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Deflation, Population Decline and Sustainability of Public Debt

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Deflation, Population Decline and Sustainability of Public Debt[1]

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Abstract

This paper investigates theoretical relationships between population growth rate and inflation/deflation, with a focus on fiscal crisis as a possible channel through which those relationships manifest themselves. An upper bound for public debt that is compatible with the no Ponzi-game condition and a dynamic general equilibrium is derived, which depends on the growth rates of money supply and population. In plausible scenarios, a slight decrease in population growth rate decreases the upper bound for public debt, which motivates a government that intends to maintain sustainable public debt to adjust money supply growth rate. In this scenario, decreases in population growth rate can cause inflation or deflation.

Key Words: Sustainability of public debt, monetary policy, population growth rate, dynamic general equilibrium

JEL Classification Numbers: E60, H60

1 Introduction

Recently, several industrialized countries have suffered declining population and fiscal crisis. Especially in Japan, these two negative factors, together with deflation, are said to be the causes of the country's extended economic malaise. Although these factors seem to interconnect and amplify each other, their correlations are so complicated that it is difficult to understand their theoretical cause and effect relationships without formal model analyses.

Neumeyer and Yano (1995) used a two-country model with money in consumers' utility functions in which international connections arise from cross national bond holdings to investigate how monetary policies exert effects across national borders. Although Neumeyer and Yano carefully considered government inter-temporal budget constraint, they did not examine the upper bound for public debt that is compatible with the no Ponzi-game (NPG) condition and a dynamic general equilibrium (DGE). Analyzing the Neumeyer-Yano model, Kondo and Kitaura (2012) explicitly derived the upper bound for public debt, and reported that it could work as a channel for worldwide monetary disturbances. They identified a possible situation in which foreign inflation or deflation may decrease this upper bound and so damage the fiscal health of the domestic government. In this situation, the decrease in the upper bound for public debt motivates the domestic government to expand or reduce money supply, resulting in "imported inflation or deflation". In other words, fiscal crisis could be a channel that links inflation rates worldwide. However, the analyses of Kondo and Kitaura did not consider demographic change, and thus how population decline aggravates government fiscal position. Kondo (2012) studied how declining fertility affects sustainability of public debt. However, his model did not include money, or interactions between inflation/deflation and fiscal crisis.

The focus of this paper is to theoretically examine interactions between population decline and deflation (or inflation) through sustainability of public debt. This paper extends the neoclassical monetary model of Neumeyer and Yano to include population growth. An upper bound for public debt that is compatible with both the NPG condition and DGE is explicitly derived, which depends through complex relationships on the growth rates of population and money supply. This paper thus aims to reveal how monetary policies and population movements are interconnected.

The next step in the theoretical examination involves comparative analyses. In some plausible scenarios, a slight decrease in population growth rate impacts government fiscal position by reducing the upper bound of public debt, because decreased population growth rate implies fewer future tax payers. Thus, an unexpected decrease in the population growth rate may motivate a government to use monetary policy instruments to maintain its fiscal position, which intensifies inflation or deflation.

Which of inflation or deflation will occur depends on level of primary deficit and monetary policy stance. (i) Under an expansionary monetary policy and primary surplus (or small primary deficit), a declining population places downwards pressure on the upper bound for public debt. This then motivates a government to print more money to maintain the upper bound. Consequently, inflation accelerates, or deflation slows. (ii) In contrast, under a contractionary monetary policy and large primary deficit, a declining population may result in accelerated deflation or slowed inflation.

The analyses of this paper postulate that consumers obtain utility from real money balances. Models of the monetary economy, called money-in-utility models, have a long tradition. Brock (1974) analyzed money-in-utility models, and showed that the hyper-deflationary path may be one of equilibrium. Obstfeld and Rogoff (1983) proposed a set of conditions that prevent hyperinflation in a money-in-utility model. Matsuyama (1991) demonstrated that if consumers obtain utility from real consumption and real money balances through complex relationships, then a model of monetary economy can even explain chaotic price dynamics. Fukuda (1993) showed that the timing of consumers obtaining utility from real money can create complex price dynamics. By using a closed economy version of Neumeyer and Yano's model, Kondo (2007) theoretically examined how the upper bound for public debt that is derived as the NPG condition depends on various factors including inflation rate. Kondo and Kitaura (2009) showed that deflation can help governments by raising the upper bound for public debt if the primary deficit is sufficiently large. However, their analysis did not consider population decline. Yakita (1989) employed an overlapping generations model, and studied optimal inflation and taxation rates. Although he incorporated both money and population growth into the model, the upper bound for sustainable public debt was not derived. Later, the same author (Yakita, 2006) investigated the effect of a slight change in life expectancy on economic growth and inflation rates through portfolio selection, where consumers made choices between physical capital and money. However, he did not consider public sector liabilities in the 2006 paper.

Reflecting fiscal crises in many industrialized countries, numerous papers have addressed the sustainability of public debt. Chalk (2000) also used an overlapping generations model to point out the importance of the initial level of public debt for fiscal sustainability, also considering levels of primary deficit, economic growth rate, long-term interest rate and the initial level of physical capital. Bräuninger (2005) explored the relationship of government budget deficit with gross domestic product (GDP) ratio and economic growth rates. Yakita (2008) and Arai (2011) both incorporated productive public spending into overlapping generations models. These authors demonstrated the existence of a critical level of public debt; the debt does not diverge provided the debt to GDP ratio remains below the critical level. However, these papers did not take money into account, and did not investigate theoretical relationships among population growth rates, monetary policies and sustainability of public debt. Yakita and Arai assumed constant population size. Bräuninger (2005) adopted the AK-type production structure, and population size did not decisively influence the assessment of the sustainability of public debt in his model.

