Structure of antichain partitions of a submodular function and the essential coalition partition of a decomposable convex game

Takeshi Naitoh*

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Abstract We consider a generalization of Narayanan's theorem describing that the family $\{\Pi' \mid \Pi' \in P_E, \sum_{X\in\Pi'} f(X) = \min_{\Pi\in P_E} \sum_{X\in\Pi} f(X)\}$ forms a lattice, where f is a submodular function on 2^E and P_E is the set of all partitions of E. We extend Narayanan's result on rooted forests. Moreover we show uniqueness of essential coalition partition of decomposable convex games.
Key words: Antichain partition, Submodular function, Decomposable convex game

1 Introduction

Let f be a function on 2^E for a nonempty finite set E. A function $f : 2^E \to \mathbf{R}$ is called a *submodular function* on 2^E if

$$f(X) + f(Y) \geq f(X \cup Y) + f(X \cap Y) \tag{1.1}$$

for all $X, Y \subseteq E$, where $\mathbf{R}$ is the set of real numbers. If $-f$ is a submodular function, we call f a *supermodular function*. For the purpose of solving problems in electrical network theory, H. Narayanan [2] studied structure of the set of the partitions $P_E^* = \{\Pi' \mid \Pi' \in P_E, \overline{f}(\Pi') = \min_{\Pi\in P_E} \overline{f}(\Pi)\}$, where P_E is the set of all partitions of E and $\overline{f}(\Pi) = \sum_{X\in\Pi} f(X)$. He showed that P_E^* forms a lattice. In section 3 we show a generalization of Narayanan's Theorem on rooted forests.

Let $N = \{1, 2, \ldots, n\}$ be a set of players. The basic model of cooperative game theory comprises a set N of *players* the subsets $S \subseteq N$ of which are *coalitions*. There is a *characteristic function* $v : 2^N \to \mathbf{R}$ that assigns to each coalition S its *value* $v(S)$. We denote (N, v) an n-person game in characteristic function form where v is a real-valued function on 2^N and $v(\emptyset) = 0$. A characteristic-function game (N, v) is called a *convex game* if the characteristic function v is supermodular. A convex game (N, v) is *decomposable* if there is a decomposable partition of (N, v) except for $\{N\}$. L. S. Shapley gave a theorem on a necessary and sufficient condition for a convex game to be decomposable. In section 4 we show uniqueness of essential coalition partition of decomposable convex games.

*Faculty of Economics, Shiga University.

2 Definitions and preliminaries

A binary relation $\preceq$ on a nonempty finite set E is called a *partial order* if it satisfies reflexivity, antisymmetry, and transitivity. $(E, \preceq)$ is called a *partially ordered set* or a *poset* for short. If for each $x, y \in E$ there exist least upper bound of $\{x, y\}$ and greatest lower bound of $\{x, y\}$, then we call the poset a *lattice*, and we write the least upper bound of $\{x, y\}$ as $x \vee y$ and the greatest lower bound of $\{x, y\}$ as $x \wedge y$.

Let $P = (E, \preceq)$ be a partially ordered set with a finite set E. A poset $P = (E, \preceq)$ is called *trivial* if there exist no order-relations between pairs of elements of E. A subset $S \subseteq E$ is called *order-ideal* if $e \prec e' \in I$ implies $e \in I$. A subset S which consists of pairwise incomparable elements with respect to P is called an *antichain*.

We denote by $\mathcal{A}(\mathcal{P})$ the set of anchichains of the poset P. With any $S \subseteq E$ we associate the order-ideal generated by S via

$$id(S) := \{x \in E \mid x \leq s \text{ for some } s \in S\}.$$

Denoting by S^+ the collection of maximal elements of the poset P restricted to S. Note that S^+ is an antichain and that every antichain A arises as $A = (id(A))^+$. Hence for $A, B \in \mathcal{A}(\mathcal{P})$ two binary operations

$$\begin{aligned} A \vee B &:= (id(A) \cup id(B))^+ \\ A \wedge B &:= (id(A) \cap id(B))^+ \end{aligned}$$

are defined. We remark that $(\mathcal{A}(\mathcal{P}), \vee, \wedge)$ is a *distributive lattice*.

A function $f : \mathcal{A}(\mathcal{P}) \to \mathbf{R}$ is called a *submodular function* on $\mathcal{A}(\mathcal{P})$ if

$$f(A) + f(B) \geq f(A \vee B) + f(A \wedge B) \tag{2.1}$$

for all $A, B \in \mathcal{A}(\mathcal{P})$. If poset P is trivial, then submodular properties (1.1) and (2.1) are equivalent.

