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Risk Aversion and Expected Utility:

The Constant-Absolute-Risk Aversion Function and its Application to Oligopoly

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Risk Aversion and Expected Utility:
The Constant-Absolute-Risk Aversion Function and its Application to Oligopoly*

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Abstract:

The purpose of this paper is to carefully investigate the relationship between the concepts of risk aversion and expected utility, with a focus on the constant-risk-aversion function and its application to oligopoly theory. Whereas there is now a growing literature in risk, uncertainty and the market, the operational theory of risk-averse oligopoly has been rather underdeveloped so far. One of the reasons for such underdevelopment is that the established concept of risk aversion remains too abstract rather than reasonably operational, whence very few economists have dared to study the economic consequences of a change of risk aversion by firms.

In this paper, we attempt to combine the constant-absolute-risk aversion function developed by K. J. Arrow and J. W. Pratt, two great economists of the 20th century, and the normal distribution function invented by K.F. Gauss, a mathematical genius of the 19th century: The resulting situation may be called the KARA-NORMAL case. We intend to invent a very useful mathematical theorem for this specific yet important case, and then apply it to the theory of risk-averse oligopoly. In particular, the impact of increasing risk aversion on the outputs of duopolies are carefully examined. It is shown among other things that the comparative static results depend on the degree of risk aversion and the state of product differentiation.

Key words: Risk aversion, expected utility, oligopoly.

1. Introduction

We live in the world of risk and uncertainty. In this world, a single act may not necessarily yield a single outcome. It is very common that one act results in many outcomes: Which one really happens among those outcomes depends on the state of the world we are in.

It is said that a farmer is presumably the biggest gambler there is. Let us consider how and to what extent risk and uncertainty affect agriculture. As a matter of fact, whether or not rice harvest in fall is good or bad is determined by a variety of weather conditions in spring and summer such as temperature, sunlight hours, rainfall, typhoon and injurious insects. Besides, the business condition of sightseeing industry is more or less affected by unknown factors including economic and political affairs.

When we discuss human behavior under risk and uncertainty, there are two key concepts play a very important role. They are: risk aversion and expected utility. The purpose of this paper is to scrutinize the relationship between these concepts, whence shedding a new light on the impact of risk and uncertainty on many economic activities.

In reality, risk aversion and human behavior are closely intermingled. In his remarkable book (1970) , K.J. Arrow, a great economist and Nobel prize winner, once remarked: [1)]

> "From the time of Bernoulli on, it has been common to argue that individuals tend to display aversion to the taking of risks, and that risk aversion in turn is an explanation for many observed phenomena in the economic world."

In this paper, we would like to discuss more specifically the measure of absolute risk aversion and demonstrate how, in connection with the expected-utility hypothesis, it may be employed to obtain concrete and useful results in economic theory.

As was pointed out by above, Daniel Bernoulli (1700~82) who was a member of the famous Bernoulli family of mathematical geniuses, wrote a epoch-making paper on decision making under risk. He was the first scholar to introduce the expected utility hypothesis to solve the St. Petersburg paradox in the game of tossing coins. From the time of Daniel Bernoulli to modern times, there have been rather irregular rises and declines in the economics of risk and uncertainty until the 1970s when many economists as a group emerged in the economics profession. Among those economist were K.J. Arrow and J. W. Pratt, who intensively discussed how and to what degree individuals displayed risk aversion in economic phenomena.[2)]

More specifically, we would like to combine the constant-absolute-risk-aversion function developed by economists including K.J. Arrow and J.W. Pratt in the latter half of the 20th century, and the normal distribution function mainly invented by K.F. Gauss, a mathematical genius of the 19th century. Hopefully, such combination will produce a set of nice results in the 21st century.

The resulting situation aforementioned may be called the KARA-NORMAL case. . We intend to introduce a very powerful mathematical theorem for this specific yet important case, and then apply it to the theory of risk-averse oligopoly. In particular, the impact of increasing risk aversion on the outputs of duopolies are carefully examined. It shown among other things that the comparative static results depends on the degree of risk aversion and the state of product differentiation.[3)]

The contents of this paper are as follows. The next section will discuss how to measure risk aversion, and then focus on the CARA-NORMAL case. This constitutes the core of the paper. Section 2 will be concerned with its applications to oligopoly theory. Concluding remarks will be made in the final section 4.

2. How to measure risk aversion

2-1. Risk aversion

"He that fights and runs away may live to fight another day."

"Do not put all eggs in one basket."

As the saying goes, people tend to keep away from any possible risk. When they find that the risk in question is unavoidable, they tend to minimize it by means of risk spreading or purchasing insurance.

In literature, to stay or to run away may be the question. This is certainly the simple world of black and white. The real world where we live is more complex、 however, presumably composing a delicate layer of grey zones. Even if people face the same risk, the degree to which they avoid it may belong to a personal matter: Jack may display stronger risk aversion than Betty. The question to ask is how to measure the personal degree of risk aversion in conjunction with the traditional theory of expected utility. In what follows, I will summarize the theory of risk aversion already developed by K.J. Arrow, J. Pratt and their followers, and then attempt to invent a set of new techniques for the purpose of applications to oligopoly and many other problems.

In order to discuss the question of risk aversion, it is very convenient to assume that a person in question is asked to choose one out of the following prospects:

Prospect (A): income x with probability 1.

Prospect (B): income $x-h$ with probability 1/2,

income $x+h$ with probability 1/2.

Prospect (A) stands for a fixed amount of income x, whereas Prospect (B) indicates a random income ($x \pm h$) with an equal chance of gaining or losing h. Observe that both prospects guarantees the same amount of average income x. It is naturally expected that a man in the street is risk averse, whence he tends to prefer fixed income (A) to random income (B) . According to the traditional expected theory, this observation leads us to the following inequality:

$$U(x) > (1/2)\, U(x-h) + (1/2)\, U(x+h). \qquad (1)$$

Graphically speaking, this shows that a risk averse person has a concave utility function, and vice versa. In a similar way, it would be an easy job to see that a risk loving person has a convex utility function.

Concerning the choice between two prospects, let us keep away from fifty-fifty chance for a while, and turn our eyes to a third prospect.

Prospect (C) : income $x - h$ with probability $1 - \rho$,

income $x+h$ with probability ρ .

As is easily seen, Prospect (C) is of a more general form than Prospect (B) since ρ can take any value of the unit interval [0, 1]. Now consider the choice between Prospects (A) and (C). If the value of ρ is near zero, an ordinary person likes Prospect (C) better. If it is near unity, he or she likes Prospect (A) better.