Several methods are proposed to test whether fiscal policy is sustainable, and many empirical works dealing with this question have emerged since Hamilton and Flavin (1986).[1] Among these works, Bohn (1998) proposed a sufficient condition for the public debt to be sustainable that incorporates insightful economic intuition. When the ratio of public debt to GDP increases, a government should take corrective action that recovers the ratio of primary surplus to GDP to maintain a healthy fiscal position. By assuming the Bohn's type budget rule, Greiner (2011, 2012) investigated growth and/or welfare effects of fiscal policies within endogenous growth models. Introducing money to consumer utility functions, Greiner (2013) extended his analyses to a money-in-utility model. Although Greiner considered the sustainability of public debt in his series of papers, he assumed constant population size, and thus did not consider the economic effects of decreased population growth rate.

The reminder of this paper is organized as follows. Section 2 presents the model on which the analyses are based. Sections 3 and 4 explicitly derive the equilibrium path and the upper bound for public debt compatible with the NPG condition, respectively. Section 5 then offers main policy analyses. Finally, section 6 presents concise conclusions.

[1] See, for example, Trehan and Walsh (1988, 1991), Wilcox (1989), Fukuda and Teruyama (1994) and, more recently, Fincke and Greiner (2012).

2 Neumeyer-Yano Model

Think of a dynamic economy in which time points are discretely indexed as $t = 0, 1, \cdots$. The period between time points $t-1$ and t is called period t. There are many identical consumers and a single government. The government includes a central bank that controls the money supply. The population during period t is N_t. The population growth rate is assumed to be $n = N_{t+1}/N_t - 1$ for any $t = 1, 2, \cdots$. The agents transact consumption goods, money and bonds during each period. The government imposes a lump-sum tax τ_t in a consumption good form on every consumer, and spends the associated tax revenue g_t, during the period t. Furthermore, the government provides money M_t^g and an interest-bearing bond B_t^g. A bond generates nominal interest rate i_t during period t to $t+1$.

The representative consumer during period t has $1+n$ siblings. She also has $1+n$ children, since the population growth rate is n. Throughout this paper, uppercase letters represent aggregate variables while lowercase letters denote per capita variables, for example $G_t = N_t g_t$ and $T_t = N_t \tau_t$. The representative consumer inherits both money $m_{t-1}/(1+n)$ and bond assets with interest revenue $(1+i_{t-1})\, b_{t-1}/\,(1+n)$ at time $t-1$ from her parents, and passes on m_t and b_t to her own children. The representative consumer produces a consumption good y_t, consumes it c_t and pays tax τ_t to the government. The flow budget constraint for period t is given by

$$m_t + b_t \leq p_t(y_t - \tau_t - c_t) + \frac{m_{t-1}}{1+n} + (1+i_{t-1})\,\frac{b_{t-1}}{1+n}, \tag{1}$$

which holds for any $t = 1, 2, \cdots$. The consumer obtains utility from three factors: consumption c_t, real money balances $\widetilde{m}_t\,(= m_t/p_t)$ and her children's welfare. A period-wise utility function is represented in log-form. Hence, the utility of the consumer during period t is

$$U_t = (\log c_t + \gamma \log \widetilde{m}_t) + \beta \cdot (1+n)\, U_{t+1}, \tag{2}$$

where γ reflects her preference for real money holdings relative to consumption, $\beta \in (0,\ 1/\,(1+n))$ is a discount factor and U_{t+1} is the utility level of the representative consumer living in period $t+1$. Under this setting, the behavior of the representative consumer during the initial period 1, is summarized as the following maximizing problem:

$$\max_{\{c_t,m_t,b_t\}_{t=1}^{\infty}} \sum_{t=1}^{\infty} [\beta (1+n)]^{t-1} (\log c_t + \gamma \log \widetilde{m}_t) \tag{3}$$
$$\text{s.t. Equation (1)}$$
$$\text{given } \{p_t, i_t\}_{t=1}^{\infty}, \; m_0/(1+n_0), \; (1+i_0) b_0/(1+n_0)$$

where $n_0 = N_1/N_0 - 1$ and is fixed throughout this paper.

The government sets up a time stream of its policy variables $\{M_t^g, B_t^g, G_t, T_t\}$. The stream is subject to the flow budget constraints

$$M_t^g + B_t^g = p_t (G_t - T_t) + M_{t-1}^g + (1+i_{t-1}) B_{t-1}^g, \tag{4}$$

for any $t = 1, 2, \cdots$. For simplicity, I assume government spending G_t is not productive.

A stream of prices and allocations for goods is determined so that all markets are cleared simultaneously. In other words, the equilibrium conditions are

$$N_t c_t + G_t = N_t y_t, \quad N_t m_t = M_t^g, \quad N_t b_t = B_t^g, \tag{5}$$

for any $t = 1, 2, \cdots$. Walras's law makes the bond market clearing condition redundant, and thus ignorable for any $t = 1, 2, \cdots$. However, this paper highlights the time evolution of the bond market.

3 Equilibrium Path

This section explicitly derives an equilibrium path. The result, which is summarized as Lemma 1 and 2, will be frequently used throughout the remainder of the paper.

First, based on the maximizing problem (3), the following relationships are derived as first order conditions:

$$\frac{c_{t+1}}{c_t} = \beta (1 + r_t), \tag{6}$$
$$\delta_t \widetilde{m}_t = \gamma c_t, \tag{7}$$

which hold for any $t = 1, 2, \cdots$, where $r_t \equiv (1+i_t) p_t/p_{t+1} - 1$ is the real interest rate and $\delta_t \equiv i_t/(1+i_t)$ is the value of the real money balance. While there is no need to explain real interest rate, the value of the real money balance δ_t may need an explanatory note. From the consumers' point of view, the value of the real money balance is the opportunity cost of holding real money. It can be transformed as

$$\delta_t = \frac{1}{1+r_t}\left(r_t + \frac{p_{t+1}-p_t}{p_{t+1}}\right). \tag{8}$$

The first term in the round brackets in (8) is interest payment cost, and the second term reflects inflation cost. Thus, δ_t is clearly the present (the period t) value of the opportunity cost of holding the real money balance. Similarly, from the government's point of view, it represents the real value of seigniorage.

