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Some Properties of
Generalized Homothetic Robust Epstein-Zin Utility

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Abstract

In the context of Knightian uncertainty, homothetic robust Epstein-Zin (HREZ) utility is a promising one among robust utilities. This paper investigates some properties of “generalized homothetic robust Epstein-Zin (GHREZ) utility,” in which the coefficient of relative ambiguity aversion in HREZ utility is generalized to the matrix of relative ambiguity aversions. The study demonstrates that GHREZ utility is “generalized stochastic differential utility (SDU)”, while HREZ utility is SDU. The paper then presents properties of ambiguity aversions of GHREZ utility. The study introduces concepts of directional ambiguity aversions, and demonstrates properties of directional ambiguity aversions of GHREZ utility.

Keywords: Ambiguity, Directional risk, Homothetic robust utility, Generalized stochastic differential utility

1 Introduction

The importance of Knightian uncertainty has been reconfirmed by the global financial crisis. Agents with robust utility, as proposed by Hansen and Sargent (2001), consider the “base probability” to be the most likely probability, while also accounting for other probabilities because the true probability is unknown. Since robust utility lacks homotheticity, Maenhout (2004) proposes homothetic robust (HR) utility, which is characterized by a subjective discount rate, relative risk aversion, and relative ambiguity aversion. HR utility is used in robust portfolio studies, including Maenhout (2006), Liu (2010), Branger, Larsen, and Munk (2013), Munk and Rubtsov (2014), and Yi, Viens, Law, and Li (2015).¹ HR utility can be interpreted as homothetic robust constant relative risk aversion (CRRA) utility, because it converges to

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¹Kikuchi and Kusuda (2024) generalize HR utility so that relative ambiguity aversion depends on age; however, the utility functional is no longer homothetic.

CRRA utility as ambiguity aversion approaches zero. CRRA utility does not separate the relative risk aversion from the elasticity of intertemporal substitution (EIS). Epstein-Zin (EZ) utility (Epstein and Zin (1989)) generalizes CRRA utility by separating these properties while retaining homotheticity. HREZ utility, introduced by Maenhout (2004), converges to EZ utility as ambiguity aversion approaches zero. HREZ utility is characterized by the subjective discount rate, EIS, relative risk aversion, and relative ambiguity aversion. Skiadas (2003) and Kusuda (2025) show that HREZ utility is SDU (Duffie and Epstein (1992)). In particular, Kusuda (2025) demonstrates various properties of HREZ utility, including that HREZ utility is SDU and homothetic. HREZ utility is a promising candidate among robust utilities because it is a generalization of the promising HR and EZ utilities.

Kikuchi and Kusuda (2025) note that HREZ utility ignores the fact that ambiguity about a risk generally depends on the investor's knowledge of that risk. For example, knowledge of foreign risks is generally less than knowledge of domestic risks; thus, ambiguity for foreign risks is greater than ambiguity for domestic risks. In homothetic robust utility, however, relative ambiguity aversion is common to all risks. Kikuchi and Kusuda (2025) propose "generalized HREZ (GHREZ) utility" in which the coefficient of relative ambiguity aversion in HREZ utility is generalized to the matrix of relative ambiguity aversions in which each relative ambiguity aversion depends on the corresponding risk. GHREZ utility is characterized by subjective discount rate β , relative risk aversion γ , EIS ψ , and matrix Θ of relative ambiguity aversions, where Θ is positive semidefinite. They examine the consumption-investment problem based on GHREZ utility under the quadratic security market model of Batbold, Kikuchi, and Kusuda (2022). This study demonstrates some properties of GHREZ utility. The main results of this study are summarized as follows.

First, I present that GHREZ utility is generalized SDU (GSDU) proposed by Lazrak and Quenez (2003) and that GHREZ utility is SDU if and only if GHREZ utility is reduced to HREZ utility. I also show that GHREZ utility is homothetic.

Second, I show that GHREZ utility is ambiguity averse and that GHREZ utility with $(\beta, \gamma, \psi, \Theta^*)$ is more ambiguity averse than GHREZ utility with $(\beta, \gamma, \psi, \Theta)$ if $\Theta^* \geq \Theta$.

Third, I introduce a simplified and ambiguity-oriented version of the set of basic concepts of "aversion to directional risk," proposed by Lazrak and Quenez (2003). Let α and $d(\alpha)$ denote the "direction coefficient" and "unit direction vector." Let GHREZ utility with $(\beta, \gamma, \psi, \Theta)$. Then, $d(\alpha)' \Theta d(\alpha)$ is interpreted as the ambiguity aversion in the direction vector $d(\alpha)$, where $d(\alpha)'$ is the transpose of $d(\alpha)$. I show that GHREZ utility with $(\beta, \gamma, \psi, \Theta)$ is more ambiguity averse in the direction vector $d(\alpha^*)$ than in the direction vector $d(\alpha)$ if $d(\alpha^*)' \Theta d(\alpha^*) \geq d(\alpha)' \Theta d(\alpha)$.

Fourth, I introduce the concept of "comparative ambiguity aversion"

and demonstrate that GHREZ utility with $(\beta, \gamma, \psi, \Theta^*)$ is more ambiguity averse in the direction vector $d(\alpha)$ than GHREZ utility with $(\beta, \gamma, \psi, \Theta)$ if $d(\alpha)' \Theta^* d(\alpha) \geq d(\alpha)' \Theta d(\alpha)$.