We extend the binary relation $\preceq$ of the poset $P = (E, \preceq)$ to pairs of antichains. For $A, B \in \mathcal{A}(\mathcal{P})$ we define $A \preceq B$ if for each $a \in A$ there exists an element $b \in B$ such that $a \preceq b$. If poset P is trivial, then $A \preceq B$ if and only if $A \subseteq B$.

Let $g : \mathcal{A}(\mathcal{P}) \to \mathbf{R}$ be given. Throughout our investigations we will assume that g is *normalized*, *i.e.*, $g(\emptyset) = 0$.

3 A generalization of Narayanan's Theorem on rooted forests

The purpose of this section is to introduce a generalization of Narayanan's Theorem 3.5 in [2].

For a poset $P = (E, \preceq)$, we call y an *upper neighbor* of x if $x \prec y$ and there is no element z such that $x \prec z \prec y$. A poset $P = (E, \preceq)$ is a *rooted forest* if each element of the ground set E has at most one upper neighbor. Throughout this section we assume that $P = (E, \preceq)$ is a rooted forest and that $f : \mathcal{A}(\mathcal{P}) \to \mathbf{R}$ is a normalized submodular function.

An *antichain partition* Π' of a nonempty finite set E is a set of disjoint nonempty antichains of E whose union is E. An *antichain subpartition* Π of a set E is a set of disjoint nonempty antichains of E. Thus if $E_1 \subseteq E$ and Π_1 is an antichain partition of E_1, then Π_1 is an antichain subpartition of E. We refer to an element N_i of an antichain subpartition Π as a *block* of Π. An antichain partition of $E_1(\subseteq E)$ is denoted by $\{N_1, \ldots, N_r, \Pi_0(E_1 - \bigcup_{i=1}^r N_i)\}$ if it has blocks $N_1, \ldots, N_r$ of an antichain subpartition of E_1 and blocks of antichain partition $\Pi_0(E_1 - \bigcup_{i=1}^r N_i)$ consisting of singletons in $E_1 - \bigcup_{i=1}^r N_i \neq \emptyset$, i.e., $\Pi_0(E_1 - \bigcup_{i=1}^r N_i) = \{\{e\} \mid e \in E_1 - \bigcup_{i=1}^r N_i\}$. If we write $\Pi_N = \{N, \Pi_0(\emptyset)\}$, then it means $\Pi_N = \{N\}$.

The collection of all antichain subpartitions (partitions) of E is denoted by ASP_E (AP_E). For $\Pi_1, \Pi_2 \in ASP_E$ we define $\Pi_2 \preceq^{ASP} \Pi_1$ if and only if for each block M of Π_2 there exists some block N of Π_1 such that $M \preceq N$. We should notice that ASP_E is not a lattice but an upper semilattice with $\preceq^{ASP}$. However, by defining $\{\emptyset\} \preceq^{ASP} \Pi \in ASP_E$ and $\{\emptyset\} \preceq^{ASP} \{\emptyset\}$, $ASP_E \cup \{\{\emptyset\}\}$ forms a lattice with $\preceq^{ASP}$. We denote the least upper bound of $\{\Pi_1, \Pi_2\}$ as $\Pi_1 \vee^{ASP} \Pi_2$ and the greatest lower bound of $\{\Pi_1, \Pi_2\}$ as $\Pi_1 \wedge^{ASP} \Pi_2$.

Let g be a real-valued function on $\mathcal{A}(\mathcal{P})$. For an antichain subpartition Π of E, we define

$$\overline{g}(\Pi) \equiv \sum_{X \in \Pi} g(X).$$

Lemma 3.1: *Let g be a function defined on $\mathcal{A}(\mathcal{P})$. Let Π_* be an antichain partition of E such that $\overline{g}(\Pi_*) = \min_{\Pi \in AP_E} \overline{g}(\Pi)$. Suppose that $N \in \Pi_*$ and $\widehat{\Pi}$ is an antichain partition of N. Then $g(N) \leq \sum_{M_i \in \widehat{\Pi}} g(M_i)$.* $\square$

Lemma 3.2: *Let $E_1, E_2 \in \mathcal{A}(\mathcal{P})$ and $\emptyset \neq N \subseteq E_2$. Let Π_N be an antichain partition of E_2 such that $\{N, \Pi_0(E_2 - N)\}$. If $E_2 \preceq E_1$, then*

$$\overline{f}(\Pi) + \overline{f}(\Pi_N) \geq \overline{f}(\Pi \vee^{ASP} \Pi_N) + \overline{f}(\Pi \wedge^{ASP} \Pi_N)$$

for all $\Pi \in AP_{E_1}$.