Two assumptions are made to solve the model.

Assumption 1 $y_{t+1}/y_t = g_{t+1}/g_t = \tau_{t+1}/\tau_t = 1+\theta$ for any $t = 1, 2, \cdots$.

Assumption 2 $0 \leq \tau_1 < y_1$ and $0 \leq g_1 < y_1$.

Assumption 1 means this paper restricts the analyses to a balanced growth path in which real per capita production y_t (in addition to government spending g_t and real tax levy τ_t per person) grows at the constant rate θ. Although it could be argued that the assumption of exogenously-given income endowment y_t is too restrictive, it is frequently used in the existing studies.[2] As a direct consequence from Assumption 1, aggregate variables grow at a rate of $(1+n)(1+\theta)-1$. Assumption 1 also implies that per capita real primary balance $\tau_t - g_t$ also grows at the same rate θ. Assumption 2 is self-explanatory.

The assumptions allow real variables along the equilibrium path to be derived as follows.

Lemma 1 *(Real Variables)*

The consumption and the real interest rate in the equilibrium satisfy

$$(a)\ c_t = (1+\theta)^{t-1}(y_1 - g_1), \qquad (b)\ r_t = \frac{1+\theta}{\beta} - 1,$$

for any $t = 1, 2, \cdots$.

[2]See, e.g. Matsuyama (1991), Fukuda (1993) and Zou (1997).

Proof. (a) Based on the market clearing condition and Assumption 1,

$$\frac{c_{t+1}}{c_t} = \frac{y_{t+1} - g_{t+1}}{y_t - g_t} = 1 + \theta, \tag{9}$$

for any $t = 1, 2, \cdots$. Using (9) and the market clearing condition at $t = 1$ yields

$$c_t = (1+\theta)\, c_{t-1} = \cdots = (1+\theta)^{t-1} c_1 = (1+\theta)^{t-1} (y_1 - g_1)\,.$$

(b) By (6) and (9), the desired result is obtained. ■

According to (b) of the above lemma, the real interest rate positively correlates with the productivity growth rate θ. The reason is as follows: high productivity means the provision of future goods will be more plentiful relative to current goods, and thus decreases the relative price of future goods to current goods. Meanwhile, the population growth rate n does not affect the real interest rate, although it affects GDP growth in the same manner as θ. Since the increase in the future population does not only implies more provision of future goods, but also more demand, it does not affect the intertemporal relative price system in the present model.

Next, I will solve the nominal variables (price of the consumption good, nominal interest rates, etc.), which requires some additional assumptions.

Assumption 3 $M^g_{t+1}/M^g_t = 1 + \mu$ for any $t = 0, 1, 2, \cdots$.

Assumption 4 $\beta\,(1+n) < 1 + \mu$.

Assumption 5 $\lim_{T\to\infty} \left(\prod_{t=1}^{T} 1/\,(1+i_t)\right) = 0$.

Assumption 3, which is standard in the literature, is useful in that it simplifies the analyses presented below. According to Assumption 4, a contractionary monetary policy—the case of $\mu < 0$—is admitted in this paper. However, the shrinkage of the money stock must be slow in relation to the discount rate of the consumers for the utilities of the future generations. Assumption 5 excludes equilibria in which the nominal interest rate rapidly falls to zero, although this type of monetary models may exhibit equilibria with zero interest rate.[3]

[3]See, e.g. Benhabib et al (2002).

Given the series of up-front assumptions, the nominal variables in the equilibrium can be derived explicitly.

Lemma 2 *(Nominal Variables)*

The nominal variables in the equilibrium satisfy the following for any $t = 1, 2, \cdots$:

$$(a)\ i_t = \frac{1+\mu}{\beta(1+n)} - 1, \qquad (b)\ \delta_t = \frac{1+\mu-\beta(1+n)}{1+\mu},$$

$$(c)\ p_t = \left(\frac{1+\mu}{(1+\theta)(1+n)}\right)^{t-1} \frac{1+\mu-\beta(1+n)}{\gamma N_1 (y_1 - g_1)} M_0^g,$$

$$(d)\ M_t^g = (1+\mu)^t M_0^g, \qquad (e)\ m_t = \left(\frac{1+\mu}{1+n}\right)^{t-1} \frac{(1+\mu) M_0^g}{N_1},$$

$$(f)\ \widetilde{m}_t = \gamma \frac{1+\mu}{1+\mu-\beta(1+n)} (1+\theta)^{t-1} (y_1 - g_1),$$

$$(g)\ B_t^g = \left(\frac{1+\mu}{\beta(1+n)}\right)^{t-1} (1+i_0) B_0^g - \left[\sum_{s=0}^{t-1} \left(\frac{1}{\beta(1+n)}\right)^s\right] \times \left(\mu + \frac{1+\mu-\beta(1+n)}{\gamma} \frac{\tau_1 - g_1}{y_1 - g_1}\right) (1+\mu)^{t-1} M_0^g.$$

Proof. See Appendix. ■

Some remarks regarding the population growth rate n should be made here. Equation (a) shows that the population growth rate negatively correlates with nominal interest rate i_t. The reason for this is as follows: Suppose the population growth rate slightly increases while other parameters are fixed. The representative consumer then gains more utilities from her children, and tends to transfer more assets to them. This implies increased demand for public bonds, and so the nominal interest rate decreases to maintain equilibrium.

Please review the equation (c). The inflation rate is $(1+\mu)/(1+\theta)(1+n) - 1$, and the price of the consumption good remains constant over time if and only if the money supply growth rate μ is set to equal the real GDP growth rate $(1+\theta)(1+n) - 1$. Notably, the inflation rate is negatively related to the population growth rate. A recovery of the population growth rate implies that GDP growth, and thus inflation rate decreases if money supply growth rate is fixed. Note that the value of the real money balance δ_t negatively correlates with the population growth rate (see equation (b) in Lemma 2) because it positively depends on the inflation rate (see equation (8)).

