growth rates of the consumers affect the real interest rate even in the steady state because the productivity growth rates are crucial for determining an inter-temporal relative price system; a high productivity growth rate for the whole economy implies that there is greater provision of future goods than provision of current goods. Of course, the real interest rate affects the dynamics of the asset distribution. Through that channel, the productivity growth rates of agents affect the dynamics of the asset distribution. The present study investigates these effects both theoretically and numerically (see Sections 4.2 and 4.3, respectively).

The main findings of this paper are as follows. Assume that the patient consumers own a great many real assets at the initial time point. In such a situation, if productivity growth rates of the impatient agents are low, the patient consumers accumulate more and more real assets as time passes. That is, a "winner takes all" path emerges; the patient agents take all the consumption and real assets. However, if the "losers" (impatient consumers) strive to increase their productivity growth rates, the dynamic properties of the real assets change; the patient consumers continue to accumulate their real assets until a certain time point, after which they begin to exhaust their real assets. In summary, a "winner takes all" path vanishes owing to continuous efforts by impatient consumers to improve their productivity, and a "golden days are destined to come to an end" path emerges. The critical level of the productivity growth rates of the impatient consumers, at which the dynamic properties change, is the level of the real interest rate in equilibrium. That is, the winner will take all if the productivity growth rates of the impatient consumers are lower than the real interest rate, and that path disappears if the productivity growth rates overtake the real interest rate.

Many studies concerning dynamic allocations of economies have focused on discount factors of consumers for future utilities as one of the most important parameters. Scheinkman (1976) and McKenzie (1983) proved that an optimal path converges to a steady state if the consumers discount their future utilities sufficiently weakly. Nishimura and Yano (1994) and Nishimura et al. (1994) demonstrated that if the consumers' discount factors are sufficiently near zero, optimal paths in dynamic models with infinitely living agents may behave complicatedly—even chaotically. Nishimura and Yano (1995) showed that even if the consumers discount their future utilities arbitrarily weakly, there exist parameter constellations that produce chaotic solutions as optimal paths. These authors, however, concentrated on cases with a representative agent, and did not study the effects of heterogeneity of economic agents.

Bewley (1982), Yano (1984, 1991, 1998) and Coles (1985) allowed for finitely many heterogeneous consumers, and proved that an equilibrium path will lie in a small neighborhood of a steady state if future utilities are sufficiently important to the consumers. Yano (1999) surveyed the literature up until the 1990s. More recently, Ghiglino (2005) explored the effects of heterogeneity of consumers' preference on the dynamic stability of the steady state. The effect of heterogeneity on the indeterminacy of equilibria was investigated by Ghiglino and Olszak-Duquenne (2005). Kondo (2008) extended a neighborhood turnpike theorem to a monetary model with heterogeneous consumers. However, these papers did not study cases in which discount factors and/or productivity growth rates are different across the consumers.

The remainder of this paper is structured as follows. The next section introduces a model economy. Section 3 establishes the Ramsey–Becker type of result; i.e., impatient consumers will consume almost nothing in future periods. The main analysis is included in Section 4, where paths of real assets are investigated both analytically and graphically. Section 5 concisely concludes the paper.

2 Model

Think of an economy with discretely indexed time points $t = 0, 1, 2, \cdots$. The period between time points $t-1$ and t is period t. There are two types of consumers, type α and β. They transact a consumption good and a real financial asset in each period. A real asset bears a real interest rate r_t from period t to $t+1$.

A representative consumer of the type i, where $i = \alpha, \beta$, holds the real asset a_{it-1} at the beginning of period t. ($a_{it-1} < 0$ represents real liabilities.[2]) The consumer receives a consumption good endowment y_{it} and consumes c_{it} in the period t. Budget constraints for the representative consumer of type i are given by

$$c_{it} + a_{it} \leq y_{it} + (1 + r_{t-1})a_{it-1}, \tag{1}$$

for any $t = 1, 2, \cdots$. The consumer obtains utility from the consumption. An instantaneous utility function is represented by the logarithmic function: $u_i(c) = \log c$, which is common for all consumers. The representative consumer of type i discounts his or her future utilities relative to the utility from the current consumption with a discount factor $\rho_i \in (0, 1)$. The behavior of the representative consumer of type i is summarized as a maximizing problem:

$$\begin{aligned} &\max_{\{c_{it}, a_{it}\}_{t=1}^{\infty}} \sum_{t=1}^{\infty} \rho_i^{t-1} \log c_{it} \qquad (2) \\ &\text{s.t. Equation (1)} \\ &\text{given } \ \{r_t\}_{t=0}^{\infty}, \ a_{i0} \end{aligned}$$

The two types of consumers interact in markets. The market clearing conditions are

$$c_{\alpha t} + c_{\beta t} = Y_t, \quad a_{\alpha t} + a_{\beta t} = 0, \quad \text{for any } t = 1, 2, \cdots, \tag{3}$$

where $Y_t \equiv y_{\alpha t} + y_{\beta t}$ is total products in period t. A time stream of price and allocation of resources in the economy is determined so that the conditions (2)-(3) are simultaneously satisfied.