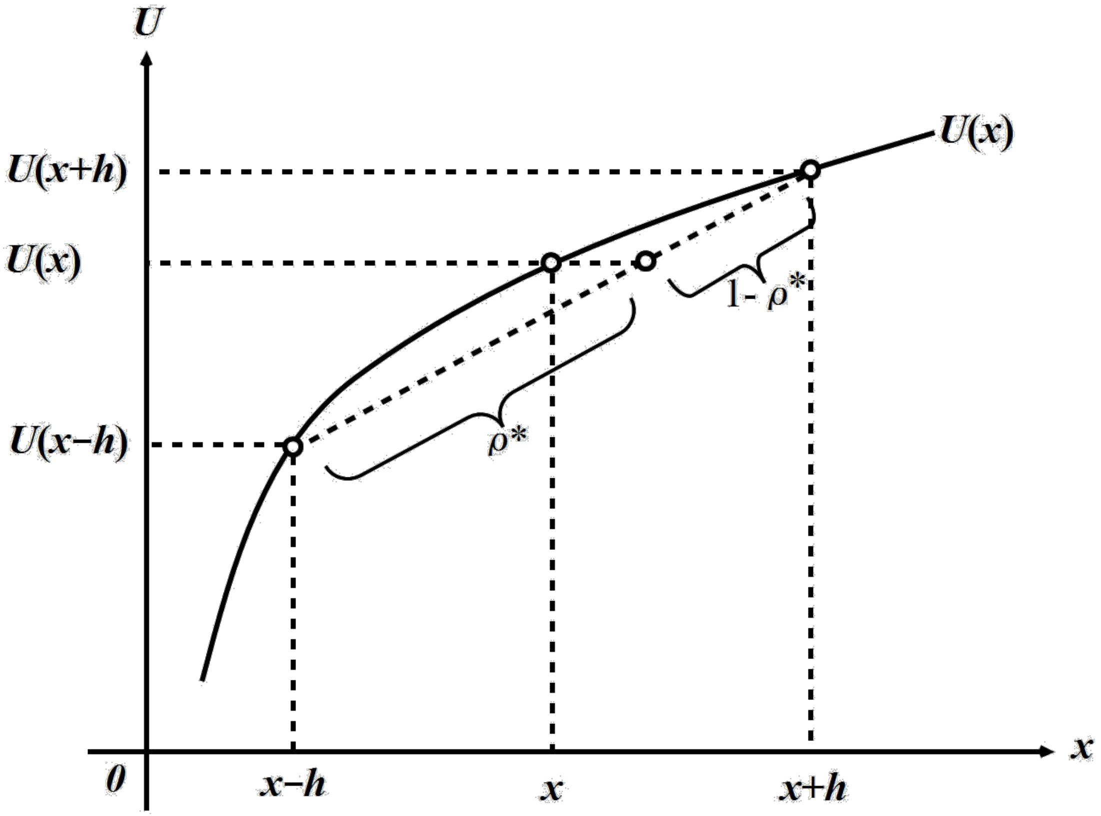
Now consider some intermediate values of ρ . When $\rho = 1/2$, a risk averse person still prefers (A) to (C) as was seen above. If we assume that the utility function is continuous and smooth, we may find a certain value ρ^* between 1/2 and 1 so that the following equality holds.

$$U(\mathrm{x}) = (1-\rho^*)\, U(x-h) + \rho^* U(x+h). \qquad (2)$$

The value of ρ^* relative to $(1-\rho^*)$ represents how much weight a risk averse person must give to a pair of gain $(x+h)$ and loss ($x-h$) so that both prospects (A) and (B) may have just the same value. It is naturally expected that a more risk averse person has a more strongly concave utility function, and thus a lager value of ρ^*. The graphic illustration of (2) can clearly be seen in Figure 1.

Figure 1　Making the two prospects equal values:

$U(x)=(1-\rho^*)\,U(x-h)+\rho^*\,U(x+h)$



In calculus, the concept of the Taylor expansion is generally a very powerful tool. There is no exception for the case of risk aversion. If we employ the theorem of Taylor expansion series, it follows from (2) that

$$U(x) = (1-\rho^*)\ \{U(x) - h\,U'(x) + (h^2/2)\,U''(x) + \cdots\}$$
$$+\ \rho^*\ \{U(x) + h\,U'(x) + (h^2/2)\,U''(x) + \cdots\}\ . \qquad (3)$$

If we do some calculations, it follows from (3) that

$$0 = h\,U'(x)\,(2\rho^* - 1) + (h^2/2)\ U''(x) + \cdots,$$

which leads us to obtain

$$\rho^* = (1/2) + (h/4)\ R^* + \text{terms of higher order in h}, \qquad (4)$$

in which

$$R^* = -U''(\mathrm{x}) / U'(\mathrm{x})\ . \qquad (5)$$

The value of R* represents a measure of risk aversion, being specifically named the absolute risk aversion. It is noted that the value of R* is closely connected with the minus of U'' (x) which in turn shows the degree of concavity of the utility function: Namely, a stronger risk averter has a stronger concave utility.

"So many men, so many minds."

This is a famous maxim showing that the degree of risk aversion varies person to person. Someone may be very prudent and tend to keep the maximum distance from a possible danger, whereas others may be less careful and sometimes dare to challenge the danger. There are clearly a wide intermediate range between these two extremes. Besides a man's mind is subject to change, and may change situation to situation: A usually prudent man may suddenly change his mind and display a more brave behavior than before.

If we keep these facts in minds, we need to invent a new approach to human behavior. It is high time for us to invent a operational theory of risk aversion, so that we may discuss quantitatively rather than qualitatively the welfare implications of a possible change of risk aversion. This is the problem we are going to turn to in the next section.

