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Abstract This study assumes homothetic robust Epstein-Zin (HREZ) utility and analyzes the consumption–investment problem under a quadratic security market model. In HREZ utility, characterized by relative risk aversion and relative ambiguity aversion, if the sum of these equals for two different utilities, then they are observationally indistinguishable. We show that under the worst-case probability, the market price of risk is replaced by the "investor price of uncertainty." We introduce the notions of the "worst-case premiums" and the "long-term worst-case premiums" on securities, and derive analytical expressions of the optimal robust control, investor price of uncertainty, worst-case premiums, and long-term worst-case premiums. Our numerical analysis suggests that we can identify the two different HREZ utilities based on the information related to the long-term worst-case premiums.

Keywords Homothetic robust utility · Stochastic differential utility · Consumption-investment problem · Stochastic volatility · Stochastic inflation

1 Introduction

There are two key issues in studying consumption and investment problems. The first is to incorporate into security market models the stylized facts that interest rates, the market price of risk, variances and covariances of asset returns, and inflation rate are stochastic and mean-reverting. The second issue is to assume a utility that takes into account Knightian uncertainty, as recognized during the recent global financial crisis.

This paper is a substantial extension of part of Batbold, Kikuchi, and Kusuda [3].

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Regarding the first issue, Batbold, Kikuchi, and Kusuda [2] examine the consumption–investment problem for long-term investors with constant relative risk aversion (CRRA) utility under a quadratic security market (QSM) model that satisfies the above stylized facts. The class of QSM models, which is a generalization of the class of affine models (Duffie and Kan [8]), has been independently developed by Ahn, Dittmar, and Gallant [1] and Leippold and Wu [18].[1] Batbold *et al.* [2] derive an optimal portfolio decomposed into myopic, intertemporal hedging, and inflation–deflation hedging demands, and show that all three demands are nonlinear functions of the state vector. Their numerical analysis presents the nonlinearity and significance of market timing effects; the nonlinearity is attributed to the stochastic variances and covariances of asset returns, whereas the significance is attributed to inflation–deflation hedging demand in addition to myopic demand.

For the second issue, Hansen and Sargent [11] introduce robust utility. Investors with robust utility regard the "base probability" as the most likely probability; however, they also consider other probabilities because the true probability is unknown. Given that robust utility lacks homotheticity, Maenhout [21] proposes homothetic robust utility. Homothetic robust utility is characterized by the subjective discount rate, relative risk aversion, and relative ambiguity aversion. Homothetic robust utility has been applied to robust control studies and asset pricing.[2] Homothetic robust utility can be interpreted as homothetic robust CRRA utility in the sense that it converges to CRRA utility as ambiguity aversion approaches zero. CRRA utility does not separate relative risk aversion and elasticity of intertemporal substitution (EIS) ψ. Epstein-Zin utility generalizes CRRA utility and separates these properties while retaining homotheticity. Maenhout [21] also introduces homothetic robust Epstein-Zin (HREZ) utility, and Kikuchi and Kusuda [16] show the properties of HREZ utility. HREZ utility also has been applied to robust control studies and asset pricing.[3] Let β, γ, θ, and ψ denote the discount rate, relative risk aversion, and relative ambiguity aversion, and EIS of an investor's HREZ utility. Kikuchi and Kusuda [16] show that HREZ utility with $(\beta, \gamma_1, \theta_1, \psi)$ is observationally indistinguishable from that with $(\beta, \gamma_2, \theta_2, \psi)$ if $\gamma_1 + \theta_1 = \gamma_2 + \theta_2$ where $\theta_1 \neq \theta_2$. Following Kikuchi and Kusuda [13], we refer to the sum of relative risk aversion and relative ambiguity aversion as "relative uncertainty aversion." This implies that we can estimate their relative uncertainty aversion of an investor with HREZ utility from their observed portfolio, but not their relative risk and ambiguity aversions.

[1] QSM models are employed in studies of empirical analysis (Leippold and Wu [19], Kim and Singletion [17], and Kikuchi [12]), security pricing (Chen, Filipović, and Poor [7], Boyarchenko and Levendorskii [4], and Filipović, Gourier, and Mancini [9], and Kikuchi and Kusuda [16]), and optimal consumption–investment (Batbold *et al.* [2], and Kikuchi and Kusuda [13] [14] [15]).

[2] For example, see Maenhout [21], [22], Liu [20], Branger, Larsen, and Munk [5], Munk and Rubtsov [23], Yi, Viens, Law, and Li [25], and Kikuchi and Kusuda [13].

[3] See Maenhout [21], and Kikuchi and Kusuda [15], [16].

Investors with homothetic robust utility or HREZ utility first determine the "worst-case probability" of minimizing utility for a given consumption and investment and then determine the optimal consumption and investment that maximizes utility under the worst-case probability. The portfolio weights to risky securities based on homothetic robust utility decrease relative to those based on non-robust utility. This is because under the worst-case probability, the market price of risk, which is the price of investing in risky assets, decreases and the expected returns on assets decrease. Thus, if $\theta_1 > \theta_2$ (resp. $\gamma_1 < \gamma_2$) in the two HREZ utilities, the expected returns under the worst-case probability for the investor with θ_1 (resp. γ_1) is expected to be lower than that for the investor with θ_2 (resp. γ_2). If this is true, we can estimate relative risk aversion and relative ambiguity aversion through an empirical analysis of the expected returns on assets under the worst-case probability. Therefore, it is important to analyze the market price of risk and the expected returns under the worst-case probability and elucidate the structure of the optimal robust portfolio determined by such a robust control process. The purpose of this paper is to analyze and elucidate these issues. We assume an infinite-lived investor who has HREZ utility with $(\beta, \gamma, \theta, \psi)$ and address the consumption–investment problem under the QSM model of Kikuchi and Kusuda [13]. We arrive at the following main results.

First, we derive the worst-case probability for a given consumption and "investment," which is the inner product of the volatility matrix of risky securities and the vector of the fractions of wealth invested in those securities. Comparing the budget constraint under the worst-case probability with the budget constraint under the base probability, we show that the volatility of wealth is invariant, whereas the market price of risk is replaced by the "investor price of uncertainty" discounted from the market price of risk. Given that the discount from the market price of risk is permanent, this implies that investors with HREZ utility do not assume high volatility of their portfolio, but rather a low long-term return as the worst-case scenario.

Second, we derive first expressions of the optimal control and the investor price of uncertainty, both of which depend on the unknown indirect utility function and its derivatives. We show that the optimal portfolio is decomposed into the sum of the myopic demand, "intertemporal marginal indirect utility hedging demand", "intertemporal indirect utility hedging demand", and inflation–deflation hedging demand. Then, we derive second expressions of the optimal consumption and investment, as well as the investor price of uncertainty, all of which depend on the unknown function governed by a nonlinear partial differential equation (PDE). We show that the optimal robust portfolio is decomposed into the sum of three demands; in this, the intertemporal marginal indirect utility hedging demand and the intertemporal indirect utility hedging demand are integrated into the "intertemporal uncertainty hedging demand." We show that the investor price of uncertainty is a weighted average of the market price of risk and the "investor hedging value of intertemporal uncertainty." The weights are $\gamma/\mathcal{U}$ and $\theta/\mathcal{U}$ where $\mathcal{U} = \gamma + \theta$.

Third, we introduce the notion of the "worst-case real premium" on each security, which is the real premium on each security under the worst-case probability. We also introduce the notion of the "worst-case real discount" on each security. We show that the worst-case real premium on each security is a weighted average of the real premium and the real discount on the security where the weights are the same as the investor price of uncertainty. Furthermore, we introduce the notion of the "long-term real premium," "worst-case long-term real discount," and "worst-case long-term real premium." We derive analytical expressions of the long-term real premiums, and show that the worst-case long-term real premium is a weighted average of the long-term real premium and the worst-case long-term real discount.

Fourth, we derive a solution to the nonlinear PDE for the unit EIS case ($\psi = 1$), and an approximate solution for the general case ($\psi \neq 1$), and show analytical expressions of the optimal consumption and investment and the investor price of uncertainty. Investor price uncertainty is approximated to be an affine function of the state vector in which each coefficient is a function of the ratio of ambiguity aversion to uncertainty aversion. Furthermore, we derive analytical expressions of the worst-case real premiums and worst-case long-term real premiums. Therefore, we can identify γ and θ from the information related to the worst-case rates of return on securities.

Fifth, we quantitatively analyze the relationship between the worst-case long-term real premiums on the S&P500 and 10-year TIPS (treasury inflation-indexed securities) and the ratio of ambiguity aversion to uncertainty aversion in the unit EIS case. We assume the QSM model estimated by Kikuchi and Kusuda [15] and investor's HREZ utility with $(\beta, \psi) = (0.04, 1)$ and $\mathcal{U} = 4$. In the case of the S&P500, for every increase of 0.1 in $\theta/\mathcal{U}$, the worst-case long-term real premium falls significantly. Therefore, if they tell us their subjective worst-case long-term expected rate of return on the S&P500 or the difference between their subjective long-term expected rate of return and their subjective long-term worst-case long-term rate of return, then we can estimate γ and θ with high precision. In the case of the 10-year TIPS, the worst-case long-term real premiums are slightly higher than the long-term real premiums, indicating a worst-case probability insurance function for the such TIPS.

The remainder of this paper is organized as follows. In Section 2, we review the QSM model and HREZ utility. In Section 3, we derive the first and second expressions of the optimal robust control and investor price of uncertainty. In Section 4, we introduce the notion of the worst-case premiums and the worst-case long-term real premiums. In Section 5, we derive analytical expressions of the optimal robust control, investor price of uncertainty, and the worst-case long-term premiums. Section 6 provides a numerical analysis of the worst-case long-term premiums, and Section 7 conculdes this study.

2 Review of QSM Model and HREZ Utility

First, we review the QSM model, HREZ utility, and the dynamics of no-arbitrage prices. Next, we show analytical expressions of the expected real rates of return on securities and real budget constraint, followed by the consumption–investment problem.

2.1 QSM Model

We consider frictionless US markets over the period $[0,\infty)$. Investors' common subjective probability and information structure are modeled by a complete filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathrm{P})$, where $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,\infty)}$ is the natural filtration generated by an N-dimensional standard Brownian motion B_t. We denote the expectation operator under P by E and the conditional expectation operator given $\mathcal{F}_t$ by E_t.

There are markets for a consumption commodity and securities at every date $t \in [0,\infty)$, and the consumer price index p_t is observed. The traded securities are K-types of indices, the instantaneously nominal risk-free security called the money market account and a continuum of zero-coupon bonds and zero-coupon inflation-indexed bonds whose maturity dates are $(t, t+\tau^*]$, where τ^* is the longest time to maturity of the bonds. Each zero-coupon bond has a one US dollar payoff at maturity, and each zero-coupon inflation-indexed bond has a p_T US dollar payoff at maturity T.

On every date t, P_t, P_t^T, Q_t^T, and S_t^k denote the USD prices of the money market account, zero-coupon bond with maturity date T, zero-coupon inflation-indexed bond with maturity date T, and k-th index, respectively. Let A' and I_n denote the transpose of A and $n\times n$ identity matrix, respectively.

We assume the following QSM model introduced by Kikuchi and Kusuda [13].

Assumption 1 *Let $(\rho_0, \iota_0, \delta_{0k}, \sigma_{0k})$, $(\lambda, \rho, \iota, \sigma_p, \delta_k, \sigma_k)$, and $(\mathcal{R}, \Delta_k, \Sigma_k)$ denote scalers, N-dimensional vectors, and $N\times N$ positive-definite symmetric matrices, respectively, where $k \in \{1,\cdots,K\}$.*

1. *State vector process X_t is N-dimensional and satisfies the following SDE:*
$$dX_t = -\mathcal{K}X_t\,dt + I_N\,dB_t, \tag{2.1}$$
where $\mathcal{K}$ is an $N\times N$ lower triangular matrix.
2. *Market price λ_t of risk and instantaneous nominal risk-free rate r_t are provided as*
$$\lambda_t = \lambda + \Lambda X_t, \qquad r_t = \rho_0 + \rho' X_t + \frac{1}{2}X_t'\mathcal{R}X_t, \tag{2.2}$$
where Λ is an $N\times N$ matrix.
3. *Consumer price index p_t satisfies*
$$\frac{dp_t}{p_t} = \mu^p(X_t)\,dt + \sigma^p(X_t)'dB_t, \qquad p_0 = 1, \tag{2.3}$$
where $\mu^p(X_t)$ and $\sigma^p(X_t)$ are given by
$$\mu^p(X_t) = \iota_0 + \iota' X_t + \frac{1}{2}X_t'\mathcal{I}X_t, \tag{2.4}$$
$$\sigma^p(X_t) = \sigma_p + \Sigma_p X_t, \tag{2.5}$$
where $\mathcal{I}$ is an $N\times N$positive-semidefinite symmetric matrix.

4. *The dividend of the k-th index is given by*

$$D_t^k = \left(\delta_{0k} + \delta_k' X_t + \frac{1}{2} X_t' \Delta_k X_t\right) \exp\left(\sigma_{0k} t + \sigma_k' X_t + \frac{1}{2} X_t' \Sigma_k X_t\right). \quad (2.6)$$

5. *The parameters introduced above and a matrix* $\bar{\mathcal{R}}$ *defined by*

$$\bar{\mathcal{R}} = \mathcal{R} - \mathcal{I} + \Sigma_p' \Lambda + \Lambda' \Sigma_p \quad (2.7)$$

satisfy the regularity conditions shown in Appendix A.1.