The analyses presented below do not impose inter-temporal budget constraints for the consumers; i.e., the parameter constellation is not necessarily compatible with the no-Ponzi game conditions. The outcomes from the real asset market, a_{it}, may diverge to $+\infty$ or $-\infty$.

3 Ramsey–Becker Result

This section reveals the time behavior of an equilibrium path. First, the Ramsey–Becker result is established; the consumption path of the impatient

[2] The consumers in this model economy are not subject to the borrowing constraint, $a_{it} \geq 0$, in contrast to several papers concerning the Ramsey model with heterogeneous agents; e.g., Becker (1980).

consumers converges to zero.[3] From a long-term viewpoint, the presence of impatient consumers in the market economy can be ignored in contrast to the presence of the patient consumers. Next, it is demonstrated that the real interest rate is convergent. On the basis of these results, the next section will offer analyses of a steady state to which the equilibrium path converges.

For that research plan, the following assumptions are made.

Assumption 1 $0 < \rho_\beta < \rho_\alpha < 1$.

Assumption 2 $y_{\alpha t+1}/y_{\alpha t} = 1 + \theta_\alpha$ and $y_{\beta t+1}/y_{\beta t} = 1 + \theta_\beta$ for any $t = 1, 2, \cdots$.

Assumption 1 means that β-type consumers are more impatient than α-type consumers. Assumption 2 states that the productivity growth rates of all consumers are constant over time, and those of the two types of consumers are possibly different. As a result, the growth rate of the total production level is also constant. Let $Y_{t+1}/Y_t = 1 + \theta$. It is easily ascertained that

$$1 + \theta = (1 + \theta_\alpha) \frac{y_{\alpha t}}{Y_t} + (1 + \theta_\beta) \frac{y_{\beta t}}{Y_t}, \tag{4}$$

for any $t = 1, 2, \cdots$.

Additionally, to prove the main result of this section, the following condition is postulated.

Assumption 3 $(1 + \theta) \rho_\beta < \rho_\alpha \, (< 1)$.

What does Assumption 3 mean? As will be shown in Corollary 1, which is presented after Theorem 1, the real interest rate in a steady state is $(1 + \theta) / \rho_\alpha$ in the present economy. Thus, Assumption 3 implies that $\rho_\beta < 1/ (1 + r)$, where r is the real interest rate in the steady state. In other words, Assumption 3 implies that the discount factor of the β-type consumers is smaller than the "market discount factor" $1/ (1 + r)$.

Under these assumptions, the Ramsey–Becker result is established.

Theorem 1 *It holds in the equilibrium path that $c_{\beta t} \to 0$ as $t \to \infty$.*

[3] More accurately, Becker showed in his 1980 paper that the consumption levels of impatient agents converge to a wage income level, while in his 2012 paper with logarithmic utility functions, the consumptions of impatient consumers converge to zero.

Proof. Take $i = \alpha, \beta$ and $t = 1, 2, \cdots$, and fix them momentarily. A first order condition of the maximizing problem (2) yields

$$\frac{c_{it+1}}{c_{it}} = \rho_i \left(1 + r_t\right), \tag{5}$$

and thus,

$$c_{it} = \rho_i^{t-1} \left(1 + r_1\right) \cdots \left(1 + r_{t-1}\right) c_{i1}. \tag{6}$$

Since (6) holds for each $i = \alpha, \beta$, it holds that

$$\frac{c_{\beta t}}{c_{\alpha t}} = \left(\frac{\rho_\beta}{\rho_\alpha}\right)^{t-1} \frac{c_{\beta 1}}{c_{\alpha 1}}. \tag{7}$$

Using (7), (3) and Assumption 2, it follows that

$$\begin{aligned} 0 \quad &< \quad c_{\beta t} = \left(\frac{\rho_\beta}{\rho_\alpha}\right)^{t-1} \frac{c_{\beta 1}}{c_{\alpha 1}} c_{\alpha t} \\ &< \quad \left(\frac{\rho_\beta}{\rho_\alpha}\right)^{t-1} \frac{c_{\beta 1}}{c_{\alpha 1}} Y_t = \left(\frac{\rho_\beta}{\rho_\alpha}\right)^{t-1} \frac{c_{\beta 1}}{c_{\alpha 1}} \left(1 + \theta\right)^{t-1} Y_1. \end{aligned}$$

From Assumption 3, the desired result follows. ■

Theorem 1 may be interpreted as an emergence of rising inequality in the market economy because it implies that α-type consumers will take almost all consumption goods in the future (i.e., $|c_{\alpha t} - Y_t| \to 0$ as $t \to \infty$) while β-type consumers will consume almost nothing. Note that Assumption 3 is crucial for the emergence of the rising inequality. Consider a situation in which β-type consumers strive to raise their productivity. That continuous effort may break the inequality; it increases the economic growth rate θ, and thus, Assumption 3 may be broken, given the discount factors for both types of consumers ρ_i. A similar effort made by α-type consumers has the same inequality-breaking effect.