Equation (c) shows that the initial price level, p_1, is also negatively correlated with n. If the population growth rate increases, consumers tend to pass more assets to their children, and demand for consumption goods decreases. As a result, price level is depressed, which increases the real value of government liabilities during the initial period. Through this channel, population growth may negatively affect government fiscal position, as is shown in Section 5.2.

Because equation (g) seems slightly complicated, an easy explanation should be presented of how various factors affect public debt level. Based on (4), it holds that

$$B_1^g = (1 + i_0) B_0^g - \left[(M_1^g - M_0^g) + p_1 N_1 (\tau_1 - g_1) \right].$$

The first term on the right hand side is the effect of interest burden on public debt. The second term represents effects that decrease debt; the first part $(M_1^g - M_0^g)$ reflects the effect of money creation on public debt, and the second part reflects the effect of primary surplus.

4 No Ponzi-game (NPG) Condition

This section formally defines a situation where public debt is "sustainable", and derives an upper bound for public debt that is sustainable in the equilibrium path. The next section analyzes this upper bound and demonstrates the possibilities for a slight decrease in population growth rate resulting in inflation or deflation.

The formal definition of "sustainability" of public debt as used in this paper is as follows:

Definition 1 *(Sustainability of Public Debt) Let* $\{p_t, r_t, i_t\}$ *be a stream of price variables in an equilibrium. Given initial conditions* (M_0^g, B_0^g)*, policy parameters* $(\{G_t, T_t\}, \mu)$ *are said to be sustainable if the NPG condition*

$$\limsup_{T \to \infty} \left(\prod_{t=1}^{T} \frac{1}{1 + r_t} \right) \frac{D_T^g}{p_T} \leq 0 \tag{10}$$

is satisfied, where

$$D_t^g = M_{t-1}^g + (1 + i_{t-1}) B_{t-1}^g \tag{11}$$

represents the government's financial liabilities in the equilibrium condition.

As is well-known, this definition is logically equivalent to an inter-temporal budget constraint for the government

$$\frac{D_1^g}{p_1} + \sum_{t=1}^{\infty} \left(\prod_{j=1}^{t-1} \frac{1}{1+r_j} \right) G_t \leq \sum_{t=1}^{\infty} \left(\prod_{j=1}^{t-1} \frac{1}{1+r_j} \right) \left(T_t + \delta_t \frac{M_t^g}{p_t} \right), \qquad (12)$$

where $\prod_{t=1}^{0} 1/(1+r_t) = 1$. Following the literature[4], this paper adopts the condition (10) for the sustainability of public debt.

Substitute the equilibrium values presented in Lemma 1 and 2 into (10), then I obtain a condition for the initial level of public debt $(1+i_0) B_0^g$ to be compatible with (10):

$$(1+i_0) B_0^g \leq \frac{[1+\mu-\beta(1+n)](\tau_1 - g_1) + \mu\gamma(y_1 - g_1)}{\gamma[1-\beta(1+n)](y_1 - g_1)} M_0^g \equiv \varphi. \qquad (13)$$

In the equation (13), the upper bound for public debt at which the initial level of public debt is compatible with (10) is denoted by φ.[5] The following lemma is thus established.

Lemma 3 *The upper bound for public debt that is considered sustainable in the dynamic general equilibrium is given by* (13).

By (13), it holds that

$$\varphi \gtreqless 0 \iff (-\gamma <) \frac{-\mu\gamma}{1+\mu-\beta(1+n)} \lesseqgtr \frac{\tau_1 - g_1}{y_1 - g_1}. \qquad (14)$$

As is seen in (14), if the money supply growth rate μ is positive, the primary balance can be negative given positive upper bound for public debt φ, because in such situations the government can use seigniorage. The following analyses do not exclude possibilities where $\varphi < 0$. This paper assumes that even if public debt becomes unsustainable, the government has an incentive to increase the upper bound φ using monetary policy μ.

[4]See, e.g. Kondo (2007).

[5]Under the parameter constellation satisfying (13), the debt to GDP ratio $B_t^g/p_t Y_t$ path becomes bounded by above. The boundedness condition is frequently used in public finance literature as a condition for the sustainability of public debt.

5 Main Analyses

This section presents the main analyses of this paper. The focus is on the effects of unexpected changes in monetary policies and population growth rates. Such changes may damage or improve the long-term fiscal standing of the government. In other words, changes in monetary policy and population growth rate affect the upper bound for public debt φ that is compatible with the NPG condition. Further, the analyses show that interactions between monetary policies and population growth rates can cause inflation or deflation.

5.1 Effects of Monetary Policies

First, I investigate the effects of monetary policies on the long-term fiscal position of the government.

Easy calculation yields

$$\frac{\partial\varphi}{\partial\mu} = \frac{(\tau_1 - g_1) + \gamma(y_1 - g_1)}{\gamma[1 - \beta(1+n)](y_1 - g_1)} M_0^g. \tag{15}$$

Thus,

$$\frac{\partial\varphi}{\partial\mu} \gtreqqless 0 \iff -\gamma \lesseqqgtr \frac{\tau_1 - g_1}{y_1 - g_1}. \tag{16}$$

What does the condition (16) mean economically? To answer that question, note that $(\tau_1 - g_1)/(y_1 - g_1)$ can be rewritten as

$$\frac{\tau_1 - g_1}{y_1 - g_1} = \frac{\tau_t - g_t}{c_t}$$

since $y_1 - g_1 = c_1$. Furthermore, it holds, by (7), that $\gamma = \delta_t \widetilde{m}_t / c_t$. Thus, on the one hand,

$$-\gamma \lesseqqgtr \frac{\tau_1 - g_1}{y_1 - g_1} \iff -\delta_t M_t^g \lesseqqgtr p_t(T_t - G_t), \tag{17}$$

in the equilibrium. On the other hand, by (11), the flow budget constraint for the government (4) can be transformed as

$$\frac{1}{1+i_t} D_{t+1}^g - D_t^g + \delta_t M_t + p_t(T_t - G_t) = 0.$$

Thus, by (17), the following equivalence relationship holds in the equilibrium:

$$-\gamma \lesseqqgtr \frac{\tau_1 - g_1}{y_1 - g_1} \iff \frac{1}{1+i_t} D^g_{t+1} - D^g_t \lesseqqgtr 0. \quad (18)$$

To sum up, it holds in the equilibrium that

$$\frac{\partial \varphi}{\partial \mu} \gtreqqless 0 \iff \frac{1}{1+i_t} D^g_{t+1} - D^g_t \lesseqqgtr 0. \quad (19)$$

Expansionary (contractionary) monetary policy improves the government's fiscal standing under situations in which the present value of the government's liability is decreasing (increasing) as time passes.