6. *Markets are complete and arbitrage-free.*

2.2 HREZ Utility

Let f denote the normalized aggregator of the form:

$$f(c,v) = \begin{cases} \dfrac{\beta}{1-\psi^{-1}}(1-\gamma)v\left(\left(c\big((1-\gamma)v\big)^{-\frac{1}{1-\gamma}}\right)^{1-\psi^{-1}} - 1\right), & \text{if} \quad \psi \neq 1, \\ \beta(1-\gamma)v\Big(\log c - \dfrac{1}{1-\gamma}\log\big((1-\gamma)v\big)\Big), & \text{if} \quad \psi = 1, \end{cases} \quad (2.8)$$

where $\beta > 0$ is the subjective discount rate, $\gamma \in (0,1) \cup (1,\infty)$ is the relative risk aversion, and $\psi > 0$ is the EIS.

Let $\mathbb{P}$ denote the set of all equivalent probability measures[4] of P. We assume an infinite-live investor with HREZ utility.

Assumption 2 *The investor's utility is HREZ utility of the form:*

$$U(c) = \inf_{\mathrm{P}^\xi \in \mathbb{P}} \mathrm{E}^\xi\left[\int_0^\infty \left(f(c_t, V_t^\xi) + \frac{(1-\gamma)V_t^\xi}{2\theta}|\xi_t|^2\right) dt\right], \quad (2.10)$$

where c is a consumption plan such that $c = (c_t)_{t\in[0,T^*)}$ *is an adapted non-negative consumption-rate process,* E^ξ *is the expectation under* P^ξ, $\theta > 0$ *is relative ambiguity aversion, and* V_t^ξ *is the utility process, defined recursively as*

$$V_t^\xi = \mathrm{E}_t^\xi\left[\int_t^\infty \left(f(c_s, V_s^\xi) + \frac{(1-\gamma)V_s^\xi}{2\theta}|\xi_s|^2\right) ds\right]. \quad (2.11)$$

[4] A probability measure $\tilde{\mathrm{P}}$ is said to be an equivalent probability measure of P if and only if $\mathrm{P}(A) = 0 \Leftrightarrow \tilde{\mathrm{P}}(A) = 0$. It follows from Girsanov's theorem that any equivalent probability measure is characterized by a measurable process ξ_t with Novikov's integrability condition as the following Radon-Nikodym derivative:

$$\mathrm{E}_{T^*}\left[\frac{d\mathrm{P}^\xi}{d\mathrm{P}}\right] = \exp\left(\int_0^{T^*} \xi_t\, dB_t - \frac{1}{2}\int_0^{T^*} |\xi_t|^2 dt\right) \quad \forall T^* \in (0,\infty). \quad (2.9)$$

2.3 Dynamics of No-Arbitrage Prices of Securities

Let $\tau = T - t$ denote the time to maturity of bond P_t^T or inflation-indexed bond Q_t^T. Batbold *et al.* [2] show the dynamics of no-arbitrage prices of securities (For the proof, see Batbold *et al.* [2]).

Lemma 1 *Under Assumptions 1 and 2, the stochastic differential equations (SDEs) of security price processes satisfy the following:*

1. *The default-free bond with time τ to maturity:*
$$\frac{dP_t^T}{P_t^T} = \left(r_t + (\sigma(\tau) + \Sigma(\tau)X_t)'\lambda_t\right) dt + (\sigma(\tau) + \Sigma(\tau)X_t)' \, dB_t, \qquad (2.12)$$
where $(\Sigma(\tau), \sigma(\tau))$ is a solution to the system of ODEs (A.3) and (A.4).
2. *The default-free inflation-indexed bond with time τ to maturity:*
$$\frac{dQ_t^T}{Q_t^T} = \left(r_t + \left(\sigma_q(\tau) + \Sigma_q(\tau)X_t\right)'\lambda_t\right) dt + \left(\sigma_q(\tau) + \Sigma_q(\tau)X_t\right)' dB_t, \ (2.13)$$
where $(\Sigma_q(\tau), \sigma_q(\tau)) = (\bar{\Sigma}_q(\tau) + \Sigma_p, \bar{\varsigma}_q(\tau) + \sigma_p)$ and $(\bar{\Sigma}_q(\tau), \bar{\varsigma}_q(\tau))$ is a solution to the system of ODEs (A.5) and (A.6).
3. *The k-th index:*
$$\frac{dS_t^k + D_t^k dt}{S_t^k} = \left(r_t + (\sigma_k + \Sigma_k X_t)'\lambda_t\right) dt + (\sigma_k + \Sigma_k X_t)' \, dB_t, \qquad (2.14)$$
where Σ_k is a solution to Eq. (A.7) and σ_k is given by Eq. (A.8).

We define the nominal premiums on P_t^T, Q_t^T and S_t^k by

$$\nu^T(X_t)dt = \mathrm{E}_t\left[\frac{dP_t^T}{P_t^T}\right] - r_t, \quad \nu_q^T(X_t)dt = \mathrm{E}_t\left[\frac{dQ_t^T}{Q_t^T}\right] - r_t, \quad \nu^k(X_t)dt = \mathrm{E}_t\left[\frac{dS_t^k + D_t^k dt}{S_t^k}\right] - r_t. \qquad (2.15)$$

It follows from Eqs. (2.12), (2.13), (2.14), and (2.15) that the premiums on P_t^T, Q_t^T and S_t^k are expressed as

$$\begin{aligned} \nu^T(X_t) &= \left(\sigma(\tau) + \Sigma(\tau)X_t\right)'\lambda_t, \\ \nu_q^T(X_t) &= \left(\sigma_q(\tau) + \Sigma_q(\tau)X_t\right)'\lambda_t, \\ \nu^k(X_t) &= \left(\sigma_k + \Sigma_k X_t\right)'\lambda_t. \end{aligned} \qquad (2.16)$$

Considering that the quadratic model assumes that r_t is a quadratic function and of X_t and λ_t is an affine function of X_t, Eq. (2.16) shows that the expected nominal rates of return on securities are quadratic functions of the state vector.

2.4 Real Premiums and Real Budget Constraint

Let $P_t(\tau) = P_t^T$ and $Q_t(\tau) = Q_t^T$ where $\tau = T - t$.

Assumption 3 *The investor invests in $P_t, P_t(\tau_1), \cdots, P_t(\tau_{I_P}), Q_t(\tau_1^q), \cdots, Q_t(\tau_{I_Q}^q)$, and $S_t^1, \cdots, S_t^K$ at time t where $I_P + I_Q + K = N$. Let $\Phi^P(\tau)$ and $\Phi^Q(\tau^q)$ denote the portfolio weights of a default-free bond with τ-time to maturity and a default-free inflation-indexed bond with τ^q-time to maturity, respectively. Let Φ^k denote the portfolio weight of the k-th index. Let Φ_t and $\Sigma(X_t)$ denote the portfolio weight and volatility matrix at time t, respectively. Φ_t and $\Sigma(X_t)$ are expressed as*

$$\Phi_t = \begin{pmatrix} \Phi_t^P(\tau_1) \\ \vdots \\ \Phi_t^P(\tau_{I_P}) \\ \Phi_t^Q(\tau_1^q) \\ \vdots \\ \Phi_t^Q(\tau_{I_Q}^q) \\ \Phi_t^1 \\ \vdots \\ \Phi_t^K \end{pmatrix}, \quad \Sigma(X_t) = \begin{pmatrix} \big(\sigma(\tau_1) + \Sigma(\tau_1)X_t\big)' \\ \vdots \\ \big(\sigma(\tau_{I_P}) + \Sigma(\tau_{I_P})X_t\big)' \\ \big(\sigma_q(\tau_1^q) + \Sigma_q(\tau_1^q)X_t\big)' \\ \vdots \\ \big(\sigma_q(\tau_{I_Q}^q) + \Sigma_q(\tau_{I_Q}^q)X_t\big)' \\ (\sigma_1 + \Sigma_1 X_t)' \\ \vdots \\ (\sigma_K + \Sigma_K X_t)' \end{pmatrix}, \tag{2.17}$$

where Φ is an adapted process.

To derive the real budget constraint equation, we define the real market price of risk $\bar{\lambda}(X_t)$ and real instantaneous interest rate $\bar{r}(X_t)$ as

$$\bar{\lambda}(X_t) = \lambda_t - \sigma^p(X_t), \tag{2.18}$$

$$\bar{r}(X_t) = r_t - \mu^p(X_t) + \lambda_t'\sigma^p(X_t). \tag{2.19}$$

Note that $\bar{\lambda}(X_t)$ and $\bar{r}(X_t)$ are the quadratic functions of X_t.

$$\bar{\lambda}(X_t) = \bar{\lambda} + \bar{\Lambda}X_t, \tag{2.20}$$

$$\bar{r}(X_t) = \bar{\rho}_0 + \bar{\rho}'X_t + \frac{1}{2}X_t'\bar{\mathcal{R}}X_t, \tag{2.21}$$

where $\bar{\mathcal{R}}$ is given by Eq. (2.7) and

$$\bar{\lambda} = \lambda - \sigma_p, \qquad \bar{\Lambda} = \Lambda - \Sigma_p, \tag{2.22}$$

$$\bar{\rho}_0 = \rho_0 - \iota_0 + \lambda'\sigma^p, \qquad \bar{\rho} = \rho - \iota + \Lambda'\sigma_p + \Sigma_p'\lambda. \tag{2.23}$$

Let $\bar{P}_t^T, \bar{Q}_t^T, \bar{S}_t^k$, and $\bar{D}_t^k$ denote the real prices of P_t^T, Q_t^T, S_t^k, and D_t^k, respectively. We define the real premiums on P_t^T, Q_t^T and S_t^k by

$$\begin{aligned} \bar{\nu}^T(X_t)dt &= \mathrm{E}_t\left[\frac{d\bar{P}_t^T}{\bar{P}_t^T}\right] - \bar{r}(X_t)dt, \\ \bar{\nu}_q^T(X_t)dt &= \mathrm{E}_t\left[\frac{d\bar{Q}_t^T}{\bar{Q}_t^T}\right] - \bar{r}(X_t)dt, \\ \bar{\nu}^k(X_t)dt &= \mathrm{E}_t\left[\frac{d\bar{S}_t^k + \bar{D}_t^k dt}{\bar{S}_t^k}\right] - \bar{r}(X_t)dt. \end{aligned} \tag{2.24}$$

Let $\bar{W}$ denote the real wealth process and $\bar{W}_0 > 0$. We obtain the following lemma.

Lemma 2 *Under Assumption 1, the following 1 and 2 holds:*

1. *The real premiums on P_t^T, Q_t^T and S_t^k are expressed as*

$$\begin{aligned}\bar{\nu}^T(X_t) &= \big(\bar{\sigma}(\tau) + \bar{\Sigma}(\tau)X_t\big)'\bar{\lambda}(X_t),\\ \bar{\nu}_q^T(X_t) &= \big(\bar{\sigma}_q(\tau) + \bar{\Sigma}_q(\tau)X_t\big)'\bar{\lambda}(X_t),\\ \bar{\nu}^k(X_t) &= \big(\bar{\sigma}_k + \bar{\Sigma}_k X_t\big)'\bar{\lambda}(X_t).\end{aligned} \tag{2.25}$$

where $\bar{\sigma}(\tau) = \sigma(\tau)-\sigma_p$, $\bar{\Sigma}(\tau) = \Sigma(\tau)-\Sigma_p$, $\bar{\sigma}_k = \sigma_k-\sigma_p$, and $\bar{\Sigma}_k = \Sigma_k-\Sigma_p$.

2. *Under Assumptions 2 and 3, given an initial state $(\bar{W}_0, X_0)$, consumption plan c, and self-financing portfolio weight Φ, the real budget constraint equation is given by*

$$\frac{d\bar{W}_t}{\bar{W}_t} = \left(\bar{r}(X_t) + \bar{\varsigma}_t'\bar{\lambda}(X_t) - \frac{c_t}{\bar{W}_t}\right) dt + \bar{\sigma}_t'\, dB_t, \quad \bar{W}_t > 0 \quad \forall t \in (0,\infty), \tag{2.26}$$

where

$$\bar{\varsigma}_t = \Sigma(X_t)'\Phi_t - \sigma_t^p. \tag{2.27}$$

Proof See Appendix C.1.

Substituting $\bar{\lambda}(X_t) = \bar{\lambda}+\bar{\Lambda}X_t$ into Eq. (2.25), the real premiums on P_t^T, Q_t^T and S_t^k are expressed as the quadratic functions of the state vector:

$$\begin{aligned}\bar{\nu}^T(X_t) &= \bar{\sigma}(\tau)'\bar{\lambda} + \Big(\bar{\Sigma}(\tau)'\bar{\lambda} + \bar{\Lambda}'\bar{\sigma}(\tau)\Big)'X_t + \frac{1}{2}X_t'\Big(\bar{\Sigma}(\tau)'\bar{\Lambda} + \bar{\Lambda}'\bar{\Sigma}(\tau)\Big)X_t,\\ \bar{\nu}_q^T(X_t) &= \bar{\sigma}_q(\tau)'\bar{\lambda} + \Big(\bar{\Sigma}_q(\tau)'\bar{\lambda} + \bar{\Lambda}'\bar{\sigma}_q(\tau)\Big)'X_t + \frac{1}{2}X_t'\Big(\bar{\Sigma}_q(\tau)'\bar{\Lambda} + \bar{\Lambda}'\bar{\Sigma}_q(\tau)\Big)X_t,\\ \bar{\nu}^k(X_t) &= \bar{\sigma}_k'\bar{\lambda} + \Big(\bar{\Sigma}_k'\bar{\lambda} + \bar{\Lambda}'\bar{\Sigma}_k\Big)X_t + \frac{1}{2}X_t'\Big(\bar{\Sigma}_k'\bar{\Lambda} + \bar{\Lambda}'\bar{\Sigma}_k\Big)X_t.\end{aligned} \tag{2.28}$$

The real budget constraint (2.26) indicates that $(c, \bar{\varsigma})$ is the control in the optimal consumption–investment problem. $\mathbf{X} = (\bar{W}, X')'$. We call $\bar{\varsigma}$ the investment control. Control $(c, \bar{\varsigma})$ is admissible if it satisfies the real budget constraint equation (2.26) with initial state $\mathbf{X}_0$ and there are measurable functions $\hat{c}(\mathbf{x})$ and $\hat{\varsigma}(\mathbf{x})$ such that $c_t = \hat{c}(\mathbf{X}_t)$ and $\bar{\varsigma}_t = \hat{\varsigma}(\mathbf{X}_t)$ for every $t \in [0,\infty)$. Let $\mathcal{B}(\mathbf{X}_0)$ denote the set of admissible controls. Furthermore, we call ξ in Eq. (2.9) the probability control, which is admissible if it satisfies Novikov's condition and there is a measurable function $\hat{\xi}(\mathbf{x})$ such that $\hat{\xi}(\mathbf{X}_t) = \xi_t$ for every $t \in [0,\infty)$. Let $\hat{\mathbb{P}}(\mathbf{X}_0)$ denote the set of admissible probability controls.