2-2 The CARA -NORMAL case

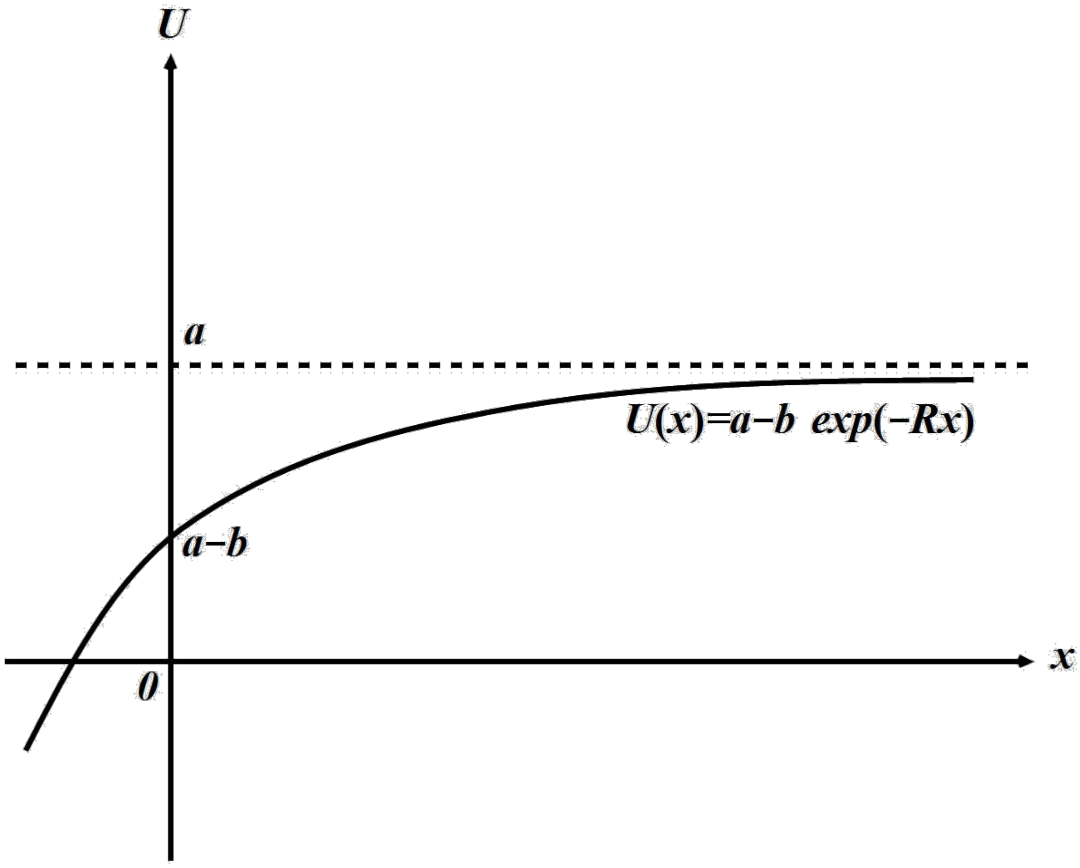
In this section, we would like to combine the two functions that are important in quantitative analysis. They are: the constant-absolute-risk-aversion function developed by K.J. Arrow and J.W. Pratt , two great social scientists of the 20th century; and the normal distribution function invented by K.F. Gauss, a mathematical genius of the 19th century. Such an academic combination across the two centuries may be named the CARA-NORMAL case, and is expected to produce a series of nice quantitative properties. [4)]

As was seen in (5), the absolute-risk-aversion function is provided by $R^*(x) = -U''(x) / U'(x)$. If we put $R^*(x) = R$ where R denotes a positive constant, and integrate both sides of this equation, we immediately obtain the following equation.

$$U(x) = a - b \exp(-Rx) . \qquad (6)$$

This is clearly what we may call the constant-absolute-risk-aversion function, or in short the CARA function. The CARA function is a sort of exponential function, and depicted in Figure 2. It is so simple and beautiful: It is increasing, concave and bounded above, with the upper bound a.

Figure 2 The constant-absolute-risk aversion function:
$U(x)=a-b\ exp(-Rx)$



Let us remind the reader of the historical fact that a simple and beautiful function was first introduced by K.F. Gauss, one of the greatest mathematicians we have ever produced——the normal or Gaussian distribution function $\boldsymbol{N}(\mu, \sigma^2)$ with average μ and variance σ^2. More specifically, the probability density function $\Phi(\tilde{\alpha})$ of a stochastic variable $\tilde{\alpha}$ is written by

$$\Phi(\tilde{\alpha}) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left[-\frac{(\tilde{\alpha}-\mu)^2}{2\sigma^2} \right] . \quad (7)$$

Now, consider the case where the utility function is given by (6) and the probability density function by (7). Then in this special KARA-NORMAL case, we can skillfully escape from the computational jangle in which we would usually be involved and got lost, and quantitatively find the concrete value of the expected utility:

$$EU(x;\tilde{\alpha}) = \int_{-\infty}^{\infty} U(x;\tilde{\alpha})\Phi(\tilde{\alpha})d\tilde{\alpha} \quad .$$

We are now in a position to establish the following mathematical results for the CARA-NORMAL case.

Theorem 1 (The CARA-NORMAL Case):

Let $\tilde{\alpha} \sim \boldsymbol{N}(\mu、\sigma^2)$ and k be a constant. Then we obtain the following properties:

(i) $E\exp[k\tilde{\alpha}] = \exp[k\mu + (1/2)k^2\sigma^2]$,

(ii) $E\exp[-k\tilde{\alpha}^2] = \dfrac{1}{\sqrt{1+2k\sigma^2}}\exp\left[-\dfrac{k\mu^2}{1+2k\sigma^2}\right]$.

It is perhaps needless to say that this theorem per se is not novel in advanced statistics. To my knowledge, however, it is hardly referred to in our economic science. Under such circumstances, I think that it is worthwhile to give a detailed proof of the theorem here.

To begin with, it is noted that the process to prove Property (i) corresponds well to the one to obtain the moment-generating function that is commonly used in statistics.

By view of (7), we can immediately have

$$E\exp[k\tilde{\alpha}] = \frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty}\exp[k\tilde{\alpha}]\exp\left[-\frac{(\tilde{\alpha}-\mu)^2}{2\sigma^2}\right]d\tilde{\alpha}$$

$$= \frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty}\exp\left[-\frac{\tilde{\alpha}^2 - 2\mu\tilde{\alpha} + \mu^2 - 2\sigma^2 k\tilde{\alpha}}{2\sigma^2}\right]d\tilde{\alpha} \quad . \tag{8}$$

We note that

$\tilde{\alpha}^2 - 2\mu\tilde{\alpha} + \mu^2 - 2\sigma^2 k\tilde{\alpha}$

$= [\tilde{\alpha} - (\mu + k\sigma^2)]^2 - 2\sigma^2[k\mu + (1/2)k^2\sigma^2]$. (9)

If we substitute (9) into (8), we obtain

$E\exp[k\tilde{\alpha}]$

$$= \frac{1}{\sqrt{2\pi}\sigma}\exp\left[k\mu + \frac{k^2\sigma^2}{2}\right]\int_{-\infty}^{\infty}\exp\left\{-\frac{\left[\tilde{\alpha} - (\mu + k\sigma^2)\right]^2}{2\sigma^2}\right\}d\tilde{\alpha} \quad . \qquad (10)$$

Taking account of the property of the normal distribution $\boldsymbol{N}(\mu + k\sigma^2, \sigma^2)$, we find that

$$\frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty}\exp\left\{-\frac{\left[\tilde{\alpha} - (\mu + k\sigma^2)\right]^2}{2\sigma^2}\right\}d\tilde{\alpha} = 1 \quad . \qquad (11)$$

By substituting Eq.(11) into (16), we immediately obtain

$E\exp[k\tilde{\alpha}] = \exp[k\mu + (1/2)k^2\sigma^2]$,

which proves Property (i).