El Karoui, Peng, and Quenez (1997) demonstrate that GSDU is time consistent and increasing in consumption. Kusuda (2025)'s proof implies the following: i) GHREZ utility is risk averse, and ii) GHREZ utility with $(\beta, \gamma^*, \psi, \Theta^*)$ is more risk averse than GHREZ utility with $(\beta, \gamma, \psi, \Theta)$ if $\gamma^* \geq \gamma$. Lazrak (2004) suggests that, within the class of GSDUs, risk aversion or ambiguity aversion is inflexibly linked to the preference for information.

The remainder of this paper is organized as follows. Section 2 demonstrates that GHREZ utility is GSDU and homothetic. Section 3 presents ambiguity aversions of GHREZ utility. Section 4 introduces the concepts of directional ambiguity aversions and demonstrates the directional ambiguity aversions of GHREZ utility. Appendix shows the proofs of the lemmas and propositions.

2 GHREZ Utility

Here, I introduce GHREZ utility and consider the consumption-investment problem under the quadratic security market model. I show that it is homothetic, but not an SDU.

2.1 Environment

I consider frictionless markets over the period $[0, T]$. Investors' common subjective probability and information structure are modeled by a complete filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, where $\mathbb{F} = (\mathcal{F}_t)_{t \in [0, T]}$ is the natural filtration generated by an N -dimensional standard Brownian motion B_t . There are markets for a consumption commodity at every date $t \in [0, T]$. The expectation operator under \mathbb{P} and the conditional expectation operator given \mathcal{F}_t are denoted by \mathbb{E} and \mathbb{E}_t , respectively. Let A' and I denote the transpose of A and $N \times N$ identity matrix, respectively.

2.2 GHREZ Utility

The normalized aggregator (Duffie and Epstein (1992)) in Epstein-Zin utility is given by

$$f(c, v) = \beta \frac{c^{1-\psi^{-1}}}{1-\psi^{-1}} ((1-\gamma)v)^{1-\frac{1-\psi^{-1}}{1-\gamma}} - \frac{\beta(1-\gamma)v}{1-\psi^{-1}}, \quad (2.1)$$

where $\beta > 0$ is the subjective discount rate, $\gamma > 1$ is the relative risk aversion, and $\psi > 0$ is the elasticity of intertemporal substitution (EIS).

The theoretical range of (γ, ψ) is $(\gamma, \psi) \in (0, 1) \times (1, \infty) \cup (1, \infty) \times (0, 1)$. Note that if $(\gamma, \psi) \in (0, 1) \times (1, \infty)$ then $\tilde{V} > 0$, and if $(\gamma, \psi) \in (1, \infty) \times (0, 1)$ then $\tilde{V} < 0$. Based on empirical analyses, I assume that $\gamma > 1 > \psi > 0$.

Let \mathbb{P}^ξ denote an equivalent martingale measure for \mathbb{P} such that the density process for \mathbb{P}^ξ is ξ . Let \mathbb{E}^ξ be the expectation operator under \mathbb{P}^ξ . Maenhout (2004) and Kusuda (2025) introduce the following HREZ utility.

$$U(c) = \inf_{\mathbb{P}^\xi \in \mathbb{P}} \mathbb{E}^\xi \left[\int_0^T \left(f(c_t, V_t^\xi) + \frac{(1-\gamma)V_t^\xi}{2\theta} |\xi_t|^2 \right) dt \right], \quad (2.2)$$

where $\theta > 0$ is the relative ambiguity aversion, and V_t is the utility process defined recursively as follows:

$$V_t^\xi = \mathbb{E}_t^\xi \left[\int_t^T \left(f(c_s, V_s^\xi) + \frac{(1-\gamma)V_s^\xi}{2\theta} |\xi_s|^2 \right) ds \right]. \quad (2.3)$$

Definition 1. *GHREZ utility is defined by*

$$U(c) = \inf_{\mathbb{P}^\xi \in \mathbb{P}} \mathbb{E}^\xi \left[\int_0^T \left(f(c_t, V_t^\xi) + \frac{(1-\gamma)V_t^\xi}{2} \xi_t' \Theta^{-1} \xi_t \right) dt \right], \quad (2.4)$$

where Θ is a positive semidefinite matrix, and V_t is defined by

$$V_t^\xi = \mathbb{E}_t^\xi \left[\int_t^T \left(f(c_s, V_s^\xi) + \frac{(1-\gamma)V_s^\xi}{2} \xi_s' \Theta^{-1} \xi_s \right) ds \right]. \quad (2.5)$$

I refer to Θ as the *relative ambiguity aversion matrix*. GHREZ utility is reduced to HREZ utility if and only if $\Theta = \theta I$ for some $\theta \geq 0$.