Given $\mathcal{F}_t$ and $\mathbf{X}_t$, the investor's robust consumption–investment problem and value function are recusively defined as

$$V_t = \sup_{(c,\bar{\varsigma})\in\mathcal{B}(\mathbf{X}_t)} \inf_{\mathrm{P}^\xi\in\hat{\mathbb{P}}(\mathbf{X}_t)} \mathrm{E}^\xi\left[\int_t^\infty \left(f(c_s, V_s) + \frac{(1-\gamma)V_s}{2\theta}\,|\xi_s|^2\right) ds\right]. \tag{2.29}$$

The recursive definition of the above value function is justified by the fact that HREZ utility is consistent. Given $\mathcal{F}_t$ and $\mathbf{X}_t = \mathbf{x}$, the indirect utility function is defined as $J(\mathbf{x}) = V_t$.

3 Optimal Robust Control and Investor Price of Uncertainty

We derive the first expressions of the optimal robust control and investor price of uncertainty, and show that the discount from the market price of risk to the investor price of uncertainty increases with investment control and decreases with "'investor hedging value of intertemporal uncertainty." Then, the second expressions of the optimal robust control and investor price of uncertainty are derived, depending on the unknown function governed by a nonlinear PDE. We show that the investor price of uncertainty is a weighted average of the market price of risk and the investor hedging value of intertemporal uncertainty.

3.1 Worst-Case Probability and Investor Price of Uncertainty

Given that the standard Brownian motion under P^ξ is given by $B_t^\xi = B_t - \int_0^t \xi_s\, ds$, the SDE for $\mathbf{X}_t$ under P^ξ is rewritten as

$$d\mathbf{X}_t = \left(\begin{pmatrix} \bar{W}_t(\bar{r}_t + \bar{\varsigma}_t'\bar{\lambda}_t) - c_t \\ -\mathcal{K}X_t \end{pmatrix} + \begin{pmatrix} \bar{W}_t\bar{\varsigma}_t' \\ I_N \end{pmatrix} \xi_t\right) dt + \begin{pmatrix} \bar{W}_t\bar{\varsigma}_t' \\ I_N \end{pmatrix} dB_t^\xi. \tag{3.1}$$

The Hamilton-Jacobi-Bellman (HJB) equation for problem (2.29) is then expressed as

$$0 = \sup_{(\hat{c},\hat{\varsigma})\in\mathbb{R}_+\times\mathbb{R}^N} \inf_{\hat{\xi}\in\mathbb{R}^N} \left\{ \begin{pmatrix} w\Big(\bar{r}(x) + \hat{\varsigma}'\bar{\lambda}(x)\Big) - \hat{c} \\ -\mathcal{K}x \end{pmatrix}' \begin{pmatrix} J_w \\ J_x \end{pmatrix} + \hat{\xi}' \begin{pmatrix} w\hat{\varsigma}' \\ I_N \end{pmatrix}' \begin{pmatrix} J_w \\ J_x \end{pmatrix} \right.$$
$$\left. + \frac{1}{2}\,\mathrm{tr}\left[\begin{pmatrix} w\hat{\varsigma}' \\ I_N \end{pmatrix}\begin{pmatrix} w\hat{\varsigma}' \\ I_N \end{pmatrix}'\begin{pmatrix} J_{ww} & J_{wx} \\ J_{xw} & J_{xx} \end{pmatrix}\right] + f(\hat{c}, J) + \frac{(1-\gamma)J}{2\theta}|\hat{\xi}|^2 \right\}. \tag{3.2}$$

The minimizer in the HJB Eq. (3.2) satisfies

$$\hat{\xi} = -\frac{\theta}{(1-\gamma)J}\begin{pmatrix} w\hat{\varsigma}' \\ I_N \end{pmatrix}'\begin{pmatrix} J_w \\ J_x \end{pmatrix}. \tag{3.3}$$

Let $\bar{W}^*$ denote the optimal real wealth. Let $\mathbf{x}^* = (w^*, x')'$, and $\hat{c}^*(\mathbf{x}^*)$ and $\hat{\varsigma}^*(\mathbf{x}^*)$ denote the optimal consumption and investment controls, respectively. We refer to $\hat{\xi}^*(\mathbf{x}^*)$, defined by

$$\hat{\xi}^*(\mathbf{x}^*) = -\frac{\theta}{(1-\gamma)J}\begin{pmatrix} w^*\hat{\varsigma}^*(\mathbf{x})' \\ I_N \end{pmatrix}'\begin{pmatrix} J_w \\ J_x \end{pmatrix}, \tag{3.4}$$

as the worst-case probability.

3.2 First Expressions of Optimal Robust Control and Investor Price of Uncertainty

Under the worst-case probability, the real budget constraint (2.26) is rewritten as

$$\frac{d\bar{W}_t}{\bar{W}_t} = \left\{\bar{r}(X_t) + \hat{\varsigma}(\mathbf{X}_t^*)'\hat{\lambda}(\mathbf{X}_t^*) - \frac{\hat{c}(\mathbf{X}_t)}{\bar{W}_t}\right\} dt + \hat{\varsigma}(\mathbf{X}_t^*)'\, dB_t^{\hat{\xi}^*}, \tag{3.5}$$

where $\hat{\lambda}(\mathbf{x}^*)$ is given by

$$\hat{\lambda}(\mathbf{x}^*) = \bar{\lambda}(x) + \hat{\xi}^*(\mathbf{x}) = \bar{\lambda}(x) - \theta\left(\frac{w^*J_w}{(1-\gamma)J}\hat{\varsigma}^*(\mathbf{x}^*) + \frac{J_x}{(1-\gamma)J}\right). \tag{3.6}$$

The second term represents the discount from the market price of risk to $\hat{\lambda}(\mathbf{x}^*)$. Eq. (3.6) shows that the discount is proportional to the investor's ambiguity aversion. We refer to $\hat{\lambda}^*(\mathbf{x}^*)$ as the "investor price of uncertainty."

Remark 1 In Eq. (3.5), the real market price $\bar{\lambda}(X_t)$ of risk in the real budget constraint Eq. (2.26) is replaced with $\hat{\lambda}(\mathbf{X}_t)$, which is the investor price per unit of investment under the worst-case probability for a given control. When ambiguity is not considered, *i.e.* $\theta \searrow 0$, the price per unit of investment risk is the real market price $\bar{\lambda}(X_t)$ of risk, which is common to all investors. By contrast, $\hat{\lambda}(\mathbf{X}_t)$ varies across investors. Eq. (3.5) shows that ambiguity averse investors value the price per unit of investment below the real market price of risk under the conditional worst-case probability.

Remark 2 In Eq. (3.5), under the worst-case probability assumed by investors with HREZ utility, the investment control $\hat{\varsigma}(\mathbf{X}_t)$, which is the volatility of the wealth process, is as assumed under the base probability; however, its price $\hat{\lambda}(\mathbf{X}_t)$ is permanently discounted from the market price of risk. This implies that investors with HREZ utility do not assume high volatility of their portfolio, but rather a low long-term return as the worst-case scenario.

Let

$$\mathcal{U} = -\frac{w^* J_{ww}}{J_w} + \theta \frac{w^* J_w}{(1-\gamma)J}. \tag{3.7}$$

We obtain the following lemma.

Lemma 3 *Under Assumptions 1–3, the optimal control for the problem (2.29) satisfies*

$$\hat{c}^*(\mathbf{X}_t^*) = \begin{cases} \beta J_w^{-1}(1-\gamma)J, & \text{if} \quad \psi = 1, \\ \beta^{\psi} J_w^{-\psi}\big((1-\gamma)J\big)^{\frac{\gamma\psi-1}{\gamma-1}}, & \text{if} \quad \psi \neq 1, \end{cases} \tag{3.8}$$

$$\hat{\varsigma}^*(\mathbf{X}_t^*) = \frac{1}{\mathcal{U}}\left(\bar{\lambda}(X_t) + \frac{J_{xw}}{J_w} - \frac{\theta J_x}{(1-\gamma)J}\right), \tag{3.9}$$

where J is a solution of the following PDE:

$$0 = \frac{1}{2}\mathrm{tr}\left[J_{xx}\right] - \frac{\theta}{2(1-\gamma)J}|J_x|^2 - \frac{1}{2}\left(w^{*2}J_{ww} - \frac{\theta w^2 J_w^2}{(1-\gamma)J}\right)^{-1}|\pi(x)|^2$$
$$+\bar{r}(x)w^* J_w - (\mathcal{K}x)' J_x + \begin{cases} \beta\big\{(1-\gamma)(\log\hat{c}^* - 1) - \log\big((1-\gamma)\big)\big\}J, & \text{if} \quad \psi = 1, \\ \frac{1}{\psi-1}\hat{c}^* J_w - \frac{\beta(1-\gamma)}{1-\psi^{-1}}J, & \text{if} \quad \psi \neq 1, \end{cases} \tag{3.10}$$

where

$$\pi(x) = -w^* J_w\left(\bar{\lambda}(x) + \frac{J_{xw}}{J_w} - \frac{\theta J_x}{(1-\gamma)J}\right). \tag{3.11}$$

Proof See Appendix C.2.

Remark 3 Strictly speaking, $(\hat{c}^*(\mathbf{X}_t^*), \hat{\varsigma}^*(\mathbf{X}_t))$ is a candidate for optimal control because we do not provide a verification theorem. We tentatively call this optimal control in this study.

From Eqs. (2.27) and (3.9), optimal robust portfolio Φ_t^* satisfies

$$\Sigma(X_t)'\Phi_t^* - \sigma^p(X_t) = \frac{1}{\mathcal{U}}\left(\bar{\lambda}_t + \frac{J_{xw}}{J_w} - \theta\frac{J_x}{(1-\gamma)J}\right). \tag{3.12}$$

Thus, from Eq. (2.18), we decompose the optimal robust portfolio into the following four terms:

$$\Phi_t^* = \frac{1}{\mathcal{U}}\Sigma(X_t)'^{-1}\lambda_t + \frac{1}{\mathcal{U}}\Sigma(X_t)'^{-1}\frac{J_{xw}}{J_w} - \frac{1}{\mathcal{U}}\Sigma(X_t)'^{-1}\frac{\theta J_x}{(1-\gamma)J} + \left(1 - \frac{1}{\mathcal{U}}\right)\Sigma(X_t)'^{-1}\sigma^p(X_t). \tag{3.13}$$

The first term is the myopic demand. The fourth term insures inflation–deflation risk, as presented by Kikuchi and Kusuda [13],which is reffered to as the inflation–deflation hedging demand.

Remark 4 Note that the indirect utility function depends on both γ and θ because the PDE (3.10) depends not only on γ but also on θ. Thus, the second and third terms in Eq. (3.13) are related to the intertemporal uncertainty on marginal indirect utility and indirect utility, respectively. The second term hedges against intertemporal uncertainty in marginal indirect utility due to state changes, whereas the third term hedges against intertemporal uncertainty in indirect utility due to state changes. Therefore, we call the second term the "intertemporal marginal indirect utility hedging demand" and the third term the "intertemporal indirect utility hedging demand." Note that intertemporal indirect utility hedging demand disappears when $\theta = 0$.

From the PDE (3.10), we infer that the indirect utility function takes the form in Eq. (3.14):

$$J(\mathbf{x}) = \begin{cases} \dfrac{w^{1-\gamma}}{1-\gamma}\big(G(x)\big)^{1-\gamma}, & \text{if} \quad \psi = 1, \\ \dfrac{w^{1-\gamma}}{1-\gamma}\big(G(x)\big)^{\frac{1-\gamma}{\psi-1}}, & \text{if} \quad \psi \neq 1. \end{cases} \tag{3.14}$$

The partial derivatives of J are given by Eq.s (B.1) and (B.2) in Appendix B. Substituting Eq. (B.1) into Eq. (3.7), we obtain

$$\mathcal{U} = \gamma + \theta. \tag{3.15}$$

Following Kikuchi and Kusuda [13], we refer to $\mathcal{U}$ and $\mathcal{U}^{-1}$ as the relative uncertainty aversion and relative uncertainty tolerance, respectively.

3.3 Investor Hedging Value of Intertemporal Uncertainty

Substituting Eqs. (B.1) and (B.2) into Eqs. (3.4) and (3.6) yields

$$\hat{\xi}^*(\mathbf{x}^*) = \theta\Big(\hat{\varsigma}^*(\mathbf{x}^*) - \eta^*(\mathbf{x}^*)\Big), \tag{3.16}$$

$$\hat{\lambda}(\mathbf{x}^*) = \bar{\lambda}(x) - \theta\Big(\hat{\varsigma}^*(\mathbf{x}^*) - \eta^*(\mathbf{x}^*)\Big), \tag{3.17}$$

where

$$\eta^*(x) = -\frac{J_x}{(1-\gamma)J} = \begin{cases} -\dfrac{G_x(x)}{G(x)}, & \text{if} \quad \psi = 1, \\ -\dfrac{1}{\psi - 1}\dfrac{G_x(x)}{G(x)}, & \text{if} \quad \psi \neq 1. \end{cases} \tag{3.18}$$

Remark 5 Eq. (3.17) shows that the discount from the market price of risk to $\hat{\lambda}(\mathbf{x}^*)$ increases with investment control $\hat{\varsigma}(\mathbf{x}^*)$ and that it decrease with $\eta^*(x)$.