Using Theorem 1, the following corollary regarding the path of the real interest rate can be proved.

Corollary 1 *It holds in the equilibrium path that* $1 + r_t \to \left(1 + \theta\right)/\rho_\alpha$ *as* $t \to \infty$.

Proof. On the one hand, it holds, according to (5), that

$$\frac{c_{\alpha t+1} + c_{\beta t+1}}{c_{\alpha t} + c_{\beta t}} = \left(1 + r_t\right) \frac{\rho_\alpha c_{\alpha t} + \rho_\beta c_{\beta t}}{c_{\alpha t} + c_{\beta t}}, \tag{8}$$

for any $t = 1, 2, \cdots$. On the other hand, equation (3) and Assumption 2 yield

$$\frac{c_{\alpha t+1} + c_{\beta t+1}}{c_{\alpha t} + c_{\beta t}} = \frac{Y_{t+1}}{Y_t} = 1 + \theta, \tag{9}$$

for any $t = 1, 2, \cdots$. From (8) and (9), it holds that

$$1 + r_t = (1 + \theta) \frac{c_{\alpha t} + c_{\beta t}}{\rho_\alpha c_{\alpha t} + \rho_\beta c_{\beta t}},$$

$t = 1, 2, \cdots$. Since $c_{\beta t} \to 0$, the desired result is obtained. ■

As shown in the corollary, the long-run interest rate positively correlates with the economic growth rate θ. A slight increase in the economic growth rate implies that provision of consumption goods in future periods becomes more abundant relative to that in the current period. Thus, it increases the real interest rate. Further, note that the long-run interest rate relates with the inverse of the discount factor of the patient consumers (α-type consumers), not with the discount factor of the impatient consumers. Because the market presence of the impatient consumers becomes zero as time passes, the discount factor of that type of consumers does not affect the long-run real interest rate.

4 Steady State

This section investigates the properties of a steady state to which the equilibrium path converges. Dynamics of the real assets of the consumers in the steady state are highlighted. Especially, effects of productivity growth rates of the impatient consumers on the asset dynamics of patient consumers are theoretically examined in Section 4.2. Numerical examples, which illustrate the effects, are given in Section 4.3.

4.1 Steady-state Equilibrium

This subsection presents a steady state to which the equilibrium path converges. Using the results from the previous section, it is easy to derive the steady state. Given the initial conditions (y_{i0}, a_{i0}), the consumers' subjective discount factors ρ_i and the productivity growth rates θ_i, all endogenous variables in the steady state are determined.

Lemma 1 *Endogenous variables in the steady state are*

$$
\begin{aligned}
(A)\ c_{\alpha t} &= (1+\theta)^t (y_{\alpha 0} + y_{\beta 0}) \quad \text{and} \quad c_{\beta t} = 0; \\
(B)\ r_t &= \frac{1+\theta}{\rho_\alpha} - 1; \\
(C)\ a_{\alpha t} &= \left(\frac{1+\theta}{\rho_\alpha}\right)^t a_{\alpha 0} - \left[\sum_{s=0}^{t-1} (1+\theta_\beta)^{t-s-1} \left(\frac{1+\theta}{\rho_\alpha}\right)^s\right] (1+\theta_\beta)\, y_{\beta 0}; \\
a_{\beta t} &= \left(\frac{1+\theta}{\rho_\alpha}\right)^t a_{\beta 0} + \left[\sum_{s=0}^{t-1} (1+\theta_\beta)^{t-s-1} \left(\frac{1+\theta}{\rho_\alpha}\right)^s\right] (1+\theta_\beta)\, y_{\beta 0}.
\end{aligned}
$$

for any $t = 1, 2, \cdots$.

Proof. (A) and (B) immediately follow from the results of the previous section.