Proof: Let $\Pi = \{X_1, \ldots, X_t\}$. Since P is a rooted forest, we have $id(X_i) \cap id(X_j) = \emptyset$ for each $X_i, X_j \in \Pi$. We should notice that $E_2 \preceq E_1$ guarantees the existence of $\Pi \wedge^{ASP} \Pi_N$. Therefore

$$\begin{aligned}
\overline{f}(\Pi) &= \sum_{id(X_i) \cap N \neq \emptyset} f(X_i) + \sum_{id(X_i) \cap N = \emptyset} f(X_i), \\
\overline{f}(\Pi_N) &= f(N) + \sum_{e_i \in E_2 - N} f(\{e_i\}), \qquad (3.1) \\
\overline{f}(\Pi \vee^{ASP} \Pi_N) &= f\Big(\bigcup_{id(X_i) \cap N \neq \emptyset} X_i\Big) + \sum_{id(X_i) \cap N = \emptyset} f(X_i), \\
\overline{f}(\Pi \wedge^{ASP} \Pi_N) &= \sum_{id(X_i) \cap N \neq \emptyset} f(X_i \wedge N) + \sum_{e_i \in E_2 - N} f(\{e_i\}). \qquad (3.2)
\end{aligned}$$

Since

$$\bigcup_{id(X_i) \cap N \neq \emptyset} X_i = \bigvee_{id(X_i) \cap N \neq \emptyset} X_i$$

holds, we have

$$\begin{aligned}
&\overline{f}(\Pi) + \overline{f}(\Pi_N) - \Big(\overline{f}(\Pi \vee^{ASP} \Pi_N) + \overline{f}(\Pi \wedge^{ASP} \Pi_N)\Big) \\
&= \sum_{id(X_i)\cap N \neq \emptyset} f(X_i) + f(N) - f\Big(\bigvee_{id(X_i)\cap N\neq\emptyset} X_i\Big) - \sum_{id(X_i)\cap N\neq\emptyset} f(X_i \wedge N) \\
&\geq 0.
\end{aligned}$$

The last inequality follows from the fact that $id(X_i)$'s are pairwise disjoint and that f is a submodular function. □

Lemma 3.3: *Let $E_1, E_2 \in \mathcal{A}(\mathcal{P})$ and $\emptyset \neq E_2 \preceq E_1$. Let $\Pi_1 \in AP_{E_1}$, $\Pi_2 \in AP_{E_2}$, and $N \in \Pi_2$. For $\Pi_N = \{N, \Pi_0(E_2 - N)\}$, if $\overline{f}(\Pi_1) = \min_{\Pi\in AP_{E_1}} \overline{f}(\Pi)$ and $\overline{f}(\Pi_2) = \min_{\Pi\in AP_{E_2}} \overline{f}(\Pi)$, then $\overline{f}(\Pi_1 \vee^{ASP} \Pi_N) = \min_{\Pi\in AP_{E_1}} \overline{f}(\Pi)$ and $\overline{f}(\Pi_1 \wedge^{ASP} \Pi_N) = \overline{f}(\Pi_N)$.*

Proof: Let $\Pi_1 = \{X_1, \cdots, X_t\}$. By Lemma 3.2,

$$\overline{f}(\Pi_1) + \overline{f}(\Pi_N) \geq \overline{f}(\Pi_1 \vee^{ASP} \Pi_N) + \overline{f}(\Pi_1 \wedge^{ASP} \Pi_N). \tag{3.3}$$

On the other hand, by (3.1), (3.2), Lemma 3.1, and $N \subseteq E_2 \preceq E_1 = \bigcup_{i=1}^{t} X_i$,

$$\overline{f}(\Pi_1 \wedge^{ASP} \Pi_N) - \overline{f}(\Pi_N) = \sum_{id(X_i)\cap N\neq\emptyset} f(X_i \wedge N) - f(N) \geq 0. \tag{3.4}$$

Hence,

$$\overline{f}(\Pi_1) \geq \overline{f}(\Pi_1 \vee^{ASP} \Pi_N).$$

Since $\Pi_N \in AP_{E_2}$ and $E_2 \preceq E_1$, $\Pi_1 \vee^{ASP} \Pi_N$ is an antichain partition of E_1. Therefore,

$$\overline{f}(\Pi_1 \vee^{ASP} \Pi_N) = \min_{\Pi\in AP_{E_1}} \overline{f}(\Pi).$$