We refer to $\eta^*(X_t)$ as the "investor hedging value of intertemporal uncertainty."

3.4 Second Expressions of Optimal Robust Control and Investor Price of Uncertainty

We obtain the following proposition.

Proposition 1 *Under Assumptions 1–3, the optimal wealth, consumption, investment, and investor price of uncertainty for the problem (2.29) satisfy Eqs. (3.19), (3.20), (3.21), and (3.22), respectively.*

$$\frac{d\bar{W}_t^*}{\bar{W}_t^*} = \int_0^t \left(\bar{r}(X_s) + \hat{\varsigma}^*(X_s)'\bar{\lambda}(X_s) - \frac{\hat{c}^*(\mathbf{X}_s^*)}{\bar{W}_s^*} \right) ds + \int_0^t \hat{\varsigma}^*(X_s)'\, dB_s \Big), \tag{3.19}$$

$$\hat{c}^*(\mathbf{X}_t^*) = \begin{cases} \beta \bar{W}_t^*, & \text{if} \quad \psi = 1, \\ \dfrac{\beta^\psi}{G(X_t)} \bar{W}_t^*, & \text{if} \quad \psi \neq 1, \end{cases} \tag{3.20}$$

$$\hat{\varsigma}^*(X_t) = \frac{1}{\gamma+\theta}\bar{\lambda}(X_t) + \left(1 - \frac{1}{\gamma+\theta}\right)\eta^*(X_t), \tag{3.21}$$

$$\hat{\lambda}^*(X_t) = \frac{\gamma}{\gamma+\theta}\bar{\lambda}(X_t) + \frac{\theta}{\gamma+\theta}\eta^*(X_t), \tag{3.22}$$

where G is a solution of the following PDE:

1. *The unit EIS case*

$$\frac{1}{2}\mathrm{tr}\left[\frac{G_{xx}}{G}\right] - \left(1 - \frac{1}{2}(\gamma+\theta)^{-1}\right)\left|\frac{G_x}{G}\right|^2 - \Big(\mathcal{K}x + \big(1-(\gamma+\theta)^{-1}\big)\bar{\lambda}(x)\Big)'\frac{G_x}{G}$$
$$- \beta \log G + \frac{1}{2(\gamma+\theta)}|\bar{\lambda}(x)|^2 + \bar{r}(x) + \beta(\log\beta - 1) = 0. \tag{3.23}$$

2. *The general case:*

$$\frac{1}{2}\,\mathrm{tr}\left[\frac{G_{xx}}{G}\right] - \frac{\psi - (\gamma+\theta)^{-1}}{2(\psi-1)}\left|\frac{G_x}{G}\right|^2 - \Big(\mathcal{K}x + \big(1-(\gamma+\theta)^{-1}\big)\bar{\lambda}(x)\Big)'\frac{G_x}{G}$$
$$+ \frac{\beta^\psi}{G} + \frac{\psi-1}{2(\gamma+\theta)}|\bar{\lambda}(x)|^2 + (\psi-1)\,\bar{r}(x) - \beta\psi = 0. \tag{3.24}$$

Proof See Appendix C.3.

Remark 6 Eqs (3.23) and (3.24) show that G depends only on β, ψ, and $\mathcal{U} = \gamma + \theta$, but not on γ and θ. Thus, Eq. (3.18) shows that the investor hedging value of intertemporal uncertainty also depends only on β, ψ, and $\mathcal{U}$, but not on γ and θ. Eq. (3.21) shows that the optimal investment is the weighted average of the market price of risk and the investor hedging value of intertemporal uncertainty, and the weights are $\mathcal{U}^{-1}$ and $1 - \mathcal{U}^{-1}$, respectively. Hence, Eqs. (3.19) and (3.20) show that the optimal wealth and consumption also depends only on β, ψ, and $\mathcal{U}$, but not on γ and θ. Therefore, HREZ utility with $(\beta, \gamma_1, \theta_1, \psi)$ is observationally indistinguishable from that with $(\beta, \gamma_2, \theta_2, \psi)$ if $\gamma_1 + \theta_1 = \gamma_2 + \theta_2 = \mathcal{U}$.

Remark 7 Eq. (3.22) shows that the investor price $\hat{\lambda}^*(\mathbf{x}^*)$ of uncertainty depends only on x and not on w^*. This result follows from the homotheticity of HREZ utility. Eq. (3.22) also shows that $\hat{\lambda}^*(\mathbf{x}^*)$ is the weighted average of the market price of risk and the investor hedging value of intertemporal uncertainty where the weights are $\gamma/\mathcal{U}$ and $\theta/\mathcal{U}$, respectively.

From Eqs. (2.27) and (3.22), the optimal robust portfolio is rewritten as

$$\begin{aligned} \Phi^*(X_t) = \frac{1}{\gamma+\theta} \Sigma(X_t)'^{-1}\lambda_t + \left(1 - \frac{1}{\gamma+\theta}\right) \Sigma(X_t)'^{-1}\eta^*(X_t) \\ + \left(1 - \frac{1}{\gamma+\theta}\right) \Sigma(X_t)'^{-1}\sigma^p(X_t). \end{aligned} \tag{3.25}$$

Remark 8 The intertemporal marginal indirect utility hedging demand and intertemporal indirect utility hedging demand in Eq. (3.13) are integrated into the second term in Eq. (3.25). We refer to the second term in Eq. (3.25) as the "intertemporal uncertainty hedging demand."

4 Worst-Case Premiums and Long-Term Premiums

We introduce the notion of the worst-case real premiums and discounts, and then, the notion of the long-term real premiums and show their analytical expressions. Next, we introduce the long-term worst-case real discounts and premiums.

4.1 Worst-Case Premiums and Worst-Case Discounts

We introduce the notion of the worst-case real premiums on P_t^T, Q_t^T and S_t^k, defined by

$$\begin{aligned} \hat{\nu}^T(X_t)dt &= \mathrm{E}_t^{\hat{\xi}^*}\left[\frac{d\bar{P}_t^T}{\bar{P}_t^T}\right] - \bar{r}(X_t), \\ \hat{\nu}_q^T(X_t)dt &= \mathrm{E}_t^{\hat{\xi}^*}\left[\frac{d\bar{Q}_t^T}{\bar{Q}_t^T}\right] - \bar{r}(X_t), \\ \hat{\nu}^k(X_t)dt &= \mathrm{E}_t^{\hat{\xi}^*}\left[\frac{d\bar{S}_t^k + \bar{D}_t^k dt}{\bar{S}_t^k}\right] - \bar{r}(X_t). \end{aligned} \tag{4.1}$$

We verify that the worst-case real premiums on $\bar{P}_t^T, \bar{Q}_t^T$ and $\bar{S}_t^k$ are expressed as

$$\begin{aligned}\hat{\nu}^T(X_t) &= \big(\bar{\sigma}(\tau) + \bar{\Sigma}_q(\tau)X_t\big)'\hat{\lambda}^*(X_t),\\ \hat{\nu}_q^T(X_t) &= \big(\bar{\sigma}_q(\tau) + \bar{\Sigma}_q(\tau)X_t\big)'\hat{\lambda}^*(X_t),\\ \hat{\nu}^k(X_t) &= \big(\bar{\sigma}_k + \bar{\Sigma}_k X_t\big)'\hat{\lambda}^*(X_t).\end{aligned} \tag{4.2}$$

We introduce the notion of the worst-case real discounts on P_t^T, Q_t^T and S_t^k, defined by

$$\begin{aligned}\hat{\zeta}^T(X_t) &= \big(\bar{\sigma}(\tau) + \bar{\Sigma}_q(\tau)X_t\big)'\eta^*(X_t),\\ \hat{\zeta}_q^T(X_t) &= \big(\bar{\sigma}_q(\tau) + \bar{\Sigma}_q(\tau)X_t\big)'\eta^*(X_t),\\ \hat{\zeta}^k(X_t) &= \big(\bar{\sigma}_k + \bar{\Sigma}_k X_t\big)'\eta^*(X_t).\end{aligned} \tag{4.3}$$

Substituting Eq. (3.22) into Eq. (4.2) and substituting Eqs. (2.25) and (4.3) into the resultant equation, we obtain the second expression of the worst-case real premiums on P_t^T, Q_t^T and S_t^k.

$$\begin{aligned}\hat{\nu}^T(X_t) &= \frac{\gamma}{\gamma+\theta}\bar{\nu}^T(X_t) + \frac{\theta}{\gamma+\theta}\hat{\zeta}^T(X_t),\\ \hat{\nu}_q^T(X_t) &= \frac{\gamma}{\gamma+\theta}\bar{\nu}_q^T(X_t) + \frac{\theta}{\gamma+\theta}\hat{\zeta}_q^T(X_t),\\ \hat{\nu}^k(X_t) &= \frac{\gamma}{\gamma+\theta}\bar{\nu}^k(X_t) + \frac{\theta}{\gamma+\theta}\hat{\zeta}^k(X_t).\end{aligned} \tag{4.4}$$

Remark 9 Eq. (4.4) shows that the worst-case real premiums are the weighted average of the real premiums and the worst-case real discounts, where the weights are the same as the investor price of uncertainty.

4.2 Long-Term Premiums

We introduce the notion of the long-term real premiums on $P_t, \bar{P}_t^T, \bar{Q}_t^T$ and $\bar{S}_t^k$, defined by

$$\bar{r} = \mathrm{E}[\lim_{t\to\infty}\bar{r}(X_t)], \quad \bar{\nu}(\tau) = \mathrm{E}[\lim_{t\to\infty}\bar{\nu}^T(X_t)], \quad \bar{\nu}_q(\tau) = \mathrm{E}[\lim_{t\to\infty}\bar{\nu}_q^T(X_t)], \quad \bar{\nu}_k = \mathrm{E}[\lim_{t\to\infty}\bar{\nu}_k(X_t)]. \tag{4.5}$$

We obtain the following proposition.

Proposition 2 *Under Assumptions 1 and 2, the following 1 and 2 holds:*

1. *The stationary distribution of the state vector process is $N(0, \mathbf{\Sigma}_X)$ where $\mathbf{\Sigma}_X$ is a unique positive-definite symmetric solution to the following standard Lyapunov equation.*

$$-\mathcal{K}\mathbf{\Sigma}_X - \mathbf{\Sigma}_X\mathcal{K}' + I_N = 0. \tag{4.6}$$

2. *The long-term real instantaneous interest rate and the long-term real premiums are expressed as*

$$
\begin{aligned}
\bar{r} &= \bar{\rho}_0 + \frac{1}{2}\mathrm{tr}\left[\mathbf{\Sigma}_X \bar{\mathcal{R}}\right], \\
\bar{\nu}(\tau) &= \bar{\sigma}(\tau)'\bar{\lambda} + \frac{1}{2}\mathrm{tr}\left[\mathbf{\Sigma}_X\left(\bar{\varSigma}(\tau)'\bar{\varLambda} + \bar{\varLambda}'\bar{\varSigma}(\tau)\right)\right], \\
\bar{\nu}_q(\tau) &= \bar{\sigma}_q(\tau)'\bar{\lambda} + \frac{1}{2}\mathrm{tr}\left[\mathbf{\Sigma}_X\left(\bar{\varSigma}_q(\tau)'\bar{\varLambda} + \bar{\varLambda}'\bar{\varSigma}_q(\tau)\right)\right], \\
\bar{\nu}_k &= \bar{\sigma}_k'\bar{\lambda} + \frac{1}{2}\mathrm{tr}\left[\mathbf{\Sigma}_X\left(\bar{\varSigma}_k'\bar{\varLambda} + \bar{\varLambda}'\bar{\varSigma}_k\right)\right].
\end{aligned}
\tag{4.7}
$$

Proof From Eq. (2.1), X_t is solved as $X_t = e^{-t\mathcal{K}}X_0 + \int_0^t e^{(s-t)\mathcal{K}}\,dB_s$. Gardiner [10] shows that if all the eigenvalues of $\mathcal{K}$ have positive real parts,[5] $\mathrm{E}[X_\infty] = 0$ and the variance–covariance matrix of X_∞ is given by

$$
\mathrm{E}[X_\infty X_\infty'] = \mathrm{E}\Big[\lim_{t\to\infty}\int_0^t e^{(s-t)\mathcal{K}}e^{(s-t)\mathcal{K}'}\,ds\Big] = \mathbf{\Sigma}_X, \tag{4.8}
$$

where $\mathbf{\Sigma}_X$ is the solution to Eq. (4.6). Thus, the stationary distribution of the state vector process is $N(0, \mathbf{\Sigma}_X)$. Therefore, by substituting Eqs. (2.21) and (2.28) into (4.5), we obtain Eq. (4.7).