Notably, monetary policy that reduces (increases) money supply growth rate μ does not necessarily result in deflation (inflation) because inflation rate is determined by the combined effects of money growth rate, population growth rate and productivity growth rate.

From (16), the following theorem has been established.[6]

Theorem 1 *(Effects of Monetary Policies)*
(i) If $-\gamma < (\tau_1 - g_1)/(y_1 - g_1)$, an expansionary monetary policy improves government fiscal position in that it raises the upper bound for public debt φ that is compatible with the NPG condition and the DGE, i.e., $\partial\varphi/\partial\mu > 0$. (ii) If $(\tau_1 - g_1)/(y_1 - g_1) < -\gamma$, a contractionary monetary policy improves government fiscal standing, i.e., $\partial\varphi/\partial\mu < 0$.

5.2 Effects of Population Growth Rates

Next, I examine the effects of changes in population growth rate on the upper bound for public debt φ.

The following relationship is easily ascertained:

$$\frac{\partial \varphi}{\partial n} = \frac{(\tau_1 - g_1) + \gamma(y_1 - g_1)}{\gamma(y_1 - g_1)[1 - \beta(1+n)]^2} \mu\beta M^g_0. \quad (20)$$

Hence,

$$\text{if } \mu > 0, \text{ it holds that } \frac{\partial \varphi}{\partial n} \gtreqqless 0 \iff -\gamma \lesseqqgtr \frac{\tau_1 - g_1}{y_1 - g_1}; \quad (21)$$

$$\text{if } \mu < 0, \text{ it holds that } \frac{\partial \varphi}{\partial n} \gtreqqless 0 \iff -\gamma \gtreqqless \frac{\tau_1 - g_1}{y_1 - g_1}. \quad (22)$$

[6] Theorem 1 extends the results of a previous study by Kondo and Kitaura (2009) to a model with population growth.

What do the conditions (21) and (22) mean? It can be ascertained that (18) holds in much the same way as the previous subsection. Thus, (21) means that under the expansionary monetary policy, recovery of the population growth rate improves the government's fiscal standing under situations in which the present value of the government's liability is decreasing as time passes. The condition (22) can be interpreted in a similar way; under the contractionary monetary policy, a slight decrease in the population growth rate improves the government's fiscal standing under situations in which the present value of the government's liability is decreasing as time passes.

The signs of the derivative $\partial\varphi/\partial n$ can be concisely summarized in the following table:

	$\frac{\tau_1 - g_1}{y_1 - g_1} < -\gamma$	$-\gamma < \frac{\tau_1 - g_1}{y_1 - g_1}$
$\mu > 0$	$-$	$+$
$\mu < 0$	$+$	$-$

(23)

Table: the signs of $\partial\varphi/\partial n$

According to Theorem 1, if $-\gamma < (\tau_1 - g_1) / (y_1 - g_1)$, the government has an incentive to print more money. Therefore, it is natural to assume $\mu > 0$ in this situation, which can be considered as an "incentive compatibility condition" for a government. Under the assumptions— $-\gamma < (\tau_1 - g_1) / (y_1 - g_1)$ and $\mu > 0$, it holds that $\partial\varphi/\partial n > 0$ by (21) or (23). That is, a slight increase in the population growth rate improves government fiscal position, a phenomenon that is intuitive since increased population growth rate n implies more future tax payers.

Theorem 1 also implies that if $(\tau_1 - g_1) / (y_1 - g_1) < -\gamma$, a government that seeks a measure to raise the upper bound for public debt φ has a strong incentive to reduce the money supply. Thus, this paper assumes $\mu < 0$ in this situation. In such an incentive-compatible case, the equation (22) or the table (23) indicates that $\partial\varphi/\partial n > 0$ holds. In other words, declining population damages government fiscal position.

The results obtained in this subsection can be summarized as the following theorem.

Theorem 2 *(Effects of Population Growth Rates)*

Suppose that fiscal and monetary policies are implemented in "incentive-

compatible" ways; in other words, assume that

$$(i)\ -\gamma\ < \frac{\tau_1 - g_1}{y_1 - g_1}\ \textit{and}\ \mu > 0;\ \textit{or}$$
$$(ii)\ \frac{\tau_1 - g_1}{y_1 - g_1} < -\gamma\ \textit{and}\ \mu < 0.$$

Then, a slight decrease in the population growth rate negatively impacts government long-term fiscal position in the sense that $\partial\varphi/\partial n > 0$.

5.3 Population Growth Rates and Monetary Policies

Given the results from the previous two subsections, this subsection examines the theoretical interaction between population growth rates and monetary policies through government fiscal position. The analyses demonstrate that a slight decrease in population growth rate may contribute to inflation or deflation.

Let me note that the fiscal policies (g, τ) are fixed throughout the analyses. As this paper intends to investigate the interaction between population growth rates and monetary policies (inflation rates), an assumption that there are no adjustments to the primary surplus is required. Although this may be a strong assumption, it is difficult to analyze the interaction without fixing other factors.