2.3 GHREZ Utility as GSDU

Suppose a measurable pair (V, σ) satisfying the following backward stochastic differential equation (BSDE).

$$dV_t = - \left(f(c_t, V_t) - \frac{1}{2(1-\gamma)V_t} \sigma_t' \Theta \sigma_t \right) dt + \sigma_t' dB_t, \quad V_T = 0. \quad (2.6)$$

By the Girsanov theorem, the standard Brownian motion under \mathbb{P}^ξ is given by $B_t^\xi = B_t - \int_0^t \xi_s ds$. Thus, Eq. (2.6) is rewritten as

$$dV_t = - \left(f(c_t, V_t) - \frac{1}{2(1-\gamma)V_t} \sigma_t' \Theta \sigma_t - \sigma_t' \xi_t \right) dt + \sigma_t' dB_t^\xi, \quad V_T = 0. \quad (2.7)$$

From Eqs. (2.5) and (2.7), the following equation holds:

$$V_t^\xi - V_t = \mathbb{E}_t^\xi \left[\int_t^T \left(f(c_s, V_s^\xi) - f(c_s, V_s) + h(\sigma_s, V_s^\xi) - h(\sigma_s, V_s) + Q(\xi_s, \sigma_s, V_s^\xi) \right) ds \right], \quad (2.8)$$

where

$$h(\sigma, v) = -\frac{1}{2(1-\gamma)v} \sigma' \Theta^{-1} \sigma, \quad (2.9)$$

$$Q(\xi_s, \sigma_s, V_s^\xi) = \frac{(1-\gamma)V_s^\xi}{2} \left| \Theta^{-\frac{1}{2}} \xi_s + \frac{1}{(1-\gamma)V_s^\xi} \Theta^{\frac{1}{2}} \sigma_s \right|^2 \geq 0. \quad (2.10)$$

Then, f_{vv} and h_{vv} are calculated as

$$f_{vv}(c, v) = \beta(\gamma - \psi^{-1}) c^{1-\psi^{-1}} ((1-\gamma)v)^{-\frac{1-\psi^{-1}}{1-\gamma}-1} \begin{cases} > 0, & \text{if } \gamma > \psi^{-1}, \\ < 0, & \text{if } \gamma < \psi^{-1}, \end{cases}$$

$$h_{vv}(\sigma, v) = -\frac{1}{(1-\gamma)v^3} \sigma' \Theta^{-1} \sigma < 0.$$

Thus, f is convex (resp., concave) of $\gamma > \psi^{-1}$ (resp., $\gamma < \psi^{-1}$), and h is concave. Hence, $V_t^\xi - V_t^*$ is evaluated as

$$V_t^\xi - V_t^* \geq \begin{cases} \mathbb{E}_t^\xi \left[\int_t^T (f_v(c_s, V_s) + h_v(\sigma_s, V_s^\xi))(V_s^\xi - V_s) ds \right], & \text{if } \gamma > \psi^{-1}, \\ \mathbb{E}_t^\xi \left[\int_t^T (f_v(c_s, V_s^\xi) + h_v(\sigma_s, V_s^\xi))(V_s^\xi - V_s) ds \right], & \text{if } \gamma < \psi^{-1}. \end{cases}$$

Under certain integrability condition on h_v^* , in either case, the ‘‘stochastic Gronwall-Bellman (SGB) inequality’’² implies that $V_t^\xi \geq V_t^*$ P-*a.s.* for all $t \in [0, T]$. Therefore, the minimizer or the worst-case probability ξ^* is given by

$$\xi^* = -\frac{1}{(1-\gamma)V} \Theta \sigma, \quad (2.11)$$

and $V^{\xi^*} = V$.

From Eq. (2.7), V is expressed as

$$V_t = \mathbb{E}_t^\xi \left[\int_t^T g(c_s, V_s, \sigma_s) ds \right], \quad (2.12)$$

where g is given by

$$g(c, v, \sigma) = f(c, v) - \frac{1}{2(1-\gamma)v} \sigma' \Theta \sigma. \quad (2.13)$$

²See Appendix B in Duffie and Epstein (1992).

Eq. (2.12) shows that V is a GSDU (Lazrak and Quenez (2003)) with the intertemporal aggregator g . Hereafter, I refer to GHREZ utility V with g as GHREZ utility with $(\beta, \gamma, \psi, \Theta)$.

Remark 1. *It is clear that GHREZ utility is SDU if and only if $\Theta = \theta I$ for some $\theta \geq 0$; that is, GHREZ utility is reduced to HREZ utility.*

2.4 Homotheticity

A utility functional U is *homothetic* if for any consumption plan c and \tilde{c} , and any scalar $\lambda > 0$, $U(\lambda\tilde{c}) \geq U(\lambda c) \Leftrightarrow U(\tilde{c}) \geq U(c)$. Extending the proof by Kusuda (2025), I demonstrate the homotheticity of GHREZ utility.

Proposition 1. *GHREZ utility is homothetic.*

Proof. See Appendix A.1. □

3 Ambiguity Aversions

I analyze the ambiguity aversions of GHREZ utility based on the concepts defined by Chen and Epstein (2002).

3.1 Comparative Ambiguity Aversion

Chen and Epstein (2002) define the concept of *comparative ambiguity aversion* as follows³: Event $A \in \mathcal{F}_T$ is said to be *unambiguous* if $\tilde{P}(A) = P(A)$ for every $\tilde{P} \in \mathbb{P}$. Let \mathcal{R} denote the class of unambiguous events. Let $\mathcal{R}_t = \mathcal{R} \cap \mathcal{F}_t$ for every $t \in [0, T]$. The consumption process c is said to be *unambiguous* if c_t is \mathcal{R}_t -measurable for every $t \in [0, T]$. Let $\mathcal{C}_{\mathcal{R}}$ denote the set of all unambiguous consumption processes.