4.3 Worst-Case Long-Term Discounts and Premiums

First, we introduce the notion of the worst-case long-term real disounts on $\bar{P}_t^T, \bar{Q}_t^T$ and $\bar{S}_t^k$, defined by

$$
\hat{\zeta}(\tau) = \mathrm{E}[\lim_{t\to\infty}\hat{\zeta}^T(X_t)], \quad \hat{\zeta}_q(\tau) = \mathrm{E}[\lim_{t\to\infty}\hat{\zeta}_q^T(X_t)], \quad \hat{\zeta}_k = \mathrm{E}[\lim_{t\to\infty}\hat{\zeta}_k(X_t)]. \tag{4.9}
$$

Second, we define the worst-case long-term real premiums on $\bar{P}_t^T, \bar{Q}_t^T$ and $\bar{S}_t^k$ by

$$
\hat{\nu}^*(\tau) = \mathrm{E}[\lim_{t\to\infty}\hat{\nu}^T(X_t)], \quad \hat{\nu}_q^*(\tau) = \mathrm{E}[\lim_{t\to\infty}\hat{\nu}_q^T(X_t)], \quad \hat{\nu}_k^* = \mathrm{E}[\lim_{t\to\infty}\hat{\nu}^k(X_t)]. \tag{4.10}
$$

From Eqs. (4.4), (4.7), and (4.10), we obtain the following expressions of the worst-case long-term real premiums.

$$
\begin{aligned}
\hat{\nu}^*(\tau) &= \frac{\gamma}{\gamma+\theta}\bar{\nu}(\tau) + \frac{\theta}{\gamma+\theta}\hat{\zeta}_(\tau), \\
\hat{\nu}_q^*(\tau) &= \frac{\gamma}{\gamma+\theta}\bar{\nu}_q(\tau) + \frac{\theta}{\gamma+\theta}\hat{\zeta}_q(\tau), \\
\hat{\nu}_k^* &= \frac{\gamma}{\gamma+\theta}\bar{\nu}_k + \frac{\theta}{\gamma+\theta}\hat{\zeta}_k.
\end{aligned}
\tag{4.11}
$$

The following section presents the analytical expressions of the worst-case long-term real discounts and the worst-case long-term real premiums.

[5] This condition is satisfied, because all the eigenvalues are assume to be positive in Appendix A.1.

5 Analytical Expressions of Optimal Robust Control and Worst-Case Premiums

We derive an analytical expression of the optimal robust control for the unit EIS case ($\psi = 1$) and an approximate analytical expression for the general case ($\psi \neq 1$). Then, we show that the investor price of uncertainty is approximated as an affine function of the state vector in which each coefficient is a function of the ratio of ambiguity aversion to uncertainty aversion. We derive an analytical expression of the worst-case real premiums and worst-case long-term real premiums.

5.1 Optimal Solution for the Unit EIS Case

An analytical solution of the PDE (3.23) is expressed as:

$$G(x) = \exp\left(a_0 + a'x + \frac{1}{2}x'Ax\right), \tag{5.1}$$

where A is a symmetric matrix.
We obtain the following theorem.

Theorem 1 *Under Assumptions 1–3, the indirect utility function, optimal consumption, investment, and investor price of uncertainty for problem (2.29) satisfy Eqs. (5.2), (3.20), (5.3), and (5.4), respectively.*

$$J(\mathbf{X}_t^*) = \frac{\bar{W}_t^{*1-\gamma}}{1-\gamma} \exp\left(a_0 + a'X_t + \frac{1}{2}X_t'AX_t\right), \tag{5.2}$$

$$\hat{\varsigma}^*(X_t) = \frac{1}{\gamma+\theta}(\bar{\lambda} + \bar{\Lambda}X_t) + \left(1 - \frac{1}{\gamma+\theta}\right)\big(-(a + AX_t)\big), \tag{5.3}$$

$$\hat{\lambda}^*(X_t) = \frac{\gamma}{\gamma+\theta}\bar{\lambda}(X_t) + \frac{\theta}{\gamma+\theta}\big(-(a + AX_t)\big), \tag{5.4}$$

where (A, a, a_0) is a solution of the simultaneous Eqs. (5.5)–(5.7):

$$-\Big(\mathcal{K} + \big(1-(\gamma+\theta)^{-1}\big)\bar{\Lambda} + \frac{1}{2}\beta I_N\Big)'A - A\Big(\mathcal{K} + \big(1-(\gamma+\theta)^{-1}\big)\bar{\Lambda} + \frac{1}{2}\beta I_N\Big) - \big(1-(\gamma+\theta)^{-1}\big)A^2 + (\gamma+\theta)^{-1}\bar{\Lambda}'\bar{\Lambda} + \bar{\mathcal{R}} = 0, \tag{5.5}$$

$$\Big(\big(1-(\gamma+\theta)^{-1}\big)A + \mathcal{K} + \big(1-(\gamma+\theta)^{-1}\big)\bar{\Lambda}\Big)'a = \big(1-(\gamma+\theta)^{-1}\big)A\bar{\lambda} + (\gamma+\theta)^{-1}\bar{\Lambda}'\bar{\lambda} + \bar{\rho}, \tag{5.6}$$

$$\beta a_0 = \frac{1}{2}\operatorname{tr}[A] + \big((\gamma+\theta)^{-1} - 1\big)\left(\frac{1}{2}|a|^2 + \bar{\lambda}'a\right) + \frac{1}{2}(\gamma+\theta)^{-1}|\bar{\lambda}|^2 + \bar{\rho}_0 + \beta(\log\beta - 1). \tag{5.7}$$

Proof See Appendix C.4.

5.2 Approximate Optimal Solution for the General Case

For the general case, that is, $\psi \neq 1$, we derive an approximate optimal solution by applying the loglinear approximation method presented by Campbell and Viceira [6] to our QSM model. Both nonlinear and nonhomogeneous terms appear in the PDE (3.24). From Eq. (3.20), the nonhomogeneous term β^{ψ}/G is expressed as $\beta^{\psi}/G = \hat{c}^*/w$. Considering that the optimal consumption-wealth ratio is stable, Campbell and Viceira [6] make a loglinear approximation of the nonhomogeneous term and derive an approximate solution. We apply the loglinear approximation to the nonhomogeneous term.

$$\frac{1}{G(x)} \approx g_0 - g_1 \log G(x), \tag{5.8}$$

where

$$g_0 = g_1(1 - \log g_1), \tag{5.9}$$

$$g_1 = \exp\Big(-\mathrm{E}\Big[\lim_{t\to\infty} \log G(X_t)\Big]\Big). \tag{5.10}$$

Substituting Eq. (5.1) into Eq. (5.10) yields

$$g_1 = \exp\left(\left[-a_0 - a'\mathrm{E}[\lim_{t\to\infty} X_t] - \frac{1}{2}\mathrm{E}[\lim_{t\to\infty} X_t' A X_t]\right]\right). \tag{5.11}$$

Considering $X_\infty \sim N(0, \mathbf{\Sigma}_X)$, we obtain

$$g_1 = g_1(A, a_0) := \exp\left(-a_0 - \frac{1}{2}\mathrm{tr}\big[\mathbf{\Sigma}_X A\big]\right). \tag{5.12}$$

In the PDE (3.24), approximating the nonhomogeneous term by Eq. (5.8) yields the following approximate PDE:

$$\frac{1}{2}\,\mathrm{tr}\left[\frac{G_{xx}}{G}\right] - \frac{\psi - (\gamma+\theta)^{-1}}{2(\psi-1)}\left|\frac{G_x}{G}\right|^2 - \Big(\mathcal{K}x + \big(1-(\gamma+\theta)^{-1}\big)\big(\bar{\lambda} + \bar{\Lambda}x\big)\Big)'\frac{G_x}{G}$$
$$-\beta^{\psi} g_1 \log G + \beta^{\psi} g_0 + \frac{\psi-1}{2(\gamma+\theta)}\left|\bar{\lambda}+\bar{\Lambda}x\right|^2 + (\psi-1)\left(\bar{\rho}_0 + \bar{\rho}'x + \frac{1}{2}x'\bar{\mathcal{R}}x\right) - \beta\psi = 0. \tag{5.13}$$

The optimal control and investor price of uncertainty based on the approximate PDE (5.13) are called the approximate optimal control and investor price of uncertainty, denoted by $(\tilde{c}^*(\mathbf{X}_t^*), \tilde{\varsigma}^*(X_t))$ and $\tilde{\lambda}^*(X_t)$. We obtain the following proposition.

Theorem 2 *Under Assumptions 1–3, if there is a unique solution to the simultaneous Eqs. (5.17)–(5.19), then the approximate optimal consumption, investment, and investor price of uncertainty for problem (2.29) satisfy Eqs. (5.14),* *(5.15), and (5.16), respectively:*

$$\tilde{c}^*(\mathbf{X}_t^*) = \tilde{W}_t^* \exp\left[-\left(a_0 + a'X_t + \frac{1}{2}X_t'AX_t\right)\right], \tag{5.14}$$

$$\tilde{\varsigma}^*(X_t) = \frac{1}{\gamma+\theta}\left(\bar{\lambda} + \bar{\Lambda}X_t\right) + \left(1 - \frac{1}{\gamma+\theta}\right)\left(-\frac{1}{\psi-1}(a + AX_t)\right), \tag{5.15}$$

$$\tilde{\lambda}^*(X_t) = \frac{\gamma}{\gamma+\theta}\bar{\lambda}(X_t) + \frac{\theta}{\gamma+\theta}\left(-\frac{1}{\psi-1}(a + AX_t)\right), \tag{5.16}$$

where (A, a, a_0) *is a unique solution of the simultaneous Eqs. (5.17)–(5.19):*

$$-\frac{1-(\gamma+\theta)^{-1}}{\psi-1}A^2 - \left(\mathcal{K} + \left(1-(\gamma+\theta)^{-1}\right)\bar{\Lambda}\right)' A - A\left(\mathcal{K} + \left(1-(\gamma+\theta)^{-1}\right)\bar{\Lambda}\right)$$
$$- \beta^{\psi} g_1(A, a_0)A + (\psi-1)\Big((\gamma+\theta)^{-1}\bar{\Lambda}'\bar{\Lambda} + \bar{\mathcal{R}}\Big) = 0, \tag{5.17}$$

$$-\frac{1-(\gamma+\theta)^{-1}}{\psi-1}Aa - \mathcal{K}'a - \left(1-(\gamma+\theta)^{-1}\right)(A\bar{\lambda} + \bar{\Lambda}'a)$$
$$- \beta^{\psi} g_1(A, a_0)a + (\psi-1)\Big((\gamma+\theta)^{-1}\bar{\Lambda}'\bar{\lambda} + \bar{\rho}\Big) = 0, \tag{5.18}$$

$$\frac{1}{2}\,\mathrm{tr}[A] - \frac{1-(\gamma+\theta)^{-1}}{2(\psi-1)}|a|^2 - \left(1-(\gamma+\theta)^{-1}\right)\bar{\lambda}'a$$
$$+\beta^{\psi} g_1(A, a_0)(1-a_0-\log g_1(A, a_0)) + (\psi-1)\left(\frac{(\gamma+\theta)^{-1}}{2}|\bar{\lambda}|^2 + \bar{\rho}_0\right) - \beta\psi = 0. \tag{5.19}$$

Proof See Appendix C.5.

Remark 10 If $g_1(A, a_0)$ is constant, then Eq. (5.17) is a Riccati differential equation, and there is a unique symmetric solution. However, given that $g_1(A, a_0)$ is a function of (A, a_0), we need to assume that there is a unique solution to the simultaneous Eqs. (5.17)–(5.19).

5.3 Analytical Expressions of the Worst-Case Long-Term Premiums

From Eqs. (5.4) and (5.16), the investor price of uncertainty is expressed as

$$\hat{\lambda}^*(X_t) = \check{\lambda}^*_(\theta/\mathcal{U}) + \check{\Lambda}^*(\theta/\mathcal{U})X_t, \tag{5.20}$$

where

$$\check{\lambda}^*(\theta/\mathcal{U}) \begin{cases} = \left(1 - \dfrac{\theta}{\mathcal{U}}\right)\bar{\lambda} + \dfrac{\theta}{\mathcal{U}}(-a), & \text{if} \quad \psi = 1, \\ \approx \left(1 - \dfrac{\theta}{\mathcal{U}}\right)\bar{\lambda} + \dfrac{\theta}{\mathcal{U}}\left(-\dfrac{a}{\psi-1}\right), & \text{if} \quad \psi \neq 1, \end{cases} \tag{5.21}$$

$$\check{\Lambda}^*(\theta/\mathcal{U}) \begin{cases} = \left(1 - \frac{\theta}{\mathcal{U}}\right)\bar{\Lambda} + \frac{\theta}{\mathcal{U}}(-A), & \text{if} \quad \psi = 1, \\ \approx \left(1 - \frac{\gamma}{\mathcal{U}}\right)\bar{\Lambda} + \frac{\theta}{\mathcal{U}}\left(-\frac{1}{\psi - 1}A\right), & \text{if} \quad \psi \neq 1, \end{cases} \tag{5.22}$$

Eq. (5.20) shows that the investor price of uncertainty is approximated as an affine function of the state vector in which each coefficient is a function of the ratio of ambiguity aversion to uncertainty aversion. Thus, from Eq. (4.2), the worst-case real premiums are also approximated to be quadratic functions of the state vector.