To examine the interaction between population growth rates and monetary policies, the following analyses use a graphical method. To draw graphs of φ as functions of μ in the (μ, φ) space, note that by (15) or (20),

$$\frac{\partial^2\varphi}{\partial\mu\partial n} = \frac{(\tau_1 - g_1) + \gamma(y_1 - g_1)}{\gamma(y_1 - g_1)[1 - \beta(1+n)]^2}\beta M_0^g. \tag{24}$$

Hence,

$$\frac{\partial^2\varphi}{\partial\mu\partial n} \gtreqqless 0 \ \text{ if and only if } \ -\gamma \lesseqqgtr \frac{\tau_1 - g_1}{y_1 - g_1}. \tag{25}$$

Next, I check the values of $\varphi|_{\mu=0}$ and $\mu|_{\varphi=0}$. Given (13), then

$$\varphi|_{\mu=0} = \frac{1}{\gamma}\frac{\tau_1 - g_1}{y_1 - g_1}M_0^g. \tag{26}$$

The level of the upper bound for public debt φ when $\mu = 0$, which is derived in (26), is independent of the population growth rate n. Furthermore, substituting the condition $\varphi = 0$ into (13) yields

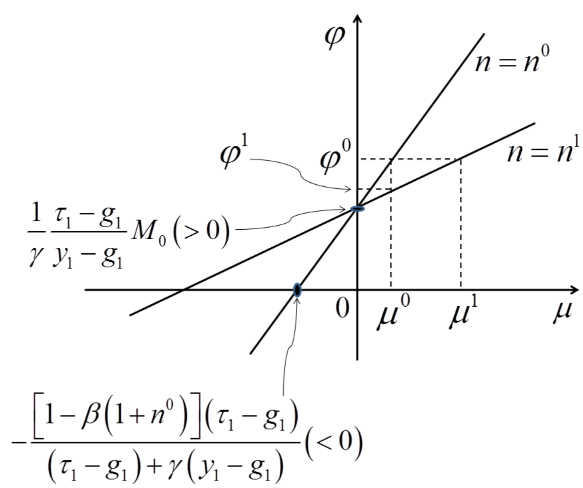
$$\mu\,|_{\varphi=0} = -\frac{[1-\beta(1+n)](\tau_1 - g_1)}{(\tau_1 - g_1) + \gamma(y_1 - g_1)}. \tag{27}$$

Two "incentive-compatible" cases (i) and (ii) introduced in the previous subsection are the focus below.

Case (i) : $-\gamma < (\tau_1 - g_1) / (y_1 - g_1)$ and $\mu > 0$.

This case is further divided into two sub-cases: the case with primary surplus $\tau_1 - g_1 > 0$ and the case with low primary deficit $(-\gamma(y_1 - g_1) <) \, \tau_1 - g_1 < 0$.

First, consider the sub-case: $\tau_1 - g_1 > 0$. Then, $\partial\varphi/\partial\mu > 0$, $\varphi\,|_{\mu=0} > 0$ and $\mu\,|_{\varphi=0} < 0$ by (16), (26) and (27), respectively. Hence, the relationship between the upper bound φ and the monetary policy parameter μ can be drawn on the (μ, φ) plane as shown below (Figure 1):

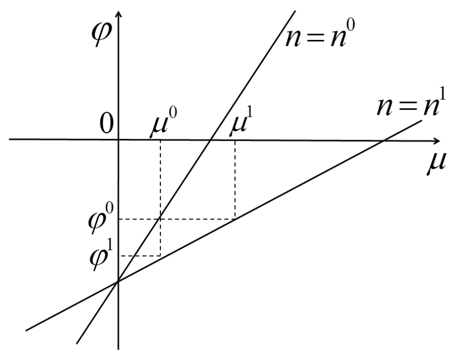


(Figure 1)

Consider a situation where the initial population growth rate is n^0 and the money supply growth rate is μ^0. The upper bound is then determined at φ^0. Suppose the population growth rate declines to $n^1 \, (< n^0)$. Then, by (25), the value of $\partial\varphi/\partial\mu$ decreases as seen in Figure 1. If the government fixes its monetary policy at $\mu = \mu^0$, its fiscal position is damaged; the upper bound for public debt φ decreases from φ^0 to $\varphi^1 \, (< \varphi^0)$. To recover to the original level φ^0, the government can increase money supply growth rate to $\mu^1 \, (> \mu^0)$, which implies an increase in inflation or a decrease in deflation.

(Which of inflation or deflation occurs depends not only on the sign of μ, but also on $\mu \lesseqgtr (1+\theta)(1+n)-1$. For this point, see Lemma 2-(c).)

Next, consider the sub-case: $(-\gamma(y_1-g_1)<)\,\tau_1-g_1<0$. In this case, it holds that $\partial\varphi/\partial\mu>0$, $\varphi|_{\mu=0}<0$ and $\mu|_{\varphi=0}>0$ by (16), (26) and (27), respectively. By (25), if the population growth rate decreases, the slope of the graph of φ as a function of μ becomes flatter. Hence, the relationship between the upper bound for public debt φ and the monetary policy parameter μ can be drawn on the (μ,φ) plane as Figure 2:



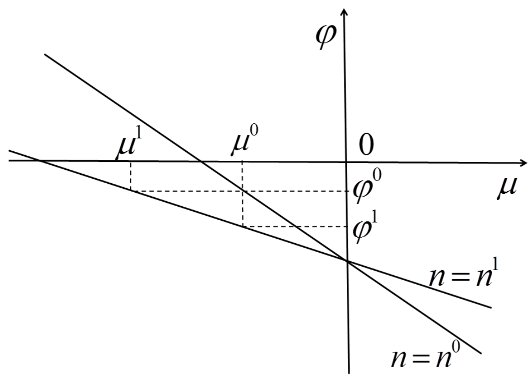
(Figure 2)

Consider a situation where the initial population growth rate is n^0 and the money supply growth rate is μ^0. Then, the upper bound for public debt is determined at φ^0. Suppose the population growth rate declines to $n^1\,(<n^0)$, then the upper bound decreases to $\varphi^1\,(<\varphi^0)$. The government wishes to maintain the upper bound for public debt at φ^0 and has an incentive to expand money supply to $\mu^1\,(>\mu^0)$, which implies increased inflation or decreased deflation.