Property (ii) will be proved in a similar fashion. If we do integral calculation, it is not hard to obtain

$$E\exp[-k\tilde{\alpha}^2] = \frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty}\exp\left[-k\tilde{\alpha}^2\right]\exp\left[-\frac{(\tilde{\alpha} - \mu)^2}{2\sigma^2}\right] d\tilde{\alpha}$$

$$= \frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty}\exp\left[-\frac{D}{2\sigma^2}\right]d\tilde{\alpha} \quad , \qquad (12)$$

where

$D = (\tilde{\alpha} - \mu)^2 + 2\sigma^2 k\tilde{\alpha}^2$

$$= (1 + 2k\sigma^2)\left(\tilde{\alpha} - \frac{\mu}{1 + 2k\sigma^2}\right)^2 + \frac{2k\sigma^2\mu^2}{1 + 2k\sigma^2} \quad . \qquad (13)$$

If we substitute (13) into (12), we obtain

$E\exp[-k\tilde{\alpha}^2]$

$$= \frac{1}{\sqrt{2\pi}\sigma}\exp\left[-\frac{k\mu^2}{1+2k\sigma^2}\right]\int_{-\infty}^{\infty}\exp\left[-\frac{\left(\tilde{\alpha}-\frac{\mu}{1+2k\sigma^2}\right)^2}{2\left(\frac{\sigma^2}{1+2k\sigma^2}\right)}\right]d\tilde{\alpha} \quad . \qquad (14)$$

If we notice the property of the normal distribution $\boldsymbol{N}(\mu/(1+2k\sigma^2), \sigma^2/(1+2k\sigma^2))$, the following equality obviously holds.

$$\frac{1}{\sqrt{2\pi}\left(\frac{\sigma}{\sqrt{1+2k\sigma^2}}\right)}\int_{-\infty}^{\infty}\exp\left[-\frac{(\tilde{\alpha}-\frac{\mu}{1+2k\sigma^2})^2}{2\left(\frac{\sigma^2}{1+2k\sigma^2}\right)}\right]d\tilde{\alpha} = 1 \quad . \qquad (15)$$

Therefore, if we take care of both equations (14) and (15), we can find

$$E\exp\left[-k\tilde{\alpha}^2\right] = \frac{1}{\sqrt{1+2k\sigma^2}}\exp\left[-\frac{k\mu^2}{1+2k\sigma^2}\right] \quad .$$

which is just the same as Property (ii) . This completes the proof of Theorem 1.

Let us get back to the CARA -NORMAL case in which the utility function $U(x)$ is given by the constant-absolute-risk aversion function, and the stochastic variable $\tilde{\alpha}$ follows the normal distribution function. For loss of generality, we may assume $a = b = 1$, so that (6) above simply reduces to

$$U(x) = 1 - \exp(-Rx) \quad . \qquad (16)$$

Then applying Theorem 1(i) to (16), we immediately have

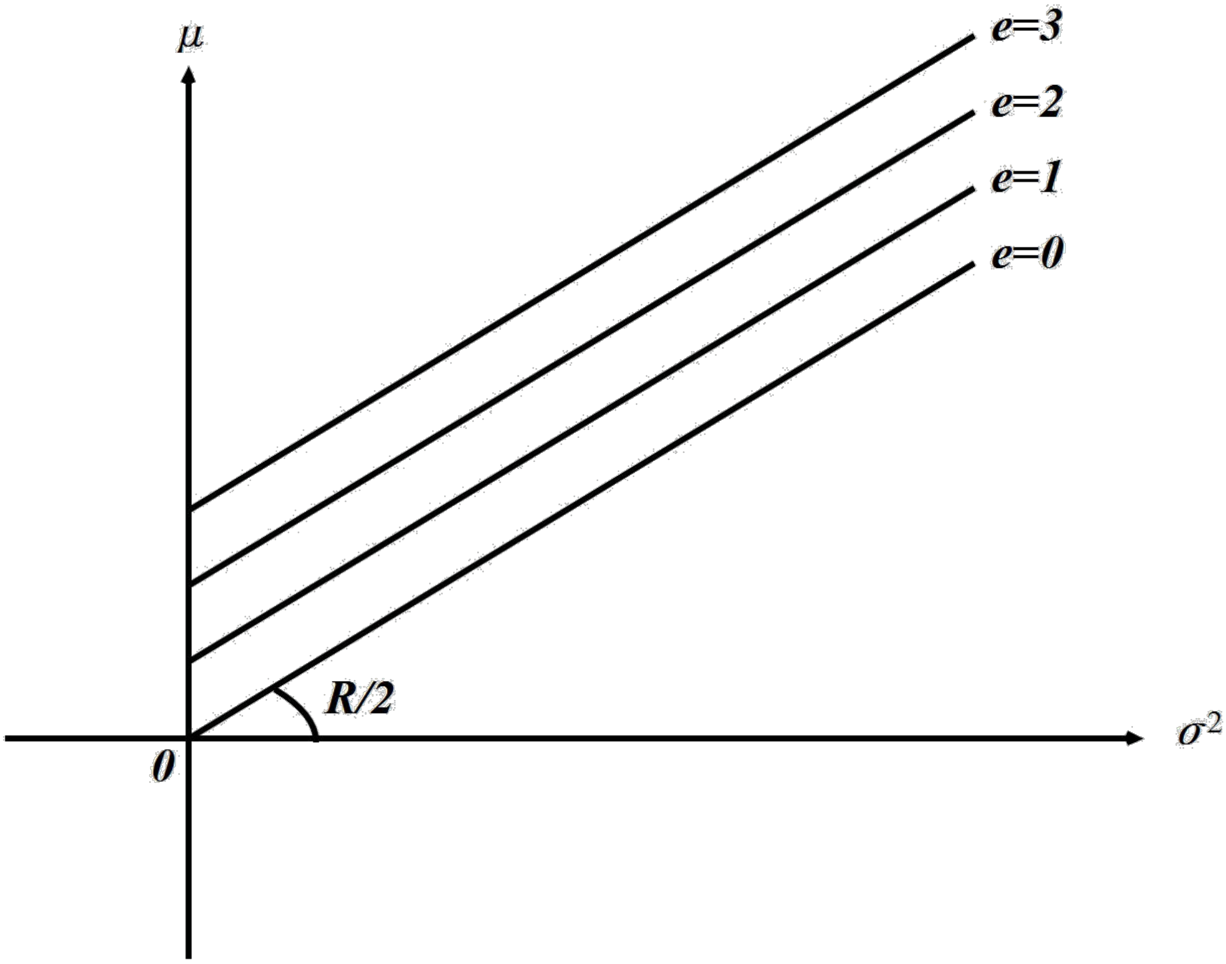
$$EU(x) = 1 - E\exp(-Rx)$$
$$= 1 - \exp[-R\mu + (1/2)R^2\sigma^2] \qquad (17)$$

In order to find the indifference curve for the CARA-NORMAL case, let us put $EU(x)$ = a constant . Then we immediately obtain $-R\mu + (1/2)R^2\sigma^2$ = a constant . We may let this constant term equal to $(-eR)$ where e is a constant. Thus it is easy to find that

$$\mu = (1/2)R\sigma^2 + e \qquad (e \text{ is a constant}) . \qquad (18)$$

Figure 3 The CARA function and its indifference curves:

$\mu = (1/2)R\sigma^2 + e$ (e is a constant)



For this special case, Figure 3 depicts a class of indifference curves when $e = 0, 1, 2, 3$. Each indifference curve is upward-sloped, and in fact a straight line with its slope $R/2$. When a man displays stronger risk aversion, he is expected to have steeper indifference lines.