$$\begin{aligned} \hat{\nu}^T(X_t) &\approx \bar{\sigma}(\tau)'\check{\lambda}^* + \left(\bar{\Sigma}(\tau)'\check{\lambda}^* + (\check{\Lambda}^*)'\bar{\sigma}(\tau)\right)' X_t + \frac{1}{2}X_t'\left(\bar{\Sigma}(\tau)'\check{\Lambda}^* + (\check{\Lambda}^*)'\bar{\Sigma}(\tau)\right)X_t, \\ \hat{\nu}_q^T(X_t) &\approx \bar{\sigma}_q(\tau)'\check{\lambda}^* + \left(\bar{\Sigma}_q(\tau)'\check{\lambda}^* + (\check{\Lambda}^*)'\bar{\sigma}_q(\tau)\right)' X_t + \frac{1}{2}X_t'\left(\bar{\Sigma}_q(\tau)'\check{\Lambda}^* + (\check{\Lambda}^*)'\bar{\Sigma}_q(\tau)\right)X_t, \\ \hat{\nu}_k(X_t) &\approx \bar{\sigma}(\tau)'\check{\lambda}^* + \left(\bar{\Sigma}_k'\check{\lambda}^* + (\check{\Lambda}^*)'\bar{\Sigma}_k\right)X_t + \frac{1}{2}X_t'\left(\bar{\Sigma}_k'\check{\Lambda}^* + (\check{\Lambda}^*)'\bar{\Sigma}_k\right)X_t, \end{aligned} \tag{5.23}$$

where $\check{\lambda}^* = \check{\lambda}^*(\theta/\mathcal{U})$ and $\check{\Lambda}^* = \check{\Lambda}^*(\theta/\mathcal{U})$. Then, the analytical expressions of the worst-case long-term real premiums are given by

$$\begin{aligned} \hat{\nu}^*(\tau) &\approx \bar{\sigma}(\tau)'\check{\lambda}^*(\theta/\mathcal{U}) + \frac{1}{2}\text{tr}\left[\mathbf{\Sigma}_X\left(\bar{\Sigma}(\tau)'\hat{\Lambda}^*(\theta/\mathcal{U}) + \check{\Lambda}^*(\theta/\mathcal{U})'\bar{\Sigma}(\tau)\right)\right], \\ \hat{\nu}_q^*(\tau) &\approx \bar{\sigma}_q(\tau)'\check{\lambda}^*(\theta/\mathcal{U}) + \frac{1}{2}\text{tr}\left[\mathbf{\Sigma}_X\left(\bar{\Sigma}_q(\tau)'\check{\Lambda}^*(\theta/\mathcal{U}) + \check{\Lambda}^*(\theta/\mathcal{U})'\bar{\Sigma}_q(\tau)\right)\right], \\ \hat{\nu}_k^* &\approx \bar{\sigma}_k'\check{\lambda}^*(\theta/\mathcal{U}) + \frac{1}{2}\text{tr}\left[\mathbf{\Sigma}_X\left(\hat{\Sigma}_k'\check{\Lambda}^*(\theta/\mathcal{U}) + \check{\Lambda}^*(\theta/\mathcal{U})'\bar{\Sigma}_k\right)\right]. \end{aligned} \tag{5.24}$$

From Eqs. (4.7) and (5.24), the difference between the long-term real premiums and the worst-case long-term real premiums are given by

$$\begin{aligned} \bar{\nu}(\tau) - \hat{\nu}^*(\tau) &\approx \bar{\sigma}(\tau)'(\bar{\lambda} - \check{\lambda}^*) + \frac{1}{2}\text{tr}\left[\mathbf{\Sigma}_X\left(\bar{\Sigma}(\tau)'(\bar{\Lambda} - \check{\Lambda}^*) + (\bar{\Lambda} - \check{\Lambda}^*)'\bar{\Sigma}(\tau)\right)\right], \\ \bar{\nu}_q(\tau) - \hat{\nu}_q^*(\tau) &\approx \bar{\sigma}_q(\tau)'(\bar{\Lambda} - \check{\Lambda}^*) + \frac{1}{2}\text{tr}\left[\mathbf{\Sigma}_X\left(\bar{\Sigma}_q(\tau)'(\bar{\Lambda} - \check{\Lambda}^*) + (\bar{\Lambda} - \check{\Lambda}^*)'\bar{\Sigma}_q(\tau)\right)\right], \\ \bar{\nu}_k - \hat{\nu}_k^* &\approx \bar{\sigma}_k'(\bar{\lambda} - \check{\lambda}^*) + \frac{1}{2}\text{tr}\left[\mathbf{\Sigma}_X\left(\hat{\Sigma}_k'(\bar{\Lambda} - \check{\Lambda}^*) + (\bar{\Lambda} - \check{\Lambda}^*)'\bar{\Sigma}_k\right)\right], \end{aligned} \tag{5.25}$$

where $\check{\lambda}^* = \check{\lambda}^*(\theta/\mathcal{U})$ and $\check{\Lambda}^* = \check{\Lambda}^*(\theta/\mathcal{U})$.

Remark 11 Suppose an investor has HREZ utility with $(\beta, \gamma, \theta, \psi)$, and we know $(\beta, \gamma + \theta, \psi)$ from their observed consumption, wealth, and portfolio. If they tell us their subjective worst-case long-term expected rate of return on the S&P500 or the difference between their subjective long-term expected rate of return and their subjective long-term worst-case long-term rate of return, then we can calculate $\theta/\mathcal{U}$ from Eq. (5.24) or (5.25). Therefore, we can estimate γ and θ.

6 Numerical Analysis of Worst-Case Premiums

We quantitatively analyze the relationship between the worst-case long-term real premiums and the ratio of ambiguity aversion to uncertainty aversion in the unit EIS case.

6.1 Basic Setting

We assume that the investor plans to invest in the 10-year TIPS $Q_t(10)$ and S&P 500 S_t^1, in addition to the money market account. Thus, the state vector in the quadratic model is two-dimensional, and the portfolio weight and volatility are given by

$$\Phi_t = \begin{pmatrix} \Phi_t^Q(10) \\ \Phi_t^1 \end{pmatrix}, \quad \Sigma(X_t) = \begin{pmatrix} \left(\sigma_q(10) + \Sigma_q(10)X_t\right)' \\ (\sigma_1 + \Sigma_1 X_t)' \end{pmatrix}. \tag{6.1}$$

We use the two-factor QSM model estimated by Kikuchi and Kusuda [15] (for details, see Appendix D). Then, from simultaneous Eqs. (A.5)–(A.6) and Eqs. (A.7)–(A.8), we obtain the following numerical solutions:

$$\bar{\Sigma}_q(10) = \begin{pmatrix} -0.04439 & 0.008236 \\ 0.008236 & -0.005102 \end{pmatrix}, \quad \bar{\sigma}_q(10) = \begin{pmatrix} 0.1231 \\ 0.02852 \end{pmatrix}, \tag{6.2}$$

$$\Sigma_1 = \begin{pmatrix} 0.2063 & 0.09206 \\ 0.09206 & 0.3009 \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 0.2892 \\ 0.8084 \end{pmatrix}. \tag{6.3}$$

We assume $(\beta, \mathcal{U}) = (0.04, 4.0)$. To use the analytical expressions, we assume $\psi = 1.0$. Then, we obtain the following numerical solutions of simultaneous Eqs. (5.5)–(5.7).

$$A = \begin{pmatrix} 0.05343 & -0.02280 \\ -0.02280 & 0.01922 \end{pmatrix}, \quad a = \begin{pmatrix} -0.2483 \\ 0..004519 \end{pmatrix}, \quad a_0 = -2.951. \tag{6.4}$$

6.2 Ratio of Ambiguity Aversion to Uncertainty Aversion and Worst-Case Long-Term Real Premiums

First, we analyze the extent to which the worst-case long-term real premiums are valued lower than the long-term real premiums, depending on $\theta/\mathcal{U}$. Let $\Delta_1^* = \bar{\nu}_1 - \hat{\nu}_1^*$. Table 1 shows the case of the S&P500.

Table 1 Ratio of ambiguity aversion to uncertainty aversion and the worst-case long-term real premiums on the S&P500.

θ	0.0	0.4	0.8	1.2	1.6	2.0	2.4	2.8	3.2	3.6	4.0
γ	4.0	3.6	3.2	2.8	2.4	2.0	1.6	1.2	0.8	0.4	0.0
$\theta/\mathcal{U}$	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
$\bar{\nu}_1$	30.3%	30.3%	30.3%	30.3%	30.3%	30.3%	30.3%	30.3%	30.3%	30.3%	30.3%
$\hat{\nu}_1^*$	30.3%	27.2%	24.1%	21.0%	17.9%	14.7%	11.6%	8.5%	5.4%	2.3%	-.0.8%
Δ_1^*	0.0%	3.1%	6.2%	9.3%	12.5%	15.6%	18.7%	21.8%	24.9%	28.0%	31.1%

For every every increase of 0.1 in $\theta/\mathcal{U}$, the worst-case long-term real premium falls significantly by about 3.1%, a strong indication of the robustness effect. Therefore, if they tell us their subjective worst-case long-term expected rate of return on the S&P500 or the difference between their subjective long-term expected rate of return and their subjective long-term worst-case long-term rate of return, then we can estimate γ and θ with high precision.

Remark 12 Note that the strong robustness effect as shown in Table 1 is also due to the fact that the long-term real premium on the S&P500 is estimated to be as high as 30.3%, which means these results are not so reliable. Although the returns are generally difficult to estimate, the results should be re-examined on the basis of plausible return estimates.

Let $\Delta^*_q(10) = \bar{\nu}_q(10) - \hat{\nu}^*_q(10)$. Table 2 shows the case of the 10-year TIPS.

Table 2 Ratio of ambiguity aversion to uncertainty aversion and the worst-case long-term real premiums on the 10-year TIPS.

θ	0.0	0.4	0.8	1.2	1.6	2.0	2.4	2.8	3.2	3.6	4.0
γ	4.0	3.6	3.2	2.8	2.4	2.0	1.6	1.2	0.8	0.4	0.0
$\theta/\mathcal{U}$	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
$\bar{\nu}_q(10)$	1.8%	1.8%	1.8%	1.8%	1.8%	1.8%	1.8%	1.8%	1.8%	1.8%	1.8%
$\hat{\nu}^*_q(10)$	1.8%	2.0%	2.3%	2.5%	2.8%	3.0%	3.3%	3.5%	3.8%	4.0%	4.3%
$\Delta^*_q(10)$	0.0%	-0.3%	-0.5%	-0.8%	-1.0%	-1.3%	-1.5%	-1.8%	-2.0%	-2.3%	-2.5%

The worst-case long-term real premiums are slightly higher than the long-term real premiums, indicating that the 10-year TIPS has the worst-case probability insurance function.

Remark 13 Campbell and Viceira [6] argue that in, an unambiguous situation, the risk-free security for long-term investors is not the money market account but a long-term inflation-indexed bond with an inflation hedging function. However, the above result shows that for long-term investors with HREZ utility, a long-term inflation-indexed bond has not only an inflation hedging function, but also a worst-case probability insurance function. The money market account, by contrast, has neither an inflation hedging function nor a worst-case probability insurance function. Traditionally, the money market account has been considered the risk-free security, and long-term inflation-indexed bonds have been considered risky securities; these terms are clearly inappropriate.

7 Conclusion

We studied the consumption–investment problem based on HREZ utility with $(\beta, \gamma, \theta, \psi)$ under the QSM model that satisfies the stylized facts in securities markets. We showed that the investor price of uncertainty is a weighted average of the market price of risk and the "investor hedging value of intertemporal uncertainty," and that the worst-case real premium on each security is a weighted average of the real premium and the real discount on the security where the weights are $\gamma/\mathcal{U}$ and $\theta/\mathcal{U}$. Next, we derived analytical expressions of the long-term real premiums, and showed that the worst-case long-term real premium on each security is a weighted average of the long-term real premium and the worst-case long-term real discount on the security.

Subsequently, after deriving a solution to the nonlinear PDE for the unit EIS case ($\psi = 1$) and an approximate solution for the general case ($\psi \neq 1$), we presented the analytical expressions of the optimal consumption and investment, as well as the investor price of uncertainty. Investor price uncertainty is approximated to be an affine function of the state vector in which each coefficient is a function of the ratio of ambiguity aversion to uncertainty aversion. We also derived the analytical expressions of the worst-case real premiums and worst-case long-term real premiums, both of which depend on the ratio of ambiguity aversion to uncertainty aversion. Therefore, we could identify γ and θ from the information related to the worst-case rates of return on securities. Our numerical analysis suggests that we can identify γ and θ with high precision, and that a long-term inflation-indexed bond had not only an inflation hedging function, but also a worst-case probability insurance function.

A QSM Model

A.1 Regularity Conditions on Parameters

1. All the diagonal elements of $\mathcal{K}$ are positive.
2. $\bar{\mathcal{R}}$ is positive-definite.
3. $(\rho_0, \rho, \mathcal{R})$ and $(\delta_{0k}, \delta_k, \Delta_k)$ satisfy

$$\rho_0 = \frac{1}{2}\rho'\mathcal{R}^{-1}\rho, \tag{A.1}$$

$$\delta_{0k} = \frac{1}{2}\delta_k'\Delta_k^{-1}\delta_k. \tag{A.2}$$

Condition 1 ensures that X_t is stationary. Condition 2 ensures that the real risk-free rate has a lower bound. Conditions (A.1) and (A.2) ensure that the nominal risk-free rate and divided are non-negative, respectively.

A.2 Parameters on Return Rates of Securities

1. The default-free bond with time τ to maturity: $(\Sigma(\tau), \sigma(\tau))$ in Eq. (2.12) is a solution to the following system of ODEs.

$$\frac{d\Sigma(\tau)}{d\tau} = (\mathcal{K} + \Lambda)'\Sigma(\tau) + \Sigma(\tau)(\mathcal{K} + \Lambda) - \Sigma(\tau)^2 + \mathcal{R}, \tag{A.3}$$

$$\frac{d\sigma(\tau)}{d\tau} = -(\mathcal{K} + \Lambda - \Sigma(\tau))'\sigma(\tau) - (\Sigma(\tau)\lambda + \rho), \tag{A.4}$$

with $(\Sigma, \sigma)(0) = (0, 0)$.

2. The default-free inflation-indexed bond with time τ to maturity: $(\Sigma(\tau), \sigma(\tau))$ in Eq. (2.13) is a solution to the following system of ODEs.

$$\frac{d\bar{\Sigma}_q(\tau)}{d\tau} = (\mathcal{K} + \bar{\Lambda})'\bar{\Sigma}_q(\tau) + \bar{\Sigma}_q(\tau)(\mathcal{K} + \bar{\Lambda}) - \bar{\Sigma}_q(\tau)^2 + \bar{\mathcal{R}}, \tag{A.5}$$

$$\frac{d\bar{\sigma}_q(\tau)}{d\tau} = -(\mathcal{K} + \bar{\Lambda} - \bar{\Sigma}_q(\tau))'\bar{\sigma}_q(\tau) - (\bar{\Sigma}_q(\tau)\bar{\lambda} + \bar{\rho}), \tag{A.6}$$

with $(\bar{\Sigma}_q, \bar{\sigma}_q)(0) = (0, 0)$.