Definition 2. *Let U and U^* be GHREZ utilities with corresponding unambiguous classes \mathcal{R} and \mathcal{R}^* of unambiguous events. U^* is said to be more ambiguity averse than U if $\mathcal{R} \supset \mathcal{R}^*$ and if for every $c \in \mathcal{C}$ and every $\tilde{\mathcal{R}}$ -unambiguous consumption plan $c^{\mathcal{R}} \in \mathcal{C}_{\mathcal{R}^*}$, the following holds:*

$$U(c) \leq U(c^{\mathcal{R}}) \implies U^*(c) \leq U^*(c^{\mathcal{R}}). \quad (3.1)$$

Let $\mathcal{C}_{\mathcal{R}}$ denote the set of all unambiguous consumption processes.

Lemma 1. *Let U be GHREZ utility with $(\beta, \gamma, \psi, \Theta)$. Let $c^{\mathcal{R}} \in \mathcal{C}_{\mathcal{R}}$ and V_t denote the utility process of $U(c^{\mathcal{R}})$. Then, V_t satisfies*

$$V_t = E_t \left[\int_t^T f(c_s^{\mathcal{R}}, V_s) ds \right], \quad (3.2)$$

³For formal arguments, see Epstein (1999) and Epstein and Zhang (2001)

where f is given by Eq. (2.1). Function f is concave in its consumption argument. In its utility argument, f is convex (resp., concave) if $\gamma > \psi^{-1}$ (resp., $\gamma < \psi^{-1}$), and linear if $\gamma = \psi^{-1}$.

Proof. See Appendix A.2. □

Proposition 2. Let U and U^* be GHREZ utilities with $(\beta, \gamma, \psi, \Theta)$ and $(\beta, \gamma, \psi, \Theta^*)$. If $\Theta^* \geq \theta I \geq \Theta$ for some $\theta > 0$, then U^* is more ambiguity averse than U .

Proof. See Appendix A.3. □

3.2 Ambiguity Aversion

Kusuda (2025) interprets the concept of *probabilistically sophisticated utility for timeless prospects*⁴ introduced by Chen and Epstein (2002) as the following definition for utilities defined in continuous-time settings.

Definition 3. For GHREZ utility with $(\beta, \gamma, \psi, \Theta)$, the corresponding probabilistically sophisticated utility is EZ utility with (β, γ, ψ) .

Then, ambiguity aversion is defined as follows.

Definition 4. Let U be GHREZ utility. Let \tilde{U} be the corresponding probabilistically sophisticated utility. Then, U is ambiguity averse if for every unambiguous consumption plan $c^{\mathcal{R}} \in \mathcal{C}_{\mathcal{R}}$ and every consumption plan $c \in \mathcal{C}$, the following holds:

$$\tilde{U}(c) \leq \tilde{U}(c^{\mathcal{R}}) \implies U(c) \leq U(c^{\mathcal{R}}). \quad (3.3)$$

Following Kusuda (2025), I immediately obtain the following proposition.

Proposition 3. GHREZ utility is ambiguity averse.

Proof. See Appendix A.4. □

4 Directional Ambiguity Aversions

In this section, I analyze directional ambiguity aversions of GHREZ utility.

⁴For definition, see Chen and Epstein (2002).

4.1 Basic Concepts

Following Lazrak and Quenez (2003), without loss of generality, let us assume that Brownian motion $B = (B_1, B_2)'$ is two-dimensional, *i.e.*, $N = 2$. I use the following notation.

$$\sigma = \begin{pmatrix} \sigma_1 \\ \sigma_2 \end{pmatrix}, \quad \Theta = \begin{pmatrix} \theta_{11} & \theta_{12} \\ \theta_{12} & \theta_{22} \end{pmatrix}, \quad (4.1)$$

Since the set of definitions provided by Lazrak and Quenez (2003) is highly abstract and somewhat esoteric, the following is a simplified interpretation of their set of definitions. Let $0 \leq \alpha < \pi$. I define the *unit direction vector* $d(\alpha)$ and *direction* B^α by

$$d(\alpha) = \begin{pmatrix} \cos \alpha \\ \sin \alpha \end{pmatrix}, \quad B^\alpha = d(\alpha)' B. \quad (4.2)$$

Assume that GHREZ utility U is expressed as Eq. (2.12) with Eq. (2.13). Let $\bar{\sigma} = |\sigma| = \sqrt{\sigma_1^2 + \sigma_2^2}$. Then, utility U^α is defined by the following utility process.

$$dV_t^\alpha = - \left(f(c_t, V_t^\alpha) - \frac{1}{2(1-\gamma)V_t^\alpha} d(\alpha)' \Theta d(\alpha) |\bar{\sigma}_t|^2 \right) dt + \bar{\sigma}_t dB_t^\alpha, \quad V_T^\alpha = 0. \quad (4.3)$$

Let $\sigma^\alpha = \bar{\sigma} d(\alpha)$. Eq. (4.3) is rewritten as

$$dV_t^\alpha = - \left(f(c_t, V_t^\alpha) - \frac{1}{2(1-\gamma)V_t^\alpha} (\sigma_t^\alpha)' \Theta \sigma_t^\alpha \right) dt + (\sigma_t^\alpha)' dB_t, \quad V_T^\alpha = 0. \quad (4.4)$$

Remark 2. First, note that $\text{Var}_t[V_t] = \text{Var}_t[V_t^\alpha] = |\sigma_t|^2 dt$; that is, the conditional variance of dV_t^α preserves that of dV_t . Next, for example, suppose $\alpha = 0$. Then, $B^\alpha = B_1$ and $(\sigma_t^\alpha)' \Theta \sigma_t^\alpha = d(\alpha)' \Theta d(\alpha) |\bar{\sigma}_t|^2 = \theta_{11} |\bar{\sigma}_t|^2$. This shows that $d(\alpha)' \Theta d(\alpha)$ represents the relative ambiguity aversion to uncertainty in the direction B^α . Thus, U^α can be interpreted as utility if U perceives ambiguity to uncertainty only in the direction B .