In general, the utility curve of a risk averse man is concave, whence the following Jensen's inequality holds.

$$EU(x) < U(Ex) = U(\mu). \quad (19)$$

Let us introduce a risk premium π^* so that the inequality mentioned above may become an equality.

$$EU(x) = U(\mu - \pi^*). \quad (20)$$

The value of π^* indicates the maximum amount of extra money a risk averter is willing to pay for a 100% sure income in exchange for a risky income. The relation between risk aversion and risk premium ρ^* can easily be understood in Figure 4.

In the special case of the CARA-NORMAL case, we can proceed more and do direct computations. In fact, if we make use of (17), we obtain

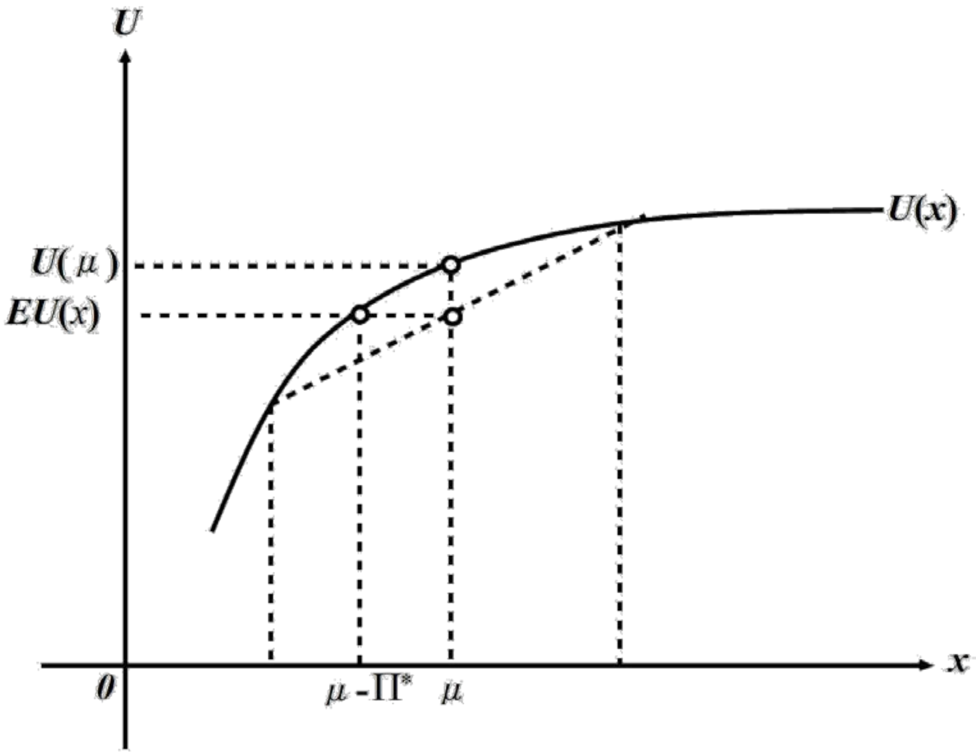
$$1-\exp[-R\mu + (1/2)R^2\sigma^2] = 1-\exp[-R(\mu - \pi^*)],$$

from which clearly follows

$$\pi^* = (1/2)R\sigma^2. \quad (21)$$

This equation has significant meaning. In the case of the CARA-NORMAL case, the amount of risk premium π^* is just equal to the half of the product of the degree of absolute risk aversion R and the value of variance σ^2 .

Figure 4 Risk aversion and risk premium:
$EU(x) = U(\mu - \pi^*)$



As is naturally expected, an increase in R or σ^2 corresponds well to a rise in π^*. It should be noticed that such a correspondence holds not only locally, but also very globally. Surely, such a global property represents one of the nicest results we can derive from the CARA-NORMAL specification.

3. Applications to Oligopoly

As the saying goes, seeing is believing. In general, if we focus on the CORA--NORMAL case, we are able to simplify the computation process very significantly with much loss of generality, and sometimes help get out of the mathematical jungle in which we would be miserably involved. This claim will be confirmed in the application of the results associated with the CORA-NORMAL specification into oligopoly theory operating under demand risk.

In order to make the matter clear, let us take care of simple duopoly. There are two firms in an industry: Firms 1 and 2. Let $x_i \geqq 0$ be the output level firm i, and $p_i \geqq 0$ its unit price level ($i = 1, 2$). Suppose that the demand equations are given by linear equations:

$$p_1 = \alpha - \beta(x_1 + \theta x_2), \quad (22)$$
$$p_2 = \alpha - \beta(x_2 + \theta x_1), \quad (23)$$

in which α stands for a common demand intercept. Besides, we assume that β is a positive constant and θ takes any value out of the unit interval $[-1, 1]$. More specifically, the value of θ is a measure of the substitutability of the two goods: Namely, the goods are substitutes, complements or independents according to whether θ is positive, negative or zero. We may assume without loss of generality that the value of β equals unity. [5)]

Suppose that the cost functions of the firms are provided by linear equations:

$$C_i(x_i) = c_i x_i \qquad (i = 1, 2). \quad (24)$$

We assume here that the unit cost c_i of each firm is a constant , and ignore the existence of fixed costs. Then in the light of (23) and (24), the profit Π_i of firm i is shown by

$$\Pi_i = p_i x_i - c_i x_i$$
$$= (\alpha_i - c_i - x_i - \theta x_j) x_i \qquad (i, j = 1, 2;\ i \neq j) \quad . \quad (25)$$

Now consider the situation in which the two firms are subject to the same demand risk. In other words, when each firm determines its production plan, it cannot foresee *ex ante* how well its product will be sold. For instance, a beer producer cannot exactly predict the amount of beer sales in the coming summer, since its production critically depends upon many whether conditions such as temperature, sunshine hours, rainfall and the like. Besides, it should be noted that product differentiation is rather common in the beer industry: The two brands of beer may be competitive, complementary or independent.

In order to introduce such demand risk, it is convenient to assume that the common demand intercept α is now a stochastic variable : Therefore , we will write $\tilde{\alpha}$ with a wave attached. In particular, let us suppose that this $\tilde{\alpha}$ follows the normal distribution $\boldsymbol{N}(\mu, \sigma^2)$ with mean μ and variance σ^2 .

Now consider the case in which each firm displays risk aversion and its utility function is given by the constant-absolute-risk aversion function. In other word, we will make full use of CARA-NORMAL specification. In this important case, the utility of the profit of firm i may be expressed by the following exponential function:

$$U_i(\Pi_i) = 1 - \exp[-R_i \Pi_i] \quad (R_i > 0;\ i = 1,2) . \qquad (26)$$

As we discussed in the last section, the important feature of the utility function (26) is that the degree of absolute risk aversion is a constant R_i . A greater (or a smaller) value of R_i represents a greater (or a smaller) degree of risk aversion on the part of firm *i*.