Moreover, by using (3.3) and (3.4), we have $\overline{f}(\Pi_N) = \overline{f}(\Pi_1 \wedge^{ASP} \Pi_N)$. □

Theorem 3.4: *Let $E_1, E_2 \in \mathcal{A}(\mathcal{P})$ and $E_2 \preceq E_1$. Let Π_1 be an antichain partition of E_1 and Π_2 an antichain partition of E_2. If $\overline{f}(\Pi_1) = \min_{\Pi\in AP_{E_1}} \overline{f}(\Pi)$ and $\overline{f}(\Pi_2) = \min_{\Pi\in AP_{E_2}} \overline{f}(\Pi)$, then $\overline{f}(\Pi_1 \vee^{ASP} \Pi_2) = \min_{\Pi\in AP_{E_1}} \overline{f}(\Pi)$ and $\overline{f}(\Pi_1 \wedge^{ASP} \Pi_2) = \min_{\Pi\in AP_{E_2}} \overline{f}(\Pi)$.*

Proof: Let us consider the two following cases, Cases (a) and (b).

Case (a): We show $\overline{f}(\Pi_1 \vee^{ASP} \Pi_2) = \min_{\Pi\in AP_{E_1}} \overline{f}(\Pi)$.
Let $\Pi_1 = \{X_1, \ldots, X_t\}$, $\Pi_2 = \{N_1, \ldots, N_r\}$, and $\Pi_{N_i} = \{N_i, \Pi_0(E_2 - N_i)\}$ $(i = 1, \ldots, r)$.
By Lemma 3.3,

$$\overline{f}(\Pi_1 \vee^{ASP} \Pi_{N_1}) = \min_{\Pi\in AP_{E_1}} \overline{f}(\Pi).$$

Repeating the application of Lemma 3.3 with $(\Pi_1 \vee^{ASP} \Pi_{N_1} \vee^{ASP} \cdots \vee^{ASP} \Pi_{N_j})$ and $\Pi_{N_{j+1}}$ for $j = 1$ to $r - 1$, we have

$$\overline{f}(\Pi_1 \vee^{ASP} \Pi_{N_1} \vee^{ASP} \cdots \vee^{ASP} \Pi_{N_r}) = \min_{\Pi\in AP_{E_1}} \overline{f}(\Pi).$$

Note that $\Pi_1 \vee^{ASP} \Pi_{N_1} \vee^{ASP} \cdots \vee^{ASP} \Pi_{N_r} = \Pi_1 \vee^{ASP} \Pi_2$.

Case (b): We show $\overline{f}(\Pi_1 \wedge^{ASP} \Pi_2) = \min_{\Pi \in AP_{E_2}} \overline{f}(\Pi)$.
Let $\Pi_1 = \{X_1, \ldots, X_t\}$, $\Pi_2 = \{N_1, \ldots, N_r\}$, and $\Pi_{N_i} = \{N_i, \Pi_0(E_2 - N_i)\}$ $(i = 1, \ldots, r)$. From Lemma 3.3, we obtain

$$\overline{f}(\Pi_{N_i}) = \overline{f}(\Pi_1 \wedge^{ASP} \Pi_{N_i}) \quad (i = 1, \ldots, r). \tag{3.5}$$

Then, by the definition of Π_{N_i} we have

$$\overline{f}(\Pi_2) = \sum_{N_i \in \Pi_2} \overline{f}(\Pi_{N_i}) - \sum_{N_i \in \Pi_2} \sum_{e_k \in E_2 - N_i} f(\{e_k\}), \tag{3.6}$$

and moreover, we obtain

$$\begin{aligned} \overline{f}(\Pi_1 \wedge^{ASP} \Pi_2) &= \sum_{N_i \in \Pi_2} \sum_{j \,:\, id(X_j) \cap N_i \neq \emptyset} f(X_j \wedge N_i) \\ &= \sum_{N_i \in \Pi_2} \overline{f}(\Pi_1 \wedge^{ASP} \Pi_{N_i}) - \sum_{N_i \in \Pi_2} \sum_{e_k \in E_2 - N_i} f(\{e_k\}). \end{aligned} \tag{3.7}$$