4.2 Aversion to Directional Ambiguity

The concept of aversion to directional risk is proposed by Lazrak and Quenez (2003). However, they do not consider ambiguity in their framework. In this study, their concept is translated into that of *aversion to directional ambiguity*. A simple example is shown below. I assume that B_1 and B_2 represent sources of uncertainty related to the domestic and foreign economies, respectively. It is reasonable to assume that agents are less ambiguity averse to the uncertainty driven by B_1 than to that driven by B_2 . Then, the agents are said to be more ambiguity averse in the direction B_2 than in the direction B_1 .

I introduce the following definition of aversion to directional ambiguity.

Definition 5. Let U be GHREZ utility with $(\beta, \gamma, \psi, \Theta)$. U is more ambiguity averse in the direction $B^{\hat{\alpha}}$ than in the direction B^α if for every $c \in \mathcal{C}$, $U^{\hat{\alpha}}(c) \leq U^\alpha(c)$.

I obtain the following proposition.

Proposition 4. Let U be GHREZ utility with $(\beta, \gamma, \psi, \Theta)$. Assume

$$d(\hat{\alpha})'\Theta d(\hat{\alpha}) \geq d(\alpha)'\Theta d(\alpha). \quad (4.5)$$

Then, U is more ambiguity averse in the direction $B^{\hat{\alpha}}$ than in the direction B^α .

Proof. See Appendix A.5. □

4.3 Comparative Directional Ambiguity Aversion

Next, I introduce the concept of *comparative directional ambiguity aversion*.

Definition 6. Let U and U^* be GHREZ utilities with the corresponding unambiguous classes \mathcal{R} and \mathcal{R}^* of unambiguous events. U^* is said to be more ambiguity averse in the direction B^α than U if $\mathcal{R} \supset \mathcal{R}^*$ and if for every $c \in \mathcal{C}$ and every \mathcal{R}^* -unambiguous consumption plan $c^{\mathcal{R}} \in \mathcal{C}_{\mathcal{R}^*}$, the following holds:

$$U^\alpha(c) \leq U^\alpha(c^{\mathcal{R}}) \implies U^{*\alpha}(c) \leq U^{*\alpha}(c^{\mathcal{R}}). \quad (4.6)$$

Proposition 5. Let U and U^* be GHREZ utilities with $(\beta, \gamma, \psi, \Theta)$ and $(\beta, \gamma, \psi, \Theta^*)$. Assume

$$d(\alpha)'\Theta^* d(\alpha) \geq d(\alpha)'\Theta d(\alpha). \quad (4.7)$$

Then, U^* is more ambiguity averse in the direction B^α than U .

Proof. See Appendix A.6. □

A Proofs

A.1 Proof of Proposition 1

Let V_t be the utility process given by Eq. (2.6). Define the ordinally equivalent utility (OEU) process \hat{V}_t of V_t as $\hat{V}_t = \hat{\varphi}(V_t)$ where $\hat{\varphi}$ is given by

$$\hat{\varphi}(v; \gamma) = ((1 - \gamma)v)^{\frac{1}{1-\gamma}}. \quad (A.1)$$

Then, from Ito's lemma, the BSDE for \hat{V}_t is calculated as follows.

$$\begin{aligned}
d\hat{V}_t &= \frac{d\hat{\varphi}}{dv}(V_t)dV_t + \frac{1}{2} \frac{d^2\hat{\varphi}}{dv^2}(V_t)(dV_t)^2 \\
&= ((1-\gamma)V_t)^{\frac{1}{1-\gamma}-1} \left\{ - \left(f(c_t, V_t) - \frac{1}{2(1-\gamma)V_t} \sigma'_t \Theta \sigma_t \right) dt + \sigma'_t dB_t + \frac{1}{2} \frac{\gamma}{V_t} |\sigma_t|^2 dt \right\} \\
&= - \left(\hat{f}(c_t, \hat{V}_t) - \frac{1}{2\hat{V}_t} \hat{\sigma}'_t (\gamma I + \Theta) \hat{\sigma}_t \right) dt + \hat{\sigma}'_t dB_t, \quad d\hat{V}_T = 0,
\end{aligned} \tag{A.2}$$

where $\hat{\sigma}_t = ((1-\gamma)V_t)^{\frac{1}{1-\gamma}-1} \sigma_t$ and

$$\hat{f}(c, v) = \frac{\beta}{1-\psi^{-1}} c^{1-\psi^{-1}} v^{\psi^{-1}} - \frac{\beta}{1-\psi^{-1}} v. \tag{A.3}$$

Let $\lambda > 0$. From Eq. (A.2), the BSDE for $\lambda \hat{V}_t$ is calculated as

$$\begin{aligned}
d(\lambda \hat{V}_t) &= - \left(\lambda \hat{f}(c_t, \hat{V}_t) - \frac{\lambda}{2\hat{V}_t} \hat{\sigma}'_t (\gamma I + \Theta) \hat{\sigma}_t \right) dt + (\lambda \hat{\sigma}_t)' dB_t \\
&= - \left(\hat{f}(\lambda c_t, \lambda \hat{V}_t) - \frac{1}{2(\lambda \hat{V}_t)} (\lambda \hat{\sigma}_t)' (\gamma I + \Theta) (\lambda \hat{\sigma}_t) \right) dt + (\lambda \hat{\sigma}_t)' dB_t
\end{aligned} \tag{A.4}$$

The SDE (A.4) shows that $U(\lambda c) = \lambda U(c)$ and U is homothetic.