We are in a position to define the Cournot-Nash equilibrium (x_{1^*}, x_{2^*}) under demand risk in the following way:

$$x_1^* = \arg\max_{x_1} E_{\tilde{a}} U\left[\Pi_1(x_1, x_2^*; \tilde{\alpha})\right]$$

$$x_2^* = \arg\max_{x_2} E_{\tilde{a}} U\left[\Pi_2(x_1^*, x_2; \tilde{\alpha})\right]$$

Once the Cournot-Nash equilibrium is reached, each firm has no incentive to deviate from it. In what follows, we will attempt to derive the numerical values of the equilibrium solution under demand risk.

In the light of (25) and (26), we can find that

$$\begin{aligned} E\,U_1(\Pi_1) &= 1 - E\exp[-R_1 x_1(\tilde{\alpha} - c_1 - \mathrm{x}_1 - \theta\,\mathbf{x_2})] \\ &= 1 - E\exp[-R_1 x_1 \tilde{\alpha}] \times \exp[R_1 \mathrm{x}_1(c_1 + x_1 + \theta\,\mathbf{x_2})]. \qquad (27) \end{aligned}$$

We recall that $\tilde{\alpha}$ per se is a stochastic variable following the normal distribution $\boldsymbol{N}(\mu, \sigma^2)$. Then we can make use of Theorem 1 above to obtain

$$E\exp[-R_1 x_1 \tilde{\alpha}] = \exp[-R_1 x_1 \mu + (1/2)R_1^2 x_1^2 \sigma^2]\,. \qquad (28)$$

By substituting (28) into (27), we obtain

$$E\,U_1(\Pi_1) = 1-\exp\{-R_1 x_1[\mu - c_1-(1+R_1\sigma^2/2)x_1-\theta x_2]\}\,. \qquad (29)$$

If we differentiate (29) with respect to x_1 and put the resulting partial derivative just zero, we find that

$$\mu - c_1-(2+R_1\sigma^2)\;x_1-\theta x_2 = 0\,. \qquad (30)$$

In a similar fashion, if we now differentiate (29) with respect to x_2, and put the resulting partial derivative just zero, it is not difficult for us to derive

$$\mu - c_2 -(2+R_2\sigma^2)\;x_2-\theta x_1 = 0\,. \qquad (31)$$

The twin equations (30) and (31) respectively indicate firm 1's and firm 2's reaction functions under the present CARA-NORMAL specification. If we think of these equations as a system of simultaneous equations, and attempt to solve for the pair of solutions (x_1*, x_2*), then we are able to find the Cournot-Nash equilibrium pair in the following way:

$$x_1^* = \frac{(2+R_2\sigma^2)(\mu-c_1)-\theta(\mu-c_2)}{(2+R_1\sigma^2)(2+R_2\sigma^2)-\theta^2}\quad ,$$

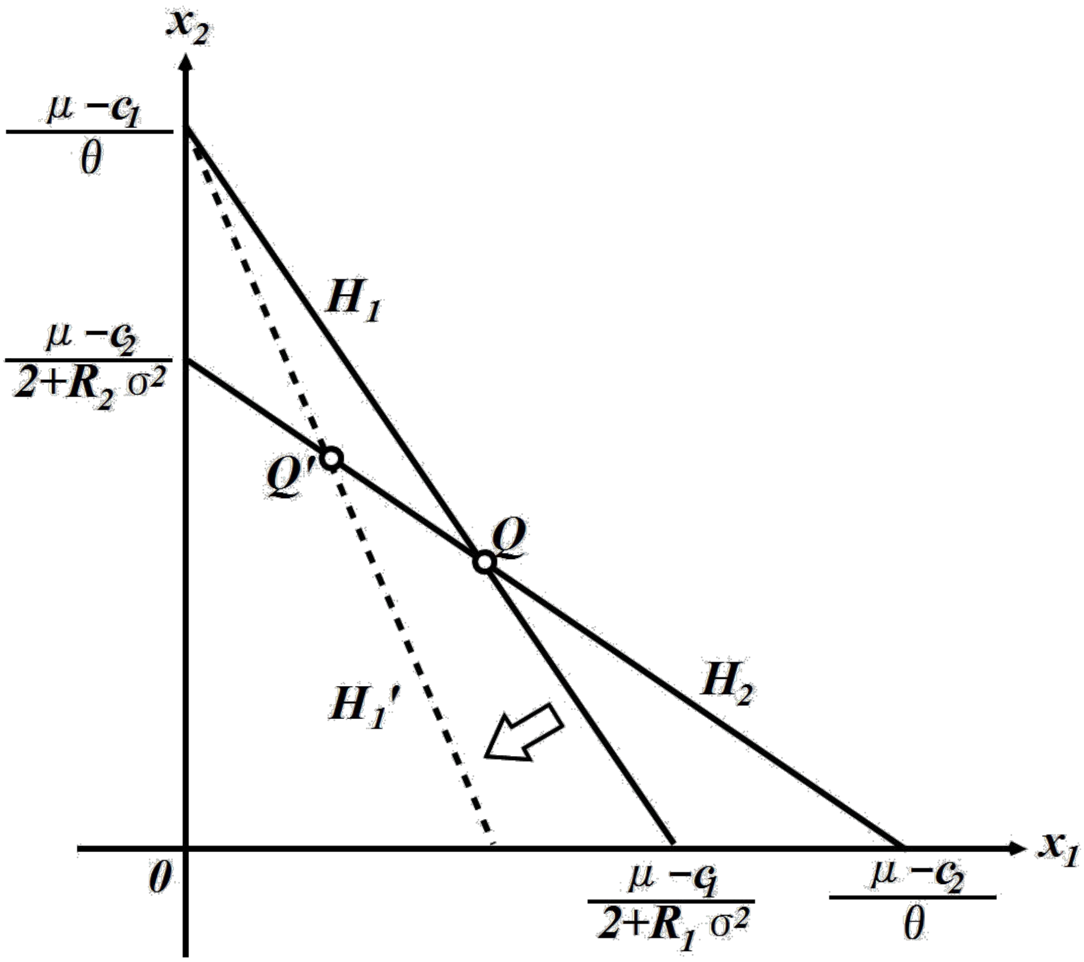
$$x_2^* = \frac{(2+R_1\sigma^2)(\mu-c_2)-\theta(\mu-c_1)}{(2+R_1\sigma^2)(2+R_2\sigma^2)-\theta^2}\quad .$$

Therefore, the amounts of equilibrium outputs depend on the five factors. They are: ① The value of μ, namely the average value of changeable demand; ② The values of c_1 and c_2 that represent the cost conditions of the two firms; ③ the value of θ, or the degree of production differentiation, ④ the value of σ^2, or the degree of the demand risk; and ⑤ the values of R_1 and R_2, or the degrees of risk aversion on the part of the two firms.

Presumably, there are a variety of comparative static analyses we are able to carry out. In this paper, however, we would like to investigate Case ⑤ only. In other words, we are interested in scrutinizing how and to what extent changes in R_1 or R_2 affect the values of x_1* and x_2*.