From (3.6) and (3.7),

$$\overline{f}(\Pi_2) - \sum_{N_i \in \Pi_2} \overline{f}(\Pi_{N_i}) = \overline{f}(\Pi_1 \wedge^{ASP} \Pi_2) - \sum_{N_i \in \Pi_2} \overline{f}(\Pi_1 \wedge^{ASP} \Pi_{N_i}). \tag{3.8}$$

Therefore, combining (3.5) and (3.8),

$$\overline{f}(\Pi_2) = \overline{f}(\Pi_1 \wedge^{ASP} \Pi_2).$$

So, we have $\overline{f}(\Pi_1 \wedge^{ASP} \Pi_2) = \min_{\Pi \in AP_{E_2}} \overline{f}(\Pi)$. □

For a rooted forest $P = (E, \preceq)$ and a submodular function $f : \mathcal{A}(\mathcal{P}) \to \mathbf{R}$, we have

$$\overline{f}(\{S\}) = \min_{\Pi \in AP_S} \overline{f}(\Pi)$$

for all $S \in \mathcal{A}(\mathcal{P})$. Hence, from Theorem 3.4 we have the following corollary.

Corollary 3.5: *For an antichain N, if $\Pi = \{X_1, \ldots, X_t\} \in P_N - \{\{N\}\}$ satisfies $\overline{f}(\Pi) = f(N)$, then*

$$\overline{f}(\Pi \wedge^{ASP} \{S\}) = \overline{f}(\{S\}) \tag{3.9}$$

for all $S \preceq N$. □

Moreover, from Theorem 3.4 we have the following corollary.

Corollary 3.6: *Let $E_i \in \mathcal{A}(\mathcal{P})$ and $\Pi_i \in AP_{E_i}$ $(i = 1, 2)$. If $\overline{f}(\Pi_i) = \min_{\Pi \in AP_{E_i}} \overline{f}(\Pi)$ $(i = 1, 2)$ and $E_1 \wedge E_2 \neq \emptyset$, then $\overline{f}(\Pi_1 \wedge \Pi_2) = \min_{\Pi \in AP_{E_1 \wedge E_2}} \overline{f}(\Pi)$.*

Proof: Since $E_1 \wedge E_2 \preceq E_i$ $(i = 1, 2)$, there exists $\Pi_i^* \in AP_{E_1 \wedge E_2}$ such that

$$\overline{f}(\Pi_i^*) = \min_{\Pi \in AP_{E_1 \wedge E_2}} \overline{f}(\Pi)$$

with $\Pi_i^* \preceq^{SP} \Pi_i$ for each Π_i. Hence, we obtain

$$\overline{f}(\Pi_1^* \wedge \Pi_2^*) = \min_{\Pi \in AP_{E_1 \wedge E_2}} \overline{f}(\Pi)$$

with $\Pi_1^* \wedge \Pi_2^* \in AP_{E_1 \wedge E_2}$.

Since $\Pi_1 \wedge \Pi_2 \in AP_{E_1 \wedge E_2}$, $\Pi_1^* \wedge \Pi_2^* \preceq^{SP} \Pi_1 \wedge \Pi_2$, and f is submodular with $f(\emptyset) = 0$, we have

$$\overline{f}(\Pi_1 \wedge \Pi_2) \leq \overline{f}(\Pi_1^* \wedge \Pi_2^*) = \min_{\Pi \in AP_{E_1 \wedge E_2}} \overline{f}(\Pi). \tag{3.10}$$

(3.10) and $\Pi_1 \wedge \Pi_2 \in AP_{E_1 \wedge E_2}$ imply $\overline{f}(\Pi_1 \wedge \Pi_2) = \min_{\Pi \in AP_{E_1 \wedge E_2}} \overline{f}(\Pi)$. □

4 Related topics in decomposable convex games

An n-person game with a coalition structure Π in the characteristic function form is a triple (N, v, Π). If $\Pi = \{N\}$, then (N, v, Π) is called an n-person game with a *grand* coalition and denotes it as (N, v) for convenience. A game (N, v, Π) is *decomposable* with respect to Π if

$$v(S) = \sum_{X_i \in \Pi} v(X_i \cap S). \tag{4.1}$$

for all $S \subseteq N$. If $\Pi \in P_N$ satisfies (4.1), then we call it a *decomposable partition of* (N, v).

L. S. Shapley [3] introduced the convex games and studied their properties. One of his results is a necessary and sufficient condition for a convex game to be decomposable with respect to given coalition structure.