(C) According to the flow budget constraints of the α-type consumers (1), (A) and (B), the real asset dynamics in the steady state must be subject to the first-order difference equation

$$
\begin{aligned}
a_{\alpha t} &= (1+r_{t-1})\, a_{\alpha t-1} + y_{\alpha t} - c_{\alpha t} && (10) \\
&= \frac{1+\theta}{\rho_\alpha} a_{\alpha t-1} - y_{\beta t} \\
&= \frac{1+\theta}{\rho_\alpha} a_{\alpha t-1} - (1+\theta_\beta)^t\, y_{\beta 0}, && (11)
\end{aligned}
$$

which holds for any $t = 1, 2, \cdots$, with the initial condition $a_{\alpha 0}$. The first equation of (C) is obtained from (11). The dynamics of $a_{\beta t}$ can be solved from the relationship $a_{\beta t} = -a_{\alpha t}$. ■

Equation (11) shows that the productivity growth rate of β-type consumers θ_β has opposite effects on the dynamics $\{a_{\alpha t}\}$. One effect appears in the first term of the equation. A slight increase in θ_β results in a higher interest rate, and thus, facilitates the accumulation of real assets $a_{\alpha t}$. The other effect relates to the second term of (11), and reduces $a_{\alpha t}$. An increase in θ_β implies more products in the future, and thus, more expenditure of α-type consumers, since they purchase all goods in the economy. That is a negative aspect of the higher productivity of β-type consumers on accumulating real assets of α-type consumers.

The interest rate in the steady state is simply denoted without a time index as $r \equiv (1+\theta)/\rho_\alpha - 1$. To guarantee that the real interest rate r is positive in the steady state, an additional assumption is required.

Assumption 4 $\rho_\alpha < 1+\theta$.

Assumptions 3 and 4 jointly imply that the economic growth rate must satisfy $\rho_\alpha < 1+\theta < \rho_\alpha/\rho_\beta$ in this paper.

4.2 Main Analyses

This subsection presents the main analyses of this paper, highlighting the dynamics of the real assets of the consumers in the steady state. Before proceeding to the analyses, it is convenient to transform equation (C) in Lemma 1 as

$$a_{\alpha t} = \left(\frac{1+\theta}{\rho_\alpha}\right)^t y_{\beta 0}\left[\frac{a_{\alpha 0}}{y_{\beta 0}} - \Gamma\left(1+\Gamma+\Gamma^2+\cdots+\Gamma^{t-1}\right)\right], \qquad (12)$$

which holds for any $t = 1, 2, \cdots$, where Γ is a variable defined as

$$\Gamma \equiv \frac{\rho_\alpha\left(1+\theta_\beta\right)}{1+\theta} = \frac{1+\theta_\beta}{1+r}\,(>0)\,. \qquad (13)$$

Note that Γ is a ratio of the gross productivity growth rates of β-type consumers $1+\theta_\beta$ to the gross real interest rate $1+r$. Further, for convenience, I define

$$\Psi_t \equiv \Gamma\left(1+\Gamma+\Gamma^2+\cdots+\Gamma^{t-1}\right), \qquad (14)$$

for any $t = 1, 2, \cdots$. Equation (12) together with (14) shows that the large/small relationship between $a_{\alpha 0}/y_{\beta 0}$ and Ψ_t is decisive for determining the direction of the real asset dynamics of α-type consumers $a_{\alpha t}$.

I point out some important properties of "the critical value" Ψ_t here. Ψ_t monotonously increases with respect to time with $\Psi_1 = \Gamma$. The definitions (13) and (14) directly show that if $\theta_\beta < r$, the critical value Ψ_t converges to a finite value $\Gamma/\left(1-\Gamma\right)$, and if $r \leq \theta_\beta$, it diverges to $+\infty$. Thus,

$$\Psi_\infty \equiv \begin{cases} \Gamma/\left(1-\Gamma\right) & \text{if } \theta_\beta < r \\ +\infty & \text{if } r \leq \theta_\beta \end{cases} \qquad (15)$$

Note that when $\theta_\beta < r$, it holds that

$$\Psi_\infty\left(=\frac{\Gamma}{1-\Gamma}\right) = \frac{\rho_\alpha\left(1+\theta_\beta\right)}{\left(1+\theta\right)-\rho_\alpha\left(1+\theta_\beta\right)}\,(>\Gamma)\,. \qquad (16)$$

As was mentioned in the introduction, the threshold level of the productivity growth rate θ_β at which the dynamics of real assets change their properties is the level of the real interest rate r. This is understood from the above equations (12)-(14); if the productivity growth rates of β-type consumers θ_β overtake the real interest rate r, then Γ exceeds 1 according to (13). As a consequence, the temporal property of the critical value Ψ_t changes, with Ψ_t becoming divergent toward $+\infty$ as shown by (15), and the dynamic property of real assets changes considerably as will be seen in the following analyses.