- Figure 5 Cournot duopoly equilibrium:

The two goods are substitutes (θ >0)

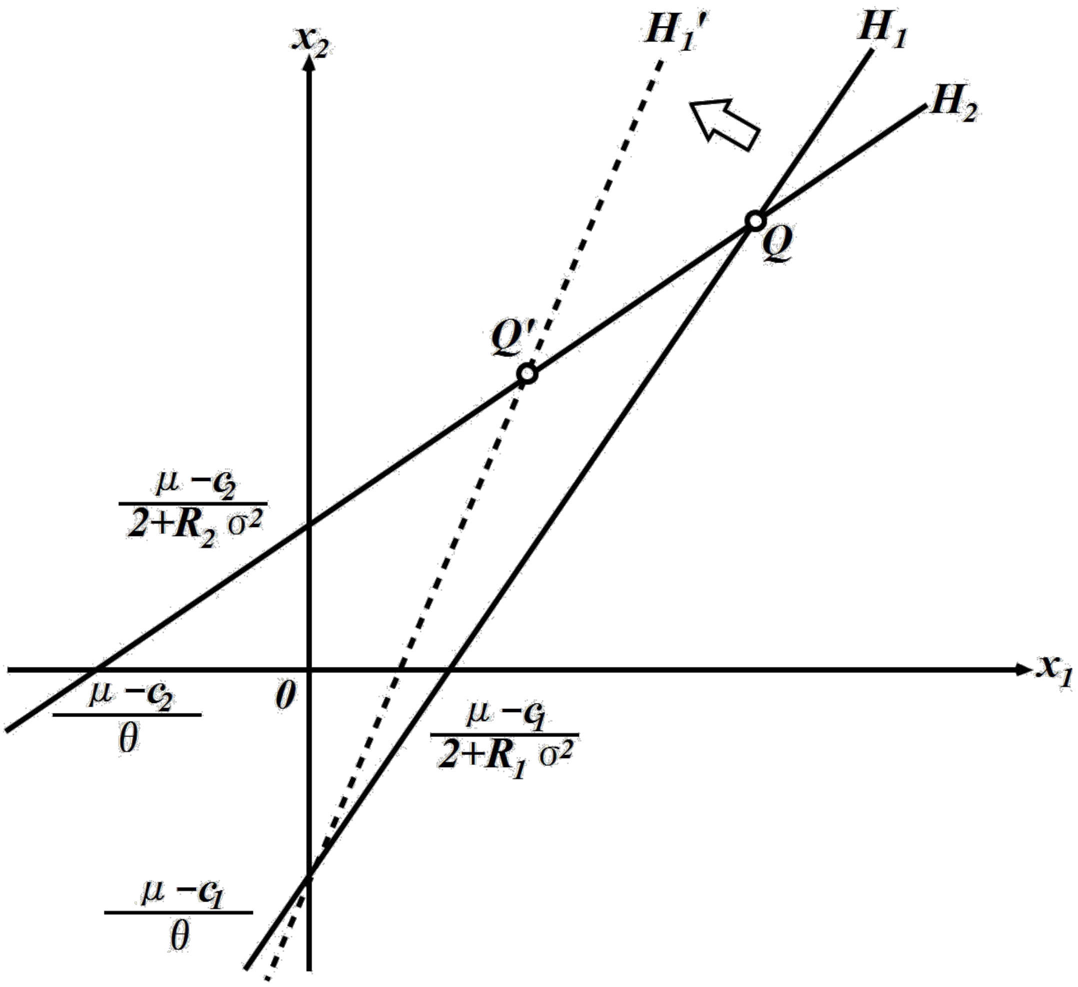


We are facing the situation of product differentiation. The two goods may be substitutes ($\theta > 0$), complements ($\theta < 0$), or independent ($\theta = 0$). Let us begin our inquiry with the case of substitutes. Then the Cournot duopoly equilibrium is shown by Point Q in Figure 5. The straight lines H_1 and H_2 respectively stand for the reaction line of firms 1 and 2, are negatively sloped.

The question to ask is how a change in R_i ($i = 1, 2$) influences the position of Point Q. Suppose that because of some reasons, firm 1 displays a stronger risk aversion than before. Then as the value of R_1 becomes greater, the reaction line H_1 evolves clockwise to H_1', thus shifting the equilibrium point from Q to Q'. These changes result ih a decrease in x_1 and an increase in x_2. Therefore, a stronger risk averse firm must decrease in its own output whereas the output of the other firm declines. In short, the two risk-averse firms producing substitutable goods are competitive rivals: One firm's gain is achievable only at the expense of the other firm.

Let us turn to the case of complements. As is seen in Figure 6, reaction lines H_1 and H_2 of firms are positively sloped.

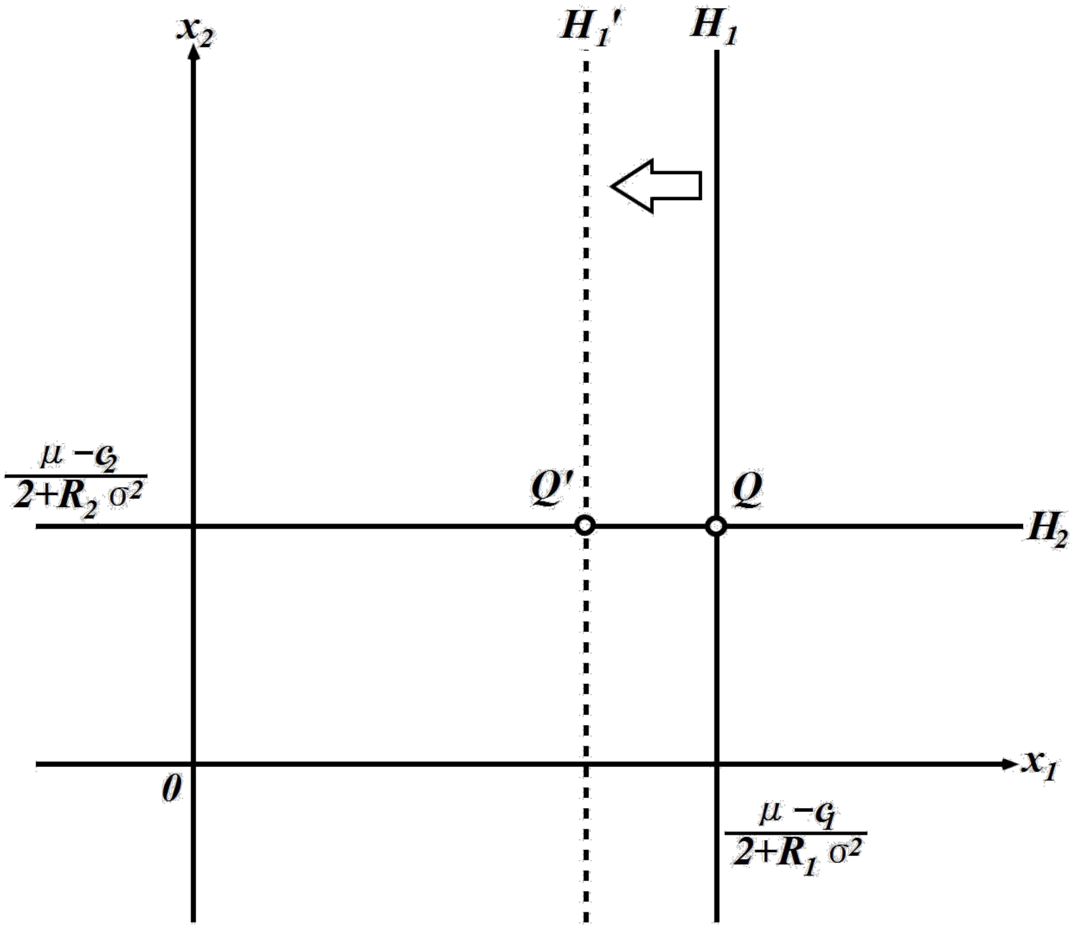
Figure 6 The case of complementary goods (θ <0)