3. The k-th index and the market portfolio: In Eq. (2.14), Σ_k is a solution to Eq. (A.7) and σ_k is given by Eq. (A.8).

$$0 = (\mathcal{K} + \Lambda)' \Sigma_k + \Sigma_k (\mathcal{K} + \Lambda) - \Sigma_k^2 + \mathcal{R} - \Delta_k, \tag{A.7}$$
$$\sigma_k = (\mathcal{K} + \Lambda - \Sigma_k)'^{-1} (\delta_k - \rho - \Sigma_k \lambda), \tag{A.8}$$

B Derivatives of the Indirect Utility Function

The partial derivatives of J with respect to w are given by

$$\begin{aligned} wJ_w &= (1-\gamma)J, \\ w^2 J_{ww} &= -\gamma(1-\gamma)J. \end{aligned} \tag{B.1}$$

The partial derivatives of J with respect to x are given by

$$\begin{aligned} J_x &= \begin{cases} (1-\gamma)J\dfrac{G_x}{G}, & \text{if} \quad \psi = 1, \\ \dfrac{1-\gamma}{\psi - 1} J \dfrac{G_x}{G}, & \text{if} \quad \psi \neq 1, \end{cases} \qquad wJ_{xw} = \begin{cases} (1-\gamma)^2 J\dfrac{G_x}{G}, & \text{if} \quad \psi = 1, \\ \dfrac{(1-\gamma)^2}{\psi - 1} J \dfrac{G_x}{G}, & \text{if} \quad \psi \neq 1, \end{cases} \\ J_{xx} &= \begin{cases} (1-\gamma)J\left(\dfrac{G_{xx}}{G} - \gamma \dfrac{G_x}{G}\dfrac{G_x'}{G}\right), & \text{if} \quad \psi = 1, \\ \dfrac{1-\gamma}{\psi - 1} J \left(\dfrac{G_{xx}}{G} + \dfrac{2-\gamma-\psi}{\psi - 1}\dfrac{G_x}{G}\dfrac{G_x'}{G}\right), & \text{if} \quad \psi \neq 1. \end{cases} \end{aligned} \tag{B.2}$$

C Proof

C.1 Proof of Lemma 2

For the proof of Lemma 2.2, see Kikuchi and Kusuda [14]. We prove Lemma 2.1, which is enough to prove the case of P_t^T. From Ito's lemma, the rate of return on P_t^T is calculated as

$$\begin{aligned} &\frac{d\bar{P}_t^T}{\bar{P}_t^T} = \frac{dP_t^T}{P_t^T} - \frac{dp_t}{p_t} - \left(\frac{dp_t}{p_t}\right)' \left(\frac{d\bar{P}_t^T}{\bar{P}_t^T}\right) \\ =&\Big(r_t + \big(\sigma(\tau) + \Sigma(\tau)X_t\big)'\lambda_t - \mu^p(X_t) - \big(\bar{\sigma}(\tau) + \bar{\Sigma}(\tau)X_t\big)'\sigma^p(X_t)\Big)dt + \big(\bar{\sigma}(\tau) + \bar{\Sigma}(\tau)X_t\big)dB_t \\ =&\Big(\bar{r}(X_t) + \big(\bar{\sigma}(\tau) + \bar{\Sigma}(\tau)X_t\big)'\big(\lambda(X_t) - \sigma^p(X_t)\big)\Big)dt + \big(\bar{\sigma}(\tau) + \bar{\Sigma}(\tau)X_t\big)dB_t \\ =&\Big(\bar{r}(X_t) + \big(\bar{\sigma}(\tau) + \bar{\Sigma}(\tau)X_t\big)'\bar{\lambda}(X_t)\Big)dt + \big(\bar{\sigma}(\tau) + \bar{\Sigma}(\tau)X_t\big)dB_t. \end{aligned} \tag{C.1}$$

Therefore, we obtain Eq. (2.25).

C.2 Proof of Lemma 3

Substituting $\hat{\xi}^*$ into the HJB Eq. (3.2) yields

$$\begin{aligned} 0 = \sup_{(\hat{c},\hat{\varsigma}) \in \mathbb{R}_+ \times \mathbb{R}^N} &\Bigg[J_t + \begin{pmatrix} w\big(\bar{r}(x) + \hat{\varsigma}' bar\lambda(x)\big) - \hat{c} \\ -\mathcal{K}x \end{pmatrix}' \begin{pmatrix} J_w \\ J_x \end{pmatrix} \\ &+ \frac{1}{2}\operatorname{tr}\left[\begin{pmatrix} w\hat{\varsigma}' \\ I_N \end{pmatrix}\begin{pmatrix} w\hat{\varsigma}' \\ I_N \end{pmatrix}'\begin{pmatrix} J_{ww} & J_{wx} \\ J_{xw} & J_{xx} \end{pmatrix}\right] + f(\hat{c}, J) - \frac{\theta}{2(1-\gamma)J}\left|\begin{pmatrix} w\hat{\varsigma}' \\ I_N \end{pmatrix}'\begin{pmatrix} J_w \\ J_x \end{pmatrix}\right|^2 \Bigg]. \end{aligned} \tag{C.2}$$

C.2.1 Proof of the Unit EIS Case

Assume $\psi = 1$. Substituting $f(c_t, J) = \beta(1-\gamma)J\log c_t - \beta v\log\big((1-\gamma)J\big)$ into the HJB Eq. (C.2) yields

$$\sup_{(\hat{c},\hat{\varsigma})\in\mathbb{R}_+\times\mathbb{R}^N}\left[J_t + \begin{pmatrix} w\left(\bar{r}(x)+\hat{\varsigma}'\bar{\lambda}(x)\right)-\hat{c} \\ -\mathcal{K}x\end{pmatrix}'\begin{pmatrix}J_w\\ J_x\end{pmatrix} + \frac{1}{2}\operatorname{tr}\left[\begin{pmatrix}w\hat{\varsigma}'\\ I_N\end{pmatrix}\begin{pmatrix}w\hat{\varsigma}'\\ I_N\end{pmatrix}'\begin{pmatrix}J_{ww} & J_{wx}\\ J_{xw} & J_{xx}\end{pmatrix}\right]\right.$$
$$\left. - \frac{\theta}{2(1-\gamma)J}\left|\begin{pmatrix}w\hat{\varsigma}'\\ I_N\end{pmatrix}'\begin{pmatrix}J_w\\ J_x\end{pmatrix}\right|^2 + \beta(1-\gamma)J\log\hat{c} - \beta J\log\big((1-\gamma)J\big)\right] = 0. \quad \text{(C.3)}$$

It is evident that the optimal control $(\hat{c}^*, \hat{\varsigma}^*)$ in the HJB Eq. (C.3) satisfies Eqs. (3.8) and (3.9). The consumption-related terms in the HJB Eq. (C.3) are computed as

$$-\hat{c}^* J_w + \beta(1-\gamma)J\log\hat{c}^* - \beta J\log\big((1-\gamma)J\big) = \beta J\big\{(1-\gamma)(\log\hat{c}^*-1) - \log\big((1-\gamma)J\big)\big\}. \quad \text{(C.4)}$$

The investment-related terms in the HJB Eq. (C.3) are computed as

$$wJ_w\bar{\lambda}(x)'\hat{\varsigma}^* + \frac{1}{2}\operatorname{tr}\left[\begin{pmatrix}w(\hat{\varsigma}^*)'\\ I_N\end{pmatrix}\begin{pmatrix}w(\hat{\varsigma}^*)'\\ I_N\end{pmatrix}'\begin{pmatrix}J_{ww} & J_{wx}\\ J_{xw} & J_{xx}\end{pmatrix}\right] - \frac{\theta}{2(1-\gamma)J}\left|\begin{pmatrix}w(\hat{\varsigma}^*)'\\ I_N\end{pmatrix}'\begin{pmatrix}J_w\\ J_x\end{pmatrix}\right|^2$$
$$= \frac{1}{2}\operatorname{tr}[J_{xx}] - \frac{\theta}{2(1-\gamma)J}|J_x|^2 - \frac{|\pi(x)|^2}{2w^2\left(J_{ww} - \dfrac{\theta J_w^2}{(1-\gamma)J}\right)}, \quad \text{(C.5)}$$

where $\pi(x)$ is given by Eq. (3.11).

Substituting the optimal control (3.8) and (3.9) into the HJB Eq. (C.3) and using Eqs. (C.4) and (C.5), we obtain the PDE (3.10).

C.2.2 Proof of the General Case

Assume $\psi \neq 1$. Substituting $f(\hat{c}, J) = \dfrac{\beta}{1-\psi^{-1}}\hat{c}_t^{1-\psi^{-1}}\big((1-\gamma)J\big)^{1-\frac{1-\psi^{-1}}{1-\gamma}} - \dfrac{\beta(1-\gamma)}{1-\psi^{-1}}J$ into the HJB Eq. (C.2) yields

$$\sup_{(\hat{c},\hat{\varsigma})\in\mathbb{R}_+\times\mathbb{R}^N}\left[\begin{pmatrix} w\left(\bar{r}_t+\hat{\varsigma}'\bar{\lambda}_t\right)-\hat{c} \\ -\mathcal{K}x\end{pmatrix}'\begin{pmatrix}J_w\\ J_x\end{pmatrix} + \frac{1}{2}\operatorname{tr}\left[\begin{pmatrix}w\hat{\varsigma}'\\ I_N\end{pmatrix}\begin{pmatrix}w\hat{\varsigma}'\\ I_N\end{pmatrix}'\begin{pmatrix}J_{ww} & J_{wx}\\ J_{xw} & J_{xx}\end{pmatrix}\right]\right.$$
$$\left. - \frac{\theta}{2(1-\gamma)J}\left|\begin{pmatrix}w\hat{\varsigma}'\\ I_N\end{pmatrix}'\begin{pmatrix}J_w\\ J_x\end{pmatrix}\right|^2 + \frac{\beta}{1-\psi^{-1}}\hat{c}^{1-\psi^{-1}}\big((1-\gamma)J\big)^{1-\frac{1-\psi^{-1}}{1-\gamma}} - \frac{\beta(1-\gamma)}{1-\psi^{-1}}J\right] = 0. \quad \text{(C.6)}$$

Optimal control $(\hat{c}^*, \hat{\varsigma}^*)$ in the HJB Eq. (C.6) satisfies Eqs. (3.8) and (3.9). The consumption-related terms in the HJB Eq. (C.6) are computed as

$$-\hat{c}^* J_w + f(\hat{c}^*, J) = \hat{c}^*\left(-J_w + \frac{1}{1-\psi^{-1}}J_w\right) - \frac{\beta(1-\gamma)}{1-\psi^{-1}}J = \frac{1}{\psi-1}\hat{c}^* J_w - \frac{\beta(1-\gamma)}{1-\psi^{-1}}J. \quad \text{(C.7)}$$

The investment-related terms in the HJB Eq. (C.6) are computed as

$$wJ_w\bar{\lambda}(x)'\hat{\varsigma}^* + \frac{1}{2}\operatorname{tr}\left[\begin{pmatrix}w(\hat{\varsigma}^*)'\\ I_N\end{pmatrix}\begin{pmatrix}w(\hat{\varsigma}^*)'\\ I_N\end{pmatrix}'\begin{pmatrix}J_{ww} & J_{wx}\\ J_{xw} & J_{xx}\end{pmatrix}\right] - \frac{\theta}{2(1-\gamma)J}\left|\begin{pmatrix}w(\hat{\varsigma}^*)'\\ I_N\end{pmatrix}'\begin{pmatrix}J_w\\ J_x\end{pmatrix}\right|^2$$
$$= \frac{1}{2}\operatorname{tr}[J_{xx}] - \frac{\theta}{2(1-\gamma)J}|J_x|^2 - \left(w^2 J_{ww} - \frac{\theta(wJ_w)^2}{(1-\gamma)J}\right)^{-1}|\pi(x)|^2, \quad \text{(C.8)}$$

where $\pi(x)$ is given by Eq. (3.11).

Substituting the optimal control (3.8) and (3.9) into the HJB Eq. (C.6) and using Eqs. (C.7) and (C.8), we obtain the PDE (3.10).