In what follows, the real asset dynamics of the α-type consumer $\{a_{\alpha t}\}$ will be analyzed with respect to possible variations of parameter constellations. Attention is paid to the initial level of α-type consumers $a_{\alpha 0}$ and the productivity growth rate of β-type consumers θ_β while the other parameters are fixed. Table (18) visually summarizes the following analyses. The rows of the table classify the dynamic properties of $a_{\alpha t}$ with regard to $a_{\alpha 0}$, while the columns refer to θ_β.

At the outset, consider the case of $a_{\alpha 0}/y_{\beta 0} < \Psi_1 \, (= \Gamma)$ (see the second row in table (18)). In this case, the real asset level of the α-type consumer $a_{\alpha t}$ is monotonously decreasing and diverges to $-\infty$ because the sign of the square-bracket term in (12) is minus for any t. An economic interpretation of this case is straightforward. The α-type consumers take all consumptions in the economy but do not own sufficient real assets at the initial time point. They cannot sustain their asset levels, and their real assets decrease because their consumption levels are too high.

In the case of $\Psi_1 < a_{\alpha 0}/y_{\beta 0}$ (see the third to fifth rows in table (18)), in contrast, the critical value Ψ_t, which increases with time, catches up or overtakes the ratio $a_{\alpha 0}/y_{\beta 0}$ according to a set of parameter values. It is convenient to classify this case with respect to the productivity growth rate of the β-type consumers θ_β.

First, the case in which θ_β is relatively low is investigated. Assume that $\theta_\beta < r$ (see the second column in (18)). Then, since $\Gamma < 1$, Ψ_t converges to a real number $\Gamma / (1 - \Gamma)$ as $t \to \infty$ with $\Psi_1 = \Gamma$ (see (15)). This case has to be divided into three sub-cases according to the level of the ratio $a_{\alpha 0}/y_{\beta 0}$.

If $(\Psi_1 <) a_{\alpha 0}/y_{\beta 0} < \Gamma / (1 - \Gamma)$ (see the third row in (18)), then the critical value Ψ_t overtakes $a_{\alpha 0}/y_{\beta 0}$. Since the part of the square brackets in (12) changes sign within a finite time interval, there is a natural number t^* such that $a_{\alpha t}$ is increasing for any $t = 1, 2, \cdots, t^* - 1$ and decreasing for any $t = t^* + 1, t^* + 2, \cdots$. This case shows a "golden days are destined to come to an end" situation; α-type consumers continue to accumulate their real assets until t^*, after which they begin to exhaust their real assets.

If $a_{\alpha 0}/y_{\beta 0} = \Gamma / (1 - \Gamma) \, (> 0)$ (see the fourth row in (18)), then Ψ_t just catches up to $a_{\alpha 0}/y_{\beta 0}$; i.e., $\Psi_t \nearrow a_{\alpha 0}/y_{\beta 0}$, where $x_t \nearrow$ (resp. $\searrow$) implies that x_t is monotonously increasing (resp. decreasing) and converges to its limit value (including $\pm\infty$). Substituting $a_{\alpha 0}/y_{\beta 0} = \Gamma/(1-\Gamma)$, (13) and (16) into (12) yields

$$a_{\alpha t} = (1+\theta_\beta)^t y_{\beta 0} \frac{\rho_\alpha (1+\theta_\beta)}{(1+\theta) - \rho_\alpha (1+\theta_\beta)}, \tag{17}$$

which holds for any $t = 1, 2, \cdots$. Note that $a_{\alpha 0} (= y_{\beta 0}\Gamma/(1-\Gamma)) > 0$ in this case. It is clear, from (17), that if $\theta_\beta > 0$, $a_{\alpha t}$ diverges to $+\infty$ and that if $\theta_\beta < 0$, $a_{\alpha t}$ monotonously decreases and converges to zero. Further, if $\theta_\beta = 0$, $a_{\alpha t}$ remains at a constant value as time passes.

If $\Gamma/(1-\Gamma) < a_{\alpha 0}/y_{\beta 0}$ (see the fifth row in (18)), the critical value Ψ_t cannot catch up to the ratio $a_{\alpha 0}/y_{\beta 0}$. Thus, there exists a positive real number $\varepsilon > 0$ such that $\Psi_t + \varepsilon < a_{\alpha 0}/y_{\beta 0}$ for any t. Thus, from (12), the real asset of a α-type consumer $a_{\alpha t}$ monotonously increases and diverge to $+\infty$. An economic interpretation of this consequence is as follows. If α-type consumers possess extremely huge real assets at the initial time point, their revenue from interest $ra_{\alpha t}$ is enough to accumulate more real assets in addition to paying for consumptions.