The equilibrium point is indicated by the intersection of the reaction lines, i.e. Point Q. An increase in R_1 will cause a shift of reaction line H_1 counter-clockwise to H_1'. Hence an increase (or decrease) in risk aversion on the part of one firm will decrease (or increase) not only the output of that firm, but also the one of the other firm. In short, in case the products are complements, the firms are in cooperative relation: All firms in an industry will thrive or decline together.

Now consider the third case in which the two goods are independent. Then as is seen in Figure7, the reaction line H_1 of firm 1 is vertical, whereas the reaction line H_2 of firm 2 is horizontal. When there is an increase in R_1, the equilibrium point will shift to the left from Q to Q' along the horizontal line H_2.

Figure 7 The case of independent goods (θ =0)



In plain English, when firm 1 becomes more bearish in the face of demand risk, the production activity of the firm has to shrink, whereas the one of firm 2 remains to the same as before. This is because x_1 and x_2 are now independent: Hence firm 1's psychological mind of being bearish or bullish has no effects at all on the production plan of firm 2. .

To sum up, the comparative static results of oligopoly models under risk are largely dependent on the state of product differentiation and the degree of risk aversion. We have shown that the CARA-NORMAL specification is powerful enough to effectively derive quantitative results.

4. Concluding Remarks

"Gone are the days when my heart was young and gay ⋯."

This is the first phase of a very popular song written and composed by Stephan Foster (1826~64), a great American composer. When I was born in Osaka, Japan, my heart was surely very young and gay. As the war broke between Japan and the U.S., however, the everyday life of my family became harder and more miserable. I still remember the tragic days when a group of American *B29* bombers dropped so many firebombs so many times over Osaka. Fortunately, my family has survived during the air attacks, yet our painful wartime memory still lingers. After the war, people lost almost everything, hence they had nothing to be afraid of. In other words, they displayed no risk aversion at all for making a living.

Since then, turbulent twenty years had passed before I became a still young and fairly ambitious student at Kobe University . In the 1960s, the Japanese society was very unstable and often disturbed by railway strikes, mining shutdowns, street demonstrations and so on. One day, the Diet Building was surrounded by so many active students, thus ceasing to do its proper function which is required to do. Those students seemed to display a sort of risk preference for political reform.

In those restless days, I happened to read a collection of nice essays written by Kiyoshi Oka (1970), a great Japanese mathematician. It was really a great inspiration for me. He once remarked:

"In the world of mathematics after the Second World War, there has been a new research direction toward 《extreme abstraction》 emerged . According to this direction, very general results were welcomed for the sake of generality, whence more specific yet more fruitful discussions were underestimated or even neglected. I am afraid that such unfortunate tendency still continues and is growing. I feel as if mathematicians became no longer human: They were just wandering from place to place in winter wilderness, could not see neither green leaves nor lovely flowers. Now I am firmly determined to drastically change the mathematical trend from the "chilly winter wilderness" to the "warm spring warmth." So I have written a series of mathematical papers with lovely spring flavor."

It was no wonder that the teachings of legend Oka gave me an incentive to drastically change my life style: I decided to go abroad for graduate study, looking for the spring warmth. [6)] When I took graduate courses at the University of Rochester, however, I was really shocked to see that many economics professors were eager to solve very abstract questions with no reference to warm human heart: They seemed to wander from place to place in the chilly winter wilderness. General equilibrium theory promoted by many Rochester professors represented the culmination of mathematical abstraction and generalization with no human heart .

After finishing my doctor thesis in mathematical economics, I got a chance to teach economic theory at the University of Pittsburgh. Fortunately, Pittsburgh was a nice place to live: Both professors and students had warm human heart. It was at that time that I said good-by to the winter wilderness and attempted to write economics papers with lovely spring flavor.

In my opinion, as pointed out by late Professor Oka, there are two different kinds of problems in every science including mathematics and economics. They are: The problems of the chilly winder color, and those of the warm spring color. Hopefully, this paper will help change the direction of economic science toward more human flavor.

To sum up, this paper aimed to investigate the relationship between risk aversion and expected utility, with a focus on the constant-risk-aversion function and iys application to oligopoly theory. Whereas there is a growing literature in risk, uncertainty, and the market, the operational theory of risk-averse oligopoly has been rather underdeveloped so far. One of the reasons for such underdevelopment is that the established concept of risk aversion remains too abstract rather than operational, whence very few economists have dared to study the consequences of a risk aversion change on oligopoly under imperfect information.

Needless to say, there remain so many unsolved problems in the related area of research. It is our sincere hope that this paper will give a spring board to step up for the promotion of risk aversion theory and its application in economic science.

Footnotes

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1) In the light of the history of economic thought, the year of 1970 is regarded as the birth year of the modern economics of risk and uncertainty. It is in that year that K.J. Arrows' outstanding essays in risk-bearing and G. Akerlof's famous paper on the lemons market were both published. See Sakai (1982, 2010).

2) See Arrow (1965,70) and Pratt(1964).

3) There is now a vast literature on the working and performance of oligopoly under information. See Sakai (1990). Unfortunately, very few papers have ever discussed the CARA-NORMAL case, however. This paper intends to further develop this special yet important case.

4) The CARA-NORMAL case was first introduced to oligopoly theory by Sakai and Yoshizumi (1991a, 91b), and later developed in related areas by several papers including Sakai and Sasaki (1996).

5) For a detailed discussion on thus point, see Sakai (1990).

6) When I began graduate courses in the United States, my life was really guided by Professor Oka's invaluable teachings: Hence I had due respect for mathematics, but had no fear for American culture whatever. I still remember the following inspiring words by Oka (1970):

"When I began my course work at the Department of Science, Kyoto University, I had too much owe for mathematics to specialize in it. I have never thought, however, that foreign cultures were fearsome and overwhelming. To my regret, the ordinary Japanese people tend to look at this matter from the opposite point of view: Although they do not show due respect for mathematics, they are so afraid of foreign cultures. I would like to emphasize that this is nothing but a terrible mistake".

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