To conclude, in the case (i), a slight decrease in population growth rate increases inflation or reduces deflation.

Case (ii) : $(\tau_1-g_1)/(y_1-g_1)<-\gamma$ and $\mu<0$.

In this case, $\partial\varphi/\partial\mu<0$, $\varphi|_{\mu=0}<0$ and $\mu|_{\varphi=0}<0$ by (16), (26) and (27), respectively. By (25), a decrease in population growth rate flattens the slope of the graph of φ as a function of μ. Hence, the relationship between the upper bound for public debt φ and the monetary policy parameter μ can be plotted on the (μ, φ) plane as Figure 3:



(Figure 3)

Consider a situation where the initial population growth rate is n^0 and the money supply growth rate is μ^0. The upper bound for public debt is φ^0. Suppose that the population growth rate declines to $n^1\,(< n^0)$. Then, the upper bound for public debt decreases to $\varphi^1\,(< \varphi^0)$ if the money supply growth rate remains μ^0. The government has an incentive to maintain the upper bound for public debt at φ^0 and may decrease the money supply growth rate to $\mu^1\,(< \mu^0)$, which increases deflation or decreases inflation.

To conclude, in the case (ii), the declines of the population growth rate may increase deflation or decrease inflation.

The following theorem summarizes the results obtained in this subsection.

Theorem 3 *(Population Growth Rates and Monetary Policies)*
(i) *In the case where* $-\gamma < (\tau_1 - g_1)/(y_1 - g_1)$ *and* $\mu > 0$*, a slight decrease in the population growth rate increases inflation or decreases deflation.* (ii) *In the case where* $(\tau_1 - g_1)/(y_1 - g_1) < -\gamma$ *and* $\mu < 0$*, a slight decrease in the population growth rate increases deflation or decreases inflation.*

Theorem 3, which is one of the main theorems in this paper, can be explained from an economic standpoint as follows; (Readers may wish to remind themselves of (18) in Section 5.1.) (i) Under situations in which the present value of the government's liability is decreasing and the money supply is increasing, population decline results in higher inflation, and (ii) under situations in which the present value of the government's liability is increasing and the money supply is decreasing, population decline results in higher deflation. Fiscal crisis could be a channel through which population movement invites price disturbance.

6 Concluding Remark

This paper reveals theoretical relationships between declining population and inflation/deflation during fiscal crises within a frictionless neoclassical framework. The first step is to derive an upper bound for public debt that is compatible with the NPG condition and a DGE. This upper bound depends on the growth rates of both population and money supply. The next step is to analyze the nature of the dependence of the upper bound for public debt on these two factors. The analytical results show why population decline results in inflation or deflation. In some plausible scenarios, a slight decrease in the population growth rate implies a lower upper bound for public debt, which may be interpreted as population decline causing a fiscal crisis. This motivates a government to implement expansionary or contractionary monetary policies. If the primary deficit is not high and monetary policy is expanding, a slight decrease in the population growth rate increases inflation. Conversely, deflation becomes severe if the primary deficit exceeds a critical level and monetary policy is contracting.

While this topic is of interest to economists and policy makers in the many developed countries experiencing population decline, price disturbances and fiscal crises, the model economy analyzed in the present study is simple. To construct a sound theoretical basis for appropriate economic policies, future studies should extend the model presented here.

For simplicity this paper ignored the accumulation of physical and human capital, but considering capital accumulation would be desirable. The effect of demographic change on public debt to capital ratio is an important research topic. Second, the model setting of the present paper should be extended to include international transactions. The effect of population decline overseas on domestic government monetary policies through international markets is an interesting question. Finally, other types of monetary models, e.g. an overlapping generations model, should be analyzed to compare their theoretical results with those of this study.

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Appendix—Proof of Lemma 2

The appendix provides a proof of Lemma 2. The proof uses the definition $\delta_t \equiv i_t / (1 + i_t)$, which is the value of real money balance.

Sublemma 1 The nominal variables in the equilibrium satisfy the following relationships:

$$
\begin{aligned}
(a)\ \frac{p_{t+1}}{p_t} &= \frac{1}{(1+\theta)(1+n)} \frac{\delta_{t+1} M_{t+1}^g}{\delta_t M_t^g}, \\
(b)\ 1 + i_t &= \frac{1}{\beta(1+n)} \frac{\delta_{t+1} M_{t+1}^g}{\delta_t M_t^g},
\end{aligned}
$$

for any $t = 1, 2, \cdots$.

Proof. (a) By the equation (7) and Lemma 1-(a), it holds that

$$\frac{\delta_{t+1}\widetilde{m}_{t+1}}{\delta_t\widetilde{m}_t}=\frac{c_{t+1}}{c_t}=1+\theta.$$

Multiplying N_{t+1}/N_t for both sides yields

$$\frac{N_{t+1}\delta_{t+1}\widetilde{m}_{t+1}}{N_t\delta_t\widetilde{m}_t}=(1+\theta)(1+n).$$

This immediately implies the desired result.

(b) By Lemma 1-(b) and (a) of this sublemma, I can obtain

$$\begin{aligned}1+i_t &\equiv (1+r_t)\frac{p_{t+1}}{p_t}\\ &= \frac{1+\theta}{\beta}\frac{1}{(1+\theta)(1+n)}\frac{\delta_{t+1}M^g_{t+1}}{\delta_t M^g_t}=\frac{1}{\beta(1+n)}\frac{\delta_{t+1}M^g_{t+1}}{\delta_t M^g_t}.\end{aligned}$$

This establishes the result. ■

The next sublemma is a crucial sub-step for proving Sublemma 3.