Next, I proceed to the case in which θ_β is relatively high. Assume that $r\,(= (1+\theta)/\rho_\alpha - 1) \leq \theta_\beta$ (see the third column in (18)). Then, the critical value Ψ_t certainly overtakes $a_{\alpha 0}/y_{\beta 0}$ because $\Gamma \geq 1$ and $\Psi_t \nearrow \infty$ as $t \to \infty$. In other words, the part of the square brackets in (12) changes sign from plus to minus. Hence, there is a natural number t^* such that $a_{\alpha t}$ is increasing for any $t = 1, 2, \cdots, t^* - 1$ and decreasing for any $t = t^* + 1, t^* + 2, \cdots$, where t^* is the critical time point at which Ψ_t overtakes $a_{\alpha 0}/y_{\beta 0}$. In this case, the phenomenon "golden days are destined to come to an end" emerges again.

The results obtained above can be summarized as the following table.

<table>
<tr><td></td><td>$\theta_\beta < r$
$(\Gamma < 1)$</td><td>$r \leq \theta_\beta$
$(1 \leq \Gamma)$</td></tr>
<tr><td>$a_{\alpha 0}/y_{\beta 0} < \Psi_1 (= \Gamma)$</td><td colspan="2">$a_{\alpha t} \searrow -\infty$</td></tr>
<tr><td>$\Psi_1 < a_{\alpha 0}/y_{\beta 0} < \Psi_\infty$</td><td colspan="2">$\exists t^* \in \mathbb{N}$ such that
$t = 1, 2, \cdots, \ t^* - 1 \Longrightarrow a_{\alpha t} \nearrow$
$t = t^* + 1, \ t^* + 2, \cdots \Longrightarrow a_{\alpha t} \searrow$</td></tr>
<tr><td>$a_{\alpha 0}/y_{\beta 0} = \Psi_\infty$</td><td>$\theta_\beta > 0 \Longrightarrow a_{\alpha t} \nearrow \infty$
$\theta_\beta = 0 \Longrightarrow a_{\alpha t}$: constant
$\theta_\beta < 0 \Longrightarrow a_{\alpha t} \searrow 0$</td><td rowspan="2">——</td></tr>
<tr><td>$\Psi_\infty < a_{\alpha 0}/y_{\beta 0}$</td><td>$a_{\alpha t} \nearrow \infty$</td></tr>
</table>

(18)

Table: Effects of $a_{\alpha 0}$ and θ_β on $\{a_{\alpha t}\}$

where $\mathbb{N}$ is the set of natural numbers.

Under Assumptions 1–4, the following theorem has been established.

Theorem 2 *In a steady state, to which an equilibrium path converges, the following holds.*

If $a_{\alpha 0}/y_{\beta 0} < \Gamma$, it holds that $a_{\alpha t} \searrow -\infty$ as $t \to \infty$, where Γ is defined in (13)*.*

Assume either of the two following cases holds: (i) $\theta_\beta < r$ and $\Gamma < a_{\alpha 0}/y_{\beta 0} < \Gamma/(1-\Gamma)$, or (ii) $r \leq \theta_\beta$ and $\Gamma < a_{\alpha 0}/y_{\beta 0}$, where r is the level of the real interest rate in the steady state. There is then a natural number t^* such that $a_{\alpha t}$ is increasing for any $t = 1, 2, \cdots, \ t^* - 1$, and decreasing for any $t = t^* + 1, \ t^* + 2 \cdots$. Further, $a_{\alpha t} \to -\infty$ as $t \to \infty$. This can be interpreted as a "golden days are destined to come to an end" situation.

Assume that $\theta_\beta < r$ and $a_{\alpha 0}/y_{\beta 0} = \Gamma/(1-\Gamma)$. The following then holds. If $\theta_\beta > 0$, $a_{\alpha t}$ is monotonically increasing and diverges to $+\infty$. If $\theta_\beta = 0$, $a_{\alpha t}$ is constant over time. Additionally, if $\theta_\beta < 0$, $a_{\alpha t}$ is monotonically decreasing and converges to zero.

If $\Gamma/(1-\Gamma) < a_{\alpha 0}/y_{\beta 0}$, then $a_{\alpha t} \nearrow \infty$ as $t \to \infty$. This can be interpreted as a "winner takes all" situation.

A remark regarding the role of productivity growth rates of consumers should be made here. Consider a situation in which α-type consumers own a great many real assets at the initial time point and the productivity growth rates of β-type consumers are lower than the market's interest rate:

$\Gamma/(1-\Gamma) < a_{\alpha 0}/y_{\beta 0}$ and $\theta_\beta < r$. Then, according to the table (18), α-type consumers will take all real assets in addition to the consumption, which exhibits a "winner takes all" situation. Even in this situation, however, if the β-type consumers work hard so that their productivity growth rates are not lower than the real interest rate ($r \leq \theta_\beta$), then the "winner takes all" path vanishes. Instead, a path of "golden days are destined to come to an end" emerges. (This scenario is graphically demonstrated in the next subsection using a numerical example (see Example 2).) Furthermore, consider a situation in which α-type consumers increase their productivity growth rates. The interest rate r then increases according to (4). If the condition $\theta_\beta < r$ is met again, the "winner takes all" scenario is revived because the amount of income of α-type consumers begins to grow at a faster pace.