A.2 Proof of Lemma 1

Subtracting Eq. (3.2) from Eq. (2.5) yields

$$V_t^\xi - V_t = \mathbb{E}_t \left[\int_t^T \left(f(c_s^\mathcal{R}, V_s^\xi) - f(c_s^\mathcal{R}, V_s) + \frac{(1-\gamma)V_s^\xi}{2} \xi'_s \Theta^{-1} \xi_s \right) ds \right]. \tag{A.5}$$

Then, the following holds:

$$f_{vv}(c, v) = \beta(\gamma - \psi^{-1}) c^{1-\psi^{-1}} ((1-\gamma)v)^{-\frac{1-\psi^{-1}}{1-\gamma}-1}. \tag{A.6}$$

Thus, in its utility argument, f^* is convex (resp., concave) if $\gamma > \psi^{-1}$ (resp., $\gamma < \psi^{-1}$), and linear if $\gamma = \psi^{-1}$. Hence, $V_t^\xi - V_t$ is evaluated from below as in the following:

$$V_t^\xi - V_t \geq \begin{cases} \mathbb{E}_t \left[\int_t^T f_v(c_s^\mathcal{R}, V_s^*) (V_s^\xi - V_s) ds \right], & \text{if } \gamma \geq \psi^{-1}, \\ \mathbb{E}_t^\xi \left[\int_t^T f_v(c_s, V_s^\xi) (V_s^\xi - V_s^*) ds \right], & \text{if } \gamma < \psi^{-1}. \end{cases} \tag{A.7}$$

Then, the SGB inequality implies $V_t^\xi \geq V_t$ P-a.s. for all $t \in [0, T]$. Therefore, the minimizer of ξ is given by $\xi_t = 0$ for every $t \in [0, T]$, and V satisfies Eq. (3.2). Finally, f is concave in its consumption argument, as shown in the following:

$$f_{cc}(c, v) = -\beta \psi^{-1} c^{-\psi^{-1}} \left((1-\gamma)v \right)^{1-\frac{1-\psi^{-1}}{1-\gamma}} < 0. \tag{A.8}$$

A.3 Proof of Proposition 2

It is obvious that $\mathcal{R} = \tilde{\mathcal{R}}$. It follows by Lemma 1 that $U(c^{\mathcal{R}}) = U^*(c^{\mathcal{R}})$ for every $c^{\mathcal{R}} \in \mathcal{C}_{\mathcal{R}}$. Let $c \in \mathcal{C}$ such that $U(c) \leq U(c^{\mathcal{R}})$. Let OEUs $\bar{U} = \varphi(U)$ and $\bar{U}^* = \varphi(U^*)$ where φ is given by

$$\varphi(v) = \frac{1}{1 - (\gamma + \theta)} ((1 - \gamma)v)^{1 - \frac{\theta}{1 - \gamma}}. \quad (\text{A.9})$$

Let

$$\bar{f}(c, \bar{v}) = \beta \frac{c^{1 - \psi^{-1}}}{1 - \psi^{-1}} ((1 - \mathcal{U})\bar{v})^{1 - \frac{1 - \psi^{-1}}{1 - \mathcal{U}}} - \frac{\beta(1 - \mathcal{U})\bar{v}}{1 - \psi^{-1}}, \quad (\text{A.10})$$

where $\mathcal{U} = \gamma + \theta$. Then, from Ito's lemma, the BSDE for \bar{V} is calculated as

$$\begin{aligned} d\bar{V}_t &= \frac{d\varphi}{dv}(V_t)dV_t + \frac{1}{2} \frac{d\varphi^2}{dv^2}(V_t)(dV_t)^2 \\ &= ((1 - \gamma)V_t)^{-\frac{\theta}{1 - \gamma}} \left\{ - \left(f(c_t, V_t) - \frac{1}{2(1 - \gamma)V_t} \sigma'_t \Theta \sigma_t \right) dt + \sigma'_t dB_t \right\} \\ &\quad + \frac{1}{2} ((1 - \gamma)V_t)^{-\frac{\theta}{1 - \gamma}} \left(- \frac{\theta}{(1 - \gamma)V_t} \right) |\sigma_t|^2 dt \\ &= - \left(\bar{f}(c_t, \bar{V}_t) - \frac{1}{2(1 - \mathcal{U})\bar{V}_t} \bar{\sigma}'_t (\Theta - \theta I) \bar{\sigma}_t \right) dt + \bar{\sigma}'_t dB_t, \end{aligned} \quad (\text{A.11})$$

where $\bar{\sigma}_t = ((1 - \gamma)V_t)^{-\frac{\theta}{1 - \gamma}} \sigma_t$. Similarly, the SDE for \bar{V}^* is calculated as

$$d\bar{V}_t^* = - \left(\bar{f}(c_t, \bar{V}_t^*) - \frac{1}{2(1 - \mathcal{U})\bar{V}_t^*} (\bar{\sigma}_t^*)' (\Theta^* - \theta I) \bar{\sigma}_t^* \right) dt + (\bar{\sigma}_t^*)' dB_t. \quad (\text{A.12})$$