Theorem 4.1 (Shapley [3]): *A convex game (N, v) is decomposable if and only if*

$$f(N) = f(X_1) + \cdots + f(X_p)$$

holds for some partition $\{X_1, X_2, \ldots, X_p\}$ of N into $p \geq 2$ nonempty subsets, where $f = -v$. □

If poset $P = (N, \preceq)$ is trivial, then $\mathcal{A}(\mathcal{P}) = 2^N$. The collection of all subpartitions (partitions) of E is denoted by SP_E (P_E). We define a partial order $\preceq^{SP}$ on SP_E by defining $\Pi_2 \preceq^{SP} \Pi_1$ if and only if each block of Π_2 is contained in some block of Π_1. The least (greatest) element of SP_E above (below) Π_1 and Π_2 in the partially ordered set SP_E is denoted by $\Pi_1 \vee^{SP} \Pi_2$ $(\Pi_1 \wedge^{SP} \Pi_2)$. Hence (3.9) implies

$$f(S) = f(X_1 \cap S) + \cdots + f(X_t \cap S)$$

for all $S \subseteq N$. Hence Corollary 3.5 is a generalization of the Shapley's Theorem 4.1.

Moreover if poset P is trivial, then from Theorem 3.4 we have the following Narayanan's Theorem 3.5 in [2].

Theorem 4.2 ([2]): *Let f be a submodular function on 2^E for a nonempty finite set E. Let $\Pi_1, \Pi_2 \in P_E$. If $\overline{f}(\Pi_1) = \overline{f}(\Pi_2) = \min_{\Pi \in P_E} \overline{f}(\Pi)$, then $\overline{f}(\Pi_1 \vee^{SP} \Pi_2) = \overline{f}(\Pi_1 \wedge^{SP} \Pi_2) = \min_{\Pi \in P_E} \overline{f}(\Pi)$.* $\square$

In a convex game (N, v) a coalition S is called *inessential* if it has a proper partition $\Pi = \{S_1, \ldots, S_t\} \in P_S - \{\{S\}\}$ such that $v(S) = \overline{v}(\Pi)$. Coalitions which are not inessential are called *essential*. For a convex game (N, v) we denote by $\mathcal{E}$ the collection of its nonempty essential coalitions.

For $X \subseteq N$ and $x = (x_1, \ldots, x_n) \in \mathbf{R}^n$, let $x(X) = \sum_{i \in X} x_i$, where $x(\phi) = 0$. The core of a convex game (N, v) is defined by

$$\mathrm{Core}(v) = \{x \mid x \in \mathbf{R}^n, x(N) = v(N), \forall X \subseteq N \colon x(X) \geq v(X)\}.$$

The collection $\mathcal{E}$ gives a description of the core as follows:

$$\{x \mid x \in \mathbf{R}^n, x(N) = v(N), \forall X \in \mathcal{E} \colon x(X) \geq v(X)\}.$$

A partition $\Pi \in P_N - \{\{N\}\}$ is called an *$\mathcal{E}$-partition* (*essential coalition partition*) of N if each block of Π is nonempty essential coalition. We note that the grand coalition N is inessential if and only if $v(N) = \sum_{E_i \in \Pi} v(E_i)$ for some $\mathcal{E}$-partition Π of N.

Theorem 4.3: *Let (N, v) be a convex game with inessential grand coalition N. Then there exists a unique $\mathcal{E}$-partition Π^* of N such that $v(N) = \overline{v}(\Pi^*)$.*

Proof: Suppose there exist two different $\mathcal{E}$-partition $\Pi_1, \Pi_2 \in P_N$ such that

$$v(N) = \overline{v}(\Pi_1) = \overline{v}(\Pi_2).$$

This implies $\overline{v}(\Pi_1) = \overline{v}(\Pi_2) = \max_{\Pi \in P_N} \overline{v}(\Pi)$. Hence from Theorem 4.2 we have

$$v(N) = \overline{v}(\Pi_1 \wedge^{SP} \Pi_2).$$

However since $\Pi_1 \wedge^{SP} \Pi_2 \preceq \Pi_1$, at least one block of Π_1 is inessential. $\square$

A convex game (N, v) is decomposable if and only if the grand coalition N is inessential. Let $P_N^* = \{\Pi' \mid \Pi' \in P_N, \overline{v}(\Pi') = \max_{\Pi \in P_N} \overline{v}(\Pi)\}$.

Corollary 4.4: *For a decomposable convex game (N, v) the $\mathcal{E}$-partition of N is the minimum partition of lattice P_N^*, i.e., $\bigwedge^{SP}_{\Pi \in P_N^*} \Pi$.* $\square$

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