4.3 Numerical Examples

In this subsection, two numerical examples are offered to illustrate the temporal evolution of the real asset level of α-type consumers $a_{\alpha t}$. Particular attention is paid to the effects of the initial level of the real assets of α-type consumers (Example 1) and the productivity growth rate of β-type consumers (Example 2) on the asset path $\{a_{\alpha t}\}$ in each example. Throughout these examples, the discount factor of a representative consumer of β-type, ρ_β, is supposed to be sufficiently small so that it satisfies Assumptions 1 and 3.

Example 1. (Effects of $a_{\alpha 0}$)

Let parameter values be $\rho_\alpha = 0.95$, $\theta_\alpha = -0.05$, $\theta_\beta = 0.05$, $y_{\alpha 0}/Y_0 = y_{\beta 0}/Y_0 = 0.5$, and $y_{\beta 0} = 1$. (In this economy, α-type consumers are patient but lazy, while β-type consumers are myopic but hard-working.) Other variables are then determined as $\theta = 0$, $r = 1/19$, $\Gamma = 399/400$ and $\Psi_\infty = \Gamma/(1-\Gamma) = 399$. Note that in this case, $\theta_\beta < r$. The critical value that determines the direction of the real asset dynamics, Ψ_t defined in (14) is

$$\Psi_t = 399 - 399\left(\frac{399}{400}\right)^t,$$

for any $t = 1, 2, \cdots$.

For this parameter constellation, the real asset level $a_{\alpha t}$ is

$$a_{\alpha t} = 399\left(\frac{399}{380}\right)^t + (a_{\alpha 0} - 399)\left(\frac{20}{19}\right)^t, \tag{19}$$

for any $t = 1, 2, \cdots$. The dynamic paths of $a_{\alpha t}$ with $a_{\alpha 0} = 50$ (the thin curve), 60 (the medium curve) and 70 (the bold curve) are drawn as the following graphs. (For visual clarity, the graphs are drawn as if the time structure is continuous.)

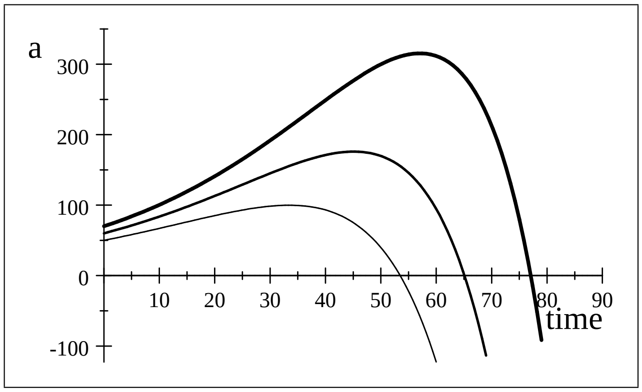

Figure 1.

Example 2. (Effects of θ_β)

Example 2 shows the effects of a productivity improvement of β-type consumers on the real asset dynamics of α-type consumers $\{a_{\alpha t}\}$.

Assume that $a_{\alpha 0} = 400$. First, let the parameter values other than $a_{\alpha 0}$ be those in Example 1. Then, according to (19), the real asset level of the α-type consumer is

$$a_{\alpha t} = 399 \left(\frac{399}{380}\right)^t + \left(\frac{20}{19}\right)^t, \tag{20}$$

for any $t = 1, 2, \cdots$, which diverges to $+\infty$.

Next, assume that the productivity growth rates of β-type consumers slightly increase to $\theta_\beta = 0.07$. The other parameter values are then affected: $\theta = 0.01$, $r = 6/95 \doteqdot 0.063$, $\Gamma = 2033/2020\,(> 1)$ and $\Psi_\infty = \infty$. In contrast to the case of the previous parameter set reflected in (20), the productivity growth rates of β-type consumers surpass the long-term interest rate in this case: $r < \theta_\beta$. The critical value defined in (14) is

$$\Psi_t = \frac{2033}{13} \left(\frac{2033}{2020}\right)^t - \frac{2033}{13},$$

for any $t = 1, 2, \cdots$, and the real asset level $a_{\alpha t}$ is

$$a_{\alpha t} = \frac{7233}{13} \left(\frac{101}{95}\right)^t - \frac{2033}{13} \left(\frac{2033}{1900}\right)^t, \tag{21}$$

for any $t = 1, 2, \cdots$.