Note that $\mathcal{U} = \gamma + \theta > \psi^{-1}$ because $\gamma \geq \psi^{-1}$ and $\theta > 0$. Then, in its utility argument, \bar{f} is convex (resp., concave) if $\gamma + \theta > \psi^{-1}$ (resp., $\gamma + \theta < \psi^{-1}$), and linear if $\gamma + \theta = \psi^{-1}$ as shown in the following:

$$\bar{f}_{vv}(c, \bar{v}) = \beta (\mathcal{U} - \psi^{-1}) c^{1 - \psi^{-1}} \left((1 - \mathcal{U})\bar{v} \right)^{-\frac{1 - \psi^{-1}}{1 - (\gamma + \theta)} - 1}. \quad (\text{A.13})$$

Let

$$q(\bar{\sigma}, \bar{\sigma}^*, \bar{v}, \bar{v}^*) = \frac{1}{2(1 - \mathcal{U})\bar{v}} \bar{\sigma}' (\theta I - \Theta) \bar{\sigma} + \frac{1}{2(1 - \mathcal{U})\bar{v}^*} (\bar{\sigma}^*)' (\Theta^* - \theta I) \bar{\sigma}^*. \quad (\text{A.14})$$

Then, $q(\bar{\sigma}, \bar{\sigma}^*, \bar{v}, \bar{v}^*) \geq 0$ because $\Theta^* \geq \theta I \geq \Theta$. From Eqs. (A.11) and (A.12), $\bar{V}_t - \bar{V}_t^*$ is evaluated as

$$\begin{aligned} \bar{V}_t - \bar{V}_t^* &= \text{E}_t \left[\int_t^T (\bar{f}(c_s, \bar{V}_s) - \bar{f}(c_s, \bar{V}_s^*) + q(\bar{\sigma}_s, \bar{\sigma}_s^*, \bar{V}_s, \bar{V}_s^*)) ds \right] \\ &\geq \begin{cases} \text{E}_t \left[\int_t^T \bar{f}_v(c_s, \bar{V}_s^*) (\bar{V}_s - \bar{V}_s^*) ds \right], & \text{if } \gamma + \theta \geq \psi^{-1}, \\ \text{E}_t \left[\int_t^T \bar{f}_v(c_s, \bar{V}_s) (\bar{V}_s - \bar{V}_s^*) ds \right], & \text{if } \gamma + \theta < \psi^{-1}. \end{cases} \end{aligned} \quad (\text{A.15})$$

The SGB inequality implies $\bar{V}_t \geq \bar{V}_t^*$ P-a.s. for all $t \in [0, T]$. Therefore, $U^*(c) \leq U(c) \leq U(c^{\mathcal{R}}) = U^*(c^{\mathcal{R}})$.

A.4 Proof of Proposition 3

Let U be GHREZ utility with $(\beta, \gamma, \psi, \Theta)$. Let \tilde{U} be the corresponding EZ utility with (β, γ, ψ) . Assume that $c^{\mathcal{R}} \in \mathcal{C}_{\mathcal{R}}$ and $c \in \mathcal{C}$ are such that $\tilde{U}(c) \leq \tilde{U}(c^{\mathcal{R}})$. First, $\tilde{U}(c^{\mathcal{R}}) = U(c^{\mathcal{R}})$ holds from Lemma 1. Since \tilde{U} is interpreted as HREZ utility with $(\beta, \gamma, \psi, 0+)$, from Proposition 2, the following holds:

$$U(c) \leq \tilde{U}(c). \quad (\text{A.16})$$

Thus, it follows from Eq. (A.16) and $\tilde{U}(c^{\mathcal{R}}) = U(c^{\mathcal{R}})$ that $U(c) - U(c^{\mathcal{R}}) = U(c) - \tilde{U}(c) + \tilde{U}(c) - \tilde{U}(c^{\mathcal{R}}) + \tilde{U}(c^{\mathcal{R}}) - U(c^{\mathcal{R}}) \leq 0$. Therefore, U is ambiguity averse.

A.5 Proof of Proposition 4

Let $c \in \mathcal{C}$. Let V^α and $V^{\hat{\alpha}}$ denote the utility processes of $U^\alpha(c)$ and $U^{\hat{\alpha}}(c)$, respectively. Let $d(\hat{\alpha})'\Theta d(\hat{\alpha}) = \theta^{\hat{\alpha}}$ and $\theta^\alpha = d(\alpha)'\Theta d(\alpha)$. Then, as $\theta^{\hat{\alpha}} \geq \theta^\alpha$ by assumption, $V_t^\alpha - V_t^{\hat{\alpha}}$ is evaluated as

$$\begin{aligned} V_t^\alpha - V_t^{\hat{\alpha}} &= \mathbb{E}_t \left[\int_t^T \left\{ f(c_s, V_s^\alpha) - f(c_s, V_s^{\hat{\alpha}}) + \left(-\frac{\theta^\alpha}{2(1-\gamma)V_t^\alpha} + \frac{\theta^{\hat{\alpha}}}{2(1-\gamma)V_t^{\hat{\alpha}}} \right) |\bar{\sigma}_t|^2 \right\} ds \right] \\ &\geq \mathbb{E}_t \left[\int_t^T \left(f(c_s, V_s^\alpha) - f(c_s, V_s^{\hat{\alpha}}) + h^\alpha(V_t^\alpha) - h^\alpha(V_t^{\hat{\alpha}}) \right) ds \right], \end{aligned} \quad (\text{A.17})$$

where

$$h^\alpha(v) = -\frac{\theta^\alpha}{2(1-\gamma)v} |\bar{\sigma}_t|^2. \quad (\text{A.18})$$