Sublemma 2 It holds in the equilibrium that

$$(a)\ \delta_t=1-\frac{\beta}{1+\theta}\frac{1+n}{1+\mu}\frac{\widetilde{m}_{t+1}}{\widetilde{m}_t},\qquad (b)\ \frac{\beta}{1+\theta}\frac{1+n}{1+\mu}=\frac{1}{1+i_t}\frac{\widetilde{m}_t}{\widetilde{m}_{t+1}},$$

for any $t=1,2,\cdots$.

Proof. (a) First, note the following relationship

$$\frac{N_{t+1}\widetilde{m}_{t+1}}{N_t\widetilde{m}_t}=\frac{M^g_{t+1}/p_{t+1}}{M^g_t/p_t}=(1+\mu)\frac{p_t}{p_{t+1}}.$$

As a direct consequence,

$$\frac{p_t}{p_{t+1}}=\frac{1+n}{1+\mu}\frac{\widetilde{m}_{t+1}}{\widetilde{m}_t}. \tag{28}$$

By using (28), the desired result can be demonstrated as follows

$$\delta_t=1-\frac{1}{1+i_t}=1-\frac{1}{1+r_t}\frac{p_t}{p_{t+1}}=1-\frac{\beta}{1+\theta}\frac{1+n}{1+\mu}\frac{\widetilde{m}_{t+1}}{\widetilde{m}_t}. \tag{29}$$

The result (b) immediately follows from the equation (29). ■

The next sublemma concerns the equilibrium level of the real money balance $\widetilde{m}_t$ and the real per capita consumption c_t.

Sublemma 3 In the equilibrium, the following holds

$$\widetilde{m}_t = \gamma c_t \frac{1+\mu}{1+\mu-\beta(1+n)},$$

for any $t = 1, 2, \cdots$.

Proof. Take $t = 1, 2, \cdots$ arbitrarily. Easy induction from (7) and Sublemma 2-(a) yields

$$\widetilde{m}_t - \left(\frac{\beta(1+n)}{(1+\theta)(1+\mu)}\right)^T \widetilde{m}_{t+T} = \gamma \sum_{s=0}^{T-1} \left(\frac{\beta(1+n)}{(1+\theta)(1+\mu)}\right)^s c_{t+s}. \tag{30}$$

Limit values of both sides of the above equation (30) as $T \to \infty$ can be obtained as follows. On the one hand, by Lemma 1 and Assumption 4,

$$\begin{aligned} RHS \text{ of } (30) &= \gamma \sum_{s=0}^{T-1} \left(\frac{\beta(1+n)}{(1+\theta)(1+\mu)}\right)^s \beta^s \left(\prod_{j=1}^{s} (1+r_{t+j-1})\right) c_t \\ &= \gamma \sum_{s=0}^{T-1} \left(\frac{\beta(1+n)}{(1+\theta)(1+\mu)}\right)^s \beta^s \left(\frac{1+\theta}{\beta}\right)^s c_t \\ &= \gamma c_t \sum_{s=0}^{T-1} \left(\frac{\beta(1+n)}{1+\mu}\right)^s \\ &\longrightarrow \gamma c_t \frac{1+\mu}{1+\mu-\beta(1+n)} \text{ as } T \to \infty. \end{aligned}$$

On the other hand, by Sublemma 2-(b) and Assumption 5,

$$\begin{aligned} LHS \text{ of } (30) &= \widetilde{m}_t - \left(\prod_{j=1}^{T} \frac{1}{1+i_{t+j-1}} \frac{\widetilde{m}_{t+j-1}}{\widetilde{m}_{t+j}}\right) \widetilde{m}_{t+T} \\ &= \widetilde{m}_t - \widetilde{m}_t \left(\prod_{j=1}^{T} \frac{1}{1+i_{t+j-1}}\right) \\ &\longrightarrow \widetilde{m}_t \text{ as } T \to \infty. \end{aligned}$$

In this way, the sublemma has been established. ∎

By using the above results including Sublemma 1-3, I can prove Lemma 2.

Proof of Lemma 2.

(a) The relationship (7) and Sublemma 3 jointly show the following:

$$\delta_t = \frac{1+\mu-\beta(1+n)}{1+\mu}, \tag{31}$$

for any $t = 1, 2, \cdots$. Thus, the result (a) is obtained as

$$1+i_t = \frac{1}{1-\delta_t} = \frac{1+\mu}{\beta(1+n)}.$$

(b) is a direct result from (a).

(c) By Sublemma 1 and Lemma 2-(a), it holds that

$$p_t = \left(\frac{1+\mu}{(1+\theta)(1+n)}\right)^{t-1} p_1, \tag{32}$$

for any $t = 1, 2, \cdots$.

To establish (c), it is sufficient to show that

$$p_1 = \frac{1+\mu-\beta(1+n)}{\gamma N_1(y_1-g_1)} M_0^g. \tag{33}$$

By the equation (7), Sublemma 3 and (31), the following holds

$$\gamma c_1 = \frac{1+\mu-\beta(1+n)}{1+\mu}\frac{m_1}{p_1}.$$

Multiplying N_1 by both sides yields

$$\gamma N_1(y_1-g_1) = \frac{1+\mu-\beta(1+n)}{1+\mu}\frac{M_1^g}{p_1}.$$

Thus, (33) holds. The equation (33) together with (32) implies the result.

Because the proofs of (d) $-$ (f) are easy, I omit them here.

(g) Substituting (a) and (c) into (4) yields

$$\begin{aligned} B_t^g &= \left(\frac{1+\mu}{\beta(1+n)}\right)^{t-1}(1+i_0)B_0^g \\ &\quad - \left[\sum_{s=0}^{t-1}\left(\frac{1}{\beta(1+n)}\right)^s\right](1+\mu)^{t-1}\left[\mu M_0^g + p_1 N_1(\tau_1-g_1)\right]. \end{aligned}$$

By using the initial price level in the equilibrium (33), I can obtain the result. ■