Lastly, let $\theta_\beta = 0.08$; i.e., the productivity growth rate of β-type consumers is further increased. The other parameter values are $\theta = 3/200 = 0.015$, $r = 13/190 \doteqdot 0.068$, $\Gamma = 1026/1015\,(> 1)$ and $\Psi_\infty = \infty$. The critical value is

$$\Psi_t = \frac{1026}{11}\left(\frac{1026}{1015}\right)^t - \frac{1026}{11},$$

for any $t = 1, 2, \cdots$, and the real asset level $a_{\alpha t}$ is

$$a_{\alpha t} = \frac{5426}{11}\left(\frac{203}{190}\right)^t - \frac{1026}{11}\left(\frac{27}{25}\right)^t, \tag{22}$$

for any $t = 1, 2, \cdots$.

The following three graphs represent the dynamic paths of $a_{\alpha t}$ with $\theta_\beta = 0.05$ (the thin curve), 0.07 (the medium curve) and 0.08 (the bold curve), which are presented in the equations (20)-(22). On the vertical axis of the graph, $2e+6$ means $2 \times 10^6 = 2,000,000$.

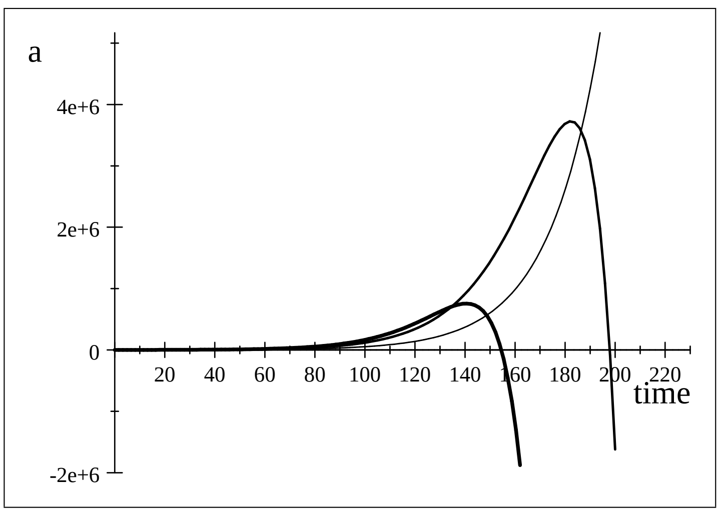

Figure 2.

As was noted immediately after presenting Theorem 2, this example shows a situation in which a "winner takes all" path vanishes and "golden days are destined to come to an end" path emerges because of the non-winners' (β-type consumers') effort to raise their productivity growth rates θ_β. However, as shown in Figure 2, a slight increase in θ_β rather increases the winners' (α-type consumers') real asset level $a_{\alpha t}$ from a short-term viewpoint. A rise in productivity growth rates of β-type consumers results in a higher interest rate because it implies that the provision of future goods becomes more abundant relative to the provision of current goods. Owing to the high interest rates, the real asset $a_{\alpha t}$ grows more quickly. This is the reason why a productivity improvement of β-type consumers facilitates the α-type's real asset accumulation at least temporarily.[4]

5 Concluding Remarks

This paper employed a simple dynamic general equilibrium model with heterogeneous consumers. The heterogeneity was characterized both by discount factors for future utilities and productivity growth rates, which were assumed to be different across two types of consumers. The paper examined the dynamics of the real asset levels in a stationary state to which an equilibrium path converges.

The analyses showed a possibility of a certain type of consumers taking all consumptions and accumulating enormous real assets. In this sense, rising inequality was explained as market outcomes. However, if other types of consumers increase their productivity faster than the real interest rate, the real asset paths of the former type of consumers will begin to decline within a finite time interval. In this sense, a "golden days are destined to come to an end" path emerges. I believe that the results and implications of the present paper would be of interest to economists and policy makers in many countries suffering from rising inequality. However, the model exploited in this research was somewhat simple, and some extensions should be made in possible future works.

First, this paper analyzed real asset dynamics without physical or human capital formation. Although this assumption was useful to simplify the analyses, it is desirable to introduce capital accumulation into a model explicitly. How are dynamics of the ratio of financial to physical assets affected by a slight change in the productivity growth rate? This presents an interesting research topic. Second, many (three, four, or more) types of consumers should be explicitly included in the analysis. Finally, the effects of endogenous changes of the discount factors of heterogeneous consumers should be analyzed.

[4] This effect has already been reported by Kondo (2012), who studied short- and long-term effects of a slight increase in productivity growth rates of consumers on public debt dynamics.

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