Given that $h_{vv}^\alpha(v) = -\frac{\theta^\alpha}{(1-\gamma)v^3} |\bar{\sigma}_t|^2 < 0$, we have

$$V_t^\alpha - V_t^{\hat{\alpha}} \geq \begin{cases} \mathbb{E}_t \left[\int_t^T (f_v(c_s, V_s^{\hat{\alpha}}) + h_v(V_s^\alpha))(V_s^\alpha - V_s^{\hat{\alpha}}) ds \right], & \text{if } \gamma \geq \psi^{-1}, \\ \mathbb{E}_t \left[\int_t^T (f_v(c_s, V_s^\alpha) + h_v(V_s^\alpha))(V_s^\alpha - V_s^{\hat{\alpha}}) ds \right], & \text{if } \gamma < \psi^{-1}. \end{cases} \quad (\text{A.19})$$

The SGB inequality implies $V^\alpha \geq V_t^{\hat{\alpha}}$. Thus, U is more ambiguity averse in the direction $B^{\hat{\alpha}}$ than in the direction B^α .

A.6 Proof of Proposition 5

By Lemma 1, $U^\alpha(c^{\mathcal{R}}) = U^{*\alpha}(c^{\mathcal{R}})$ for every $c^{\mathcal{R}} \in \mathcal{C}_{\mathcal{R}}$. Let $c \in \mathcal{C}$ such that $U^\alpha(c) \leq U^\alpha(c^{\mathcal{R}})$. Let $\theta = d(\alpha)'\Theta d(\alpha)$ and $\theta^* = d(\alpha)'\Theta^* d(\alpha)$. Let V^α and

$\bar{V}^{*\alpha}$ denote the utility processes of $U^\alpha(c)$ and $U^{*\alpha}(c)$, respectively. Define OEU's $\bar{V}_t^\alpha = \varphi(V_t^\alpha)$ and $\bar{V}_t^{*\alpha} = \varphi(V_t^{*\alpha})$ where φ is given by Eq. (A.9). Then, from Ito's lemma and Eq. (4.3), the SDE for \bar{V}^α is calculated as

$$\begin{aligned} d\bar{V}_t^\alpha &= \frac{d\varphi}{dv}(V_t^\alpha)dV_t^\alpha + \frac{1}{2}\frac{d^2\varphi}{dv^2}(V_t^\alpha)(dV_t^\alpha)^2 \\ &= ((1-\gamma)V_t^\alpha)^{-\frac{\theta}{1-\gamma}} \left\{ - \left(f(c_t, V_t^\alpha) - \frac{\theta}{2(1-\gamma)V_t^\alpha} |\bar{\sigma}_t|^2 \right) dt + \bar{\sigma}_t dB_t^\alpha \right\} \\ &\quad + \frac{1}{2} ((1-\gamma)V_t^\alpha)^{-\frac{\theta}{1-\gamma}} \left(- \frac{\theta}{(1-\gamma)V_t^\alpha} \right) |\bar{\sigma}_t|^2 dt = -\bar{f}(c_t, \bar{V}_t^\alpha) dt + \hat{\sigma}_t dB_t^\alpha, \end{aligned} \quad (\text{A.20})$$

where $\hat{\sigma}_t = ((1-\gamma)V_t^\alpha)\bar{\sigma}_t$. Similarly, the SDE for $\bar{V}^{*\alpha}$ is calculated as

$$d\bar{V}_t^{*\alpha} = - \left(\bar{f}(c_t, \bar{V}_t^{*\alpha}) - \frac{1}{2(1-\mathcal{U})\bar{V}_t^{*\alpha}} (\theta^* - \theta) |\bar{\sigma}_t^*|^2 \right) dt + \bar{\sigma}_t^* dB_t^\alpha. \quad (\text{A.21})$$

Thus, $\bar{V}_t^\alpha - \bar{V}_t^{*\alpha}$ is evaluated as

$$\begin{aligned} \bar{V}_t^\alpha - \bar{V}_t^{*\alpha} &= \mathbb{E}_t \left[\int_t^T \left(f(c_s^\mathcal{R}, V_s^\alpha) - f(c_s^\mathcal{R}, V_s^{*\alpha}) + \frac{1}{2(1-\mathcal{U})\bar{V}_t^{*\alpha}} (\theta^* - \theta) |\bar{\sigma}_t^*|^2 \right) ds \right] \\ &\geq \begin{cases} \mathbb{E}_t \left[\int_t^T f_v(c_s^\mathcal{R}, V_s^*) (V_s^\xi - V_s) ds \right], & \text{if } \gamma \geq \psi^{-1}, \\ \mathbb{E}_t^\xi \left[\int_t^T f_v(c_s, V_s^\xi) (V_s^\xi - V_s^*) ds \right], & \text{if } \gamma < \psi^{-1}. \end{cases} \end{aligned} \quad (\text{A.22})$$

Thus, the SGB inequality implies that $U^{*\alpha}(c) \leq U^\alpha(c) \leq U^\alpha(c^\mathcal{R}) = U^{*\alpha}(c^\mathcal{R})$.

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Competing Interests

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