C.3 Proof of Proposition 1

C.3.1 Proof of the Unit EIS Case

We assume $\psi \neq 1$. The optimal consumption (3.20) immediately follows from Eq. (3.8). Eq. (3.11) is rewritten as

$$\pi(x) = (\gamma - 1)J\left(\bar{\lambda}(x) - (\gamma + \theta - 1)\frac{G_x(x)}{G(x)}\right). \tag{C.9}$$

Inserting Eqs. (3.15) and the derivatives of J into Eq. (3.9), we obtain the optimal investment (3.21). Substituting the optimal investment (3.20), we obtain Eq. (3.17). From Eq. (C.9) and the derivatives of J, the first to third terms in the PDE (3.10) are calculated as

$$\begin{aligned}
&\frac{1}{2}\operatorname{tr}[J_{xx}] - \frac{\theta}{2(1-\gamma)J}|J_x|^2 - \frac{|\pi(x)|^2}{2w^2\left(J_{ww} - \dfrac{\theta J_w^2}{(1-\gamma)J}\right)} \\
=&\frac{1}{2}J\left((1-\gamma)\operatorname{tr}\left[\frac{G_{xx}}{G} - \gamma\frac{G_x}{G}\frac{G_x'}{G}\right] - (1-\gamma)\theta\left|\frac{G_x}{G}\right|^2 - \frac{\gamma-1}{\gamma+\theta}\left|\bar{\lambda}(x) - (\gamma+\theta-1)\frac{G_x}{G}\right|^2\right) \\
=&\frac{1-\gamma}{2}J\left(\operatorname{tr}\left[\frac{G_{xx}}{G} - \gamma\frac{G_x}{G}\frac{G_x'}{G}\right] - \theta\left|\frac{G_x}{G}\right|^2 + \frac{1}{\gamma+\theta}\left|\bar{\lambda}(x) - (\gamma+\theta-1)\frac{G_x}{G}\right|^2\right) \\
=&\frac{1-\gamma}{2}J\Bigg(\operatorname{tr}\left[\frac{G_{xx}}{G}\right] + \frac{1}{\gamma+\theta}|\bar{\lambda}(x)|^2 - \frac{2(\gamma+\theta-1)}{\gamma+\theta}\bar{\lambda}(x)'\frac{G_x}{G} + \left(-\gamma-\theta+\frac{(\gamma+\theta-1)^2}{\gamma+\theta}\right)\left|\frac{G_x}{G}\right|^2\Bigg) \\
=&(1-\gamma)J\Bigg(\frac{1}{2}\operatorname{tr}\left[\frac{G_{xx}}{G}\right] + \frac{1}{2(\gamma+\theta)}|\bar{\lambda}(x)|^2 - \frac{\gamma+\theta-1}{\gamma+\theta}\bar{\lambda}(x)'\frac{G_x}{G} - \frac{2(\gamma+\theta)-1}{2(\gamma+\theta)}\left|\frac{G_x}{G}\right|^2\Bigg).
\end{aligned} \tag{C.10}$$

The fourth and fifth terms in the PDE (3.10) are computed as

$$\bar{r}(x)wJ_w - (\mathcal{K}x)'J_x = (1-\gamma)J\left(\bar{r}(x) - (\mathcal{K}x)'\frac{G_x}{G}\right). \tag{C.11}$$

The sixth term in the PDE (3.10) is calculated from Eq. (3.20) as

$$\begin{aligned}
\beta J\big\{(1-\gamma)(\log\hat{c}^* - 1) - \log\big((1-\gamma)J\big)\big\} &= \beta(1-\gamma)J\big\{(\log\beta + \log w - 1) - \big(\log w + \log G\big)\big\} \\
&= \beta(1-\gamma)J\big(\log\beta - 1 - \log G\big).
\end{aligned} \tag{C.12}$$

Substituting Eqs. (C.10)–(C.12) into the PDE (3.10) and dividing by $(1-\gamma)J$ yields the PDE (3.23).

C.3.2 Proof of the General Case

We assume $\psi \neq 1$. From Eq. (3.8), the optimal consumption (3.20) is calculated as

$$\hat{c}^* = \beta^\psi\left(\frac{(1-\gamma)J}{w}\right)^{-\psi}\big((1-\gamma)J\big)^{\frac{\gamma\psi-1}{\gamma-1}} = \beta^\psi w^\psi\left(w^{1-\gamma}G^{\frac{1-\gamma}{\psi-1}}\right)^{\frac{\psi-1}{\gamma-1}} = \beta^\psi\frac{w}{G}. \tag{C.13}$$

Eq. (3.11) is rewritten as

$$\pi(x) = (\gamma-1)J\left(\bar{\lambda}(x) + \frac{\gamma+\theta-1}{1-\psi}\frac{G_x(x)}{G(x)}\right). \tag{C.14}$$

Inserting Eq. (3.15) and the derivatives of J into Eq. (3.9), we obtain the optimal investment (3.21). Substituting the optimal investment (3.20), we obtain Eq. (3.17). From Eq. (C.14) and the derivatives of J, the first to third terms in the PDE (3.10) are calculated as

$$\begin{aligned}
&\frac{1}{2}\,\mathrm{tr}\,[J_{xx}] - \frac{\theta}{2(1-\gamma)J}|J_x|^2 - \frac{1}{2}\left(w^2 J_{ww} - \frac{\theta (wJ_w)^2}{(1-\gamma)J}\right)^{-1}|\pi(x)|^2 \\
=&J\left\{\frac{1-\gamma}{2(\psi-1)}\,\mathrm{tr}\left[\frac{G_{xx}}{G} + \frac{2-\gamma-\psi}{\psi-1}\frac{G_x}{G}\frac{G_x'}{G}\right] - \frac{(1-\gamma)\theta}{2(\psi-1)^2}\left|\frac{G_x}{G}\right|^2 \right.\\
&\left. + \frac{1-\gamma}{2(\psi-1)^2(\gamma+\theta)}\left|(\psi-1)\bar{\lambda}(x) - (\gamma+\theta-1)\frac{G_x}{G}\right|^2\right\} \qquad \text{(C.15)}\\
=&\frac{1-\gamma}{\psi-1}J\left\{\frac{1}{2}\,\mathrm{tr}\left[\frac{G_{xx}}{G} + \frac{2-\gamma-\psi}{\psi-1}\frac{G_x}{G}\frac{G_x'}{G}\right] - \frac{\theta}{2(\psi-1)}\left|\frac{G_x}{G}\right|^2\right.\\
&\left. + \frac{1}{2(\psi-1)(\gamma+\theta)}\left|(\psi-1)\bar{\lambda}(x) - (\gamma+\theta-1)\frac{G_x}{G}\right|^2\right\}\\
=&\frac{1-\gamma}{\psi-1}J\left\{\frac{1}{2}\,\mathrm{tr}\left[\frac{G_{xx}}{G}\right] + \frac{\psi-1}{2(\gamma+\theta)}|\bar{\lambda}(x)|^2 - \big(1-(\gamma+\theta)^{-1}\big)\bar{\lambda}(x)'\frac{G_x}{G}\right.\\
&\left. - \frac{1}{2(\psi-1)}\Big(\gamma+\psi-2+\theta - \big(1-(\gamma+\theta)^{-1}\big)(\gamma+\theta-1)\Big)\left|\frac{G_x}{G}\right|^2\right\}\\
=&\frac{1-\gamma}{\psi-1}J\left\{\frac{1}{2}\,\mathrm{tr}\left[\frac{G_{xx}}{G}\right] + \frac{\psi-1}{2}(\gamma+\theta)^{-1}|\bar{\lambda}(x)|^2 - \big(1-(\gamma+\theta)^{-1}\big)\bar{\lambda}(x)'\frac{G_x}{G}\right.\\
&\left. - \frac{1}{2(\psi-1)}\big(\psi-(\gamma+\theta)^{-1}\big)\left|\frac{G_x}{G}\right|^2\right\}.
\end{aligned}$$

The fourth and fifth terms in the PDE (3.10) are computed as

$$\bar{r}(x)wJ_w - (\mathcal{K}x)'J_x = \frac{1-\gamma}{\psi-1}J\left(-\frac{G_\tau}{G} + (\psi-1)\bar{r}(x) - (\mathcal{K}w)'\frac{G_x}{G}\right). \qquad \text{(C.16)}$$

The sixth and seventh terms in the PDE (3.10) are calculated from Eq. (3.20).

$$\frac{1}{\psi-1}\hat{c}^* J_w - \frac{\beta(1-\gamma)}{1-\psi^{-1}}J = \frac{1}{\psi-1}\Big(\beta^\psi\frac{w}{G}\frac{(1-\gamma)J}{w} + \beta(\gamma-1)\psi J\Big) = \frac{1-\gamma}{\psi-1}J\left(\frac{\beta^\psi}{G} - \beta\psi\right). \qquad \text{(C.17)}$$

Substituting Eqs. (C.15)–(C.17) into the PDE (3.10) and dividing by $\frac{1-\gamma}{\psi-1}J$ yields the PDE (3.24).

C.4 Proof of Theorem 1

Inserting $G_x = (a + Ax)G$ into Eq. (3.18) yields

$$\eta^*(x) = -(a + Ax). \qquad \text{(C.18)}$$

By substituting Eq. (C.18) into (3.21), we obtain the optimal investment (5.3). Substituting Eqs. (2.18), (2.19), (5.1) and derivatives of G into the PDE (3.23) and noting $A' = A$ and $x'\big(\mathcal{K} + \big(1-(\gamma+\theta)^{-1}\big)\bar{\Lambda}\big)'Ax = x'A\big(\mathcal{K} + \big(1-(\gamma+\theta)^{-1}\big)\bar{\Lambda}\big)x$, we obtain

$$\begin{aligned}
&\frac{1}{2}\mathrm{tr}[A] + \frac{1}{2}\big((\gamma+\theta)^{-1}-1\big)\big(|a|^2 + 2a'Ax + x'A^2x\big)\\
&\quad - \big\{\big(1-(\gamma+\theta)^{-1}\big)\bar{\lambda} + \big(\mathcal{K} + \big(1-(\gamma+\theta)^{-1}\big)\bar{\Lambda}\big)x\big\}'a - \big(1-(\gamma+\theta)^{-1}\big)\bar{\lambda}'Ax\\
&\quad - \frac{1}{2}x'\left(\mathcal{K} + \big(1-(\gamma+\theta)^{-1}\big)\bar{\Lambda}\right)'Ax - \frac{1}{2}x'A\left(\mathcal{K} + \big(1-(\gamma+\theta)^{-1}\big)\bar{\Lambda}\right)x\\
&\quad - \beta\left(a_0 + a'x + \frac{1}{2}x'Ax\right) + \frac{1}{2}(\gamma+\theta)^{-1}\left(|\bar{\lambda}|^2 + 2\bar{\lambda}'\bar{\Lambda}x + x'\bar{\Lambda}'\bar{\Lambda}x\right)\\
&\quad + \bar{\rho}_0 + \bar{\rho}'x + \frac{1}{2}x'\bar{\mathcal{R}}x + \beta(\log\beta - 1) = 0. \qquad \text{(C.19)}
\end{aligned}$$

As Eq. (C.19) is identical on x, we obtain Eqs. (5.5)–(5.7).

C.5 Proof of Theorem 2

An analytical solution of the PDE (5.13) is expressed as Eq. (5.1). Substituting Eq. (5.1) into Eq. (3.20) yields the optimal consumption (5.14). By substituting Eq. (C.18) into (3.21), we obtain the optimal investment (5.15). Substituting G and its derivatives into the PDE (5.13) yields

$$\begin{aligned}
&\frac{1}{2}\operatorname{tr}[A] + \frac{1}{2}\left(1 - \frac{\psi-(\gamma+\theta)^{-1}}{\psi-1}\right)\left(|a|^2 + 2a'Ax + x'A^2x\right) \\
&\quad - \left\{\left(1-(\gamma+\theta)^{-1}\right)\bar{\lambda} + \left(\mathcal{K} + \left(1-(\gamma+\theta)^{-1}\right)\bar{\Lambda}\right)x\right\}' a - \left(1-(\gamma+\theta)^{-1}\right)\bar{\lambda}'Ax \\
&\quad - \frac{1}{2}x'\left(\mathcal{K} + \left(1-(\gamma+\theta)^{-1}\right)\bar{\Lambda}\right)' Ax - \frac{1}{2}x'A\left(\mathcal{K} + \left(1-(\gamma+\theta)^{-1}\right)\bar{\Lambda}\right)x \\
&- \beta^{\psi} g_1\left(\log g_1 - 1 + a_0 + a'x + \frac{1}{2}x'Ax\right) + \frac{(\psi-1)(\gamma+\theta)^{-1}}{2}\left(|\bar{\lambda}|^2 + 2\bar{\lambda}'\bar{\Lambda}x + x'\bar{\Lambda}'\bar{\Lambda}x\right) \\
&\quad + (\psi-1)\left(\bar{\rho}_0 + \bar{\rho}'x + \frac{1}{2}x'\bar{\mathcal{R}}x\right) - \beta\psi = 0. \qquad \text{(C.20)}
\end{aligned}$$

Therefore, we obtain Eqs. (5.17)–(5.19).

D Estimation Results by Kikuchi and Kusuda [15]

Kikuchi and Kusuda [15] estimate the two-factor QSM model on 262 month-end data from January 1999 to October 2020 observed in US security markets. The time-series data used for the estimation are the 6-month, 5-year, and 10-year treasury spot rates, 5-year and 10-year TIPS real spot rates, and S&P500. To reduce the estimation burden, they assume Λ is a lower triangular matrix, Σ_p is a diagonal matrix, and $\mathcal{I} = 0$. Their estimation results are

$$dX_t = -\mathcal{K}X_t dt + I_2 dB_t = -\begin{pmatrix} 0.08049 & 0 \\ -0.005062 & 0.1066 \end{pmatrix} X_t\, dt + I_2 dB_t,$$

$$\lambda_t = \lambda + \Lambda X_t = \begin{pmatrix} -0.03605 \\ 0.3500 \end{pmatrix} + \begin{pmatrix} 0.03388 & 0 \\ 0.1296 & 0.01928 \end{pmatrix} X_t,$$

$$r_t = \rho_0 + \rho' X_t + \frac{1}{2}X_t'\mathcal{R}X_t = 0.06964 + \begin{pmatrix} -0.02395 \\ 0.06803 \end{pmatrix}' X_t + \frac{1}{2}X_t' \begin{pmatrix} 0.004145 & -0.001112 \\ -0.001112 & 0.0004623 \end{pmatrix} X_t,$$

$$i_t = \iota_0 + \iota' X_t = 0.02445 + \begin{pmatrix} 0.006011 \\ 0.01658 \end{pmatrix}' X_t,$$

$$\sigma_t^p = \sigma_p + \Sigma_p X_t = \begin{pmatrix} 0.09348 \\ 0.01345 \end{pmatrix} + \begin{pmatrix} 0.05115 & 0 \\ 0 & 0.02112 \end{pmatrix} X_t,$$

$$\frac{D_t}{S_t} = \delta_0 + \delta' X_t + \frac{1}{2}X_t'\Delta X_t = 0.01482 + \begin{pmatrix} 0.0004861 \\ 0.001912 \end{pmatrix}' X_t + \frac{1}{2}X_t' \begin{pmatrix} 2.917\times 10^{-4} & 4.569\times 10^{-6} \\ 4.569\times 10^{-6} & 1.258\times 10^{-4} \end{pmatrix} X_t.$$

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Statements and Declarations

Competing Interests

The authors declare no competing interests relevant to the contents of this article.

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