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GENERATING ITERATIVE SCHEMES TO LOCATE COMMON FIXED POINTS OF NONLINEAR MAPPINGS USING SHRINKING PROJECTION METHODS

Atsumasa Kondo

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The Institute for Economic and Business Research
Faculty of Economics
SHIGA UNIVERSITY

1-1-1 BANBA, HIKONE,
SHIGA 522-8522, JAPAN

GENERATING ITERATIVE SCHEMES TO LOCATE COMMON FIXED POINTS OF NONLINEAR MAPPINGS USING SHRINKING PROJECTION METHODS

ATSUMASA KONDO

Abstract. We introduce iterative scheme generating methods (ISGMs) to find common fixed points of nonlinear mappings through shrinking projection methods, leading to strong convergence theorems. These findings extend a recent study by Kondo [Math. Ann. (2024) (published online)], which only demonstrates weak convergence. Although ISGMs combined with shrinking projection methods were explored in a prior study [A. Kondo, Carpathian J. Math. **40** (2024), 819–840], that work depended on mean-valued sequence properties. This study develops ISGMs without relying on mean-valued sequences, yielding infinitely many strong convergence theorems.

1. Introduction

Let H be a real Hilbert space equipped with an inner product $\langle\cdot, \cdot\rangle$ and the induced norm $\|\cdot\|$. Let S be a mapping from C into H, where C is a nonempty subset of H. Denote by $F(S) = \{x \in C : Sx = x\}$ the set of all fixed points of S. A mapping $S : C \to H$ is termed *nonexpansive* if $\|Sx - Sy\| \le \|x - y\|$ for all $x, y \in C$. Due to its broad applicability, researchers have widely studied constructing a sequence approximating a fixed point of a nonexpansive mapping. Recently, Kondo [20] proved the following theorem:

Theorem 1.1 ([20]). *Let C be a nonempty, closed, and convex subset of H and let $S, T : C \to C$ be quasi-nonexpansive* (2.5) *and demiclosed mappings* (2.6)*. Suppose that $F(S) \cap F(T) \neq \emptyset$. Let $P_{F(S)\cap F(T)}$ be the metric projection from H onto $F(S) \cap F(T)$. Let $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ be sequences of real numbers in the interval $[0, 1]$ such that $a_n + b_n + c_n = 1$ for all $n \in \mathbb{N}$, $\underline{\lim}_{n\to\infty} a_n b_n > 0$, and $\underline{\lim}_{n\to\infty} a_n c_n > 0$, where $\mathbb{N} = \{1, 2, \cdots\}$. Define a sequence $\{x_n\}$ in C as follows:*

$$x_1 \in C : \text{ given},$$
$$x_{n+1} = a_n y_n + b_n S z_n + c_n T w_n$$

for all $n \in \mathbb{N}$, where $\{y_n\}$, $\{z_n\}$, and $\{w_n\}$ are sequences in C that satisfy the following conditions:

$$\|y_n - q\| \le \|x_n - q\|, \ \|z_n - q\| \le \|x_n - q\|, \text{ and } \|w_n - q\| \le \|x_n - q\| \tag{1.1}$$

for all $q \in F(S) \cap F(T)$ and $n \in \mathbb{N}$ and

$$x_n - y_n \to 0, \ x_n - z_n \to 0, \text{ and } x_n - w_n \to 0. \tag{1.2}$$

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Then, the sequence $\{x_n\}$ *converges weakly to a point* $\widehat{x} \in F(S) \cap F(T)$*, where* $\widehat{x} \equiv \lim_{n\to\infty} P_{F(S)\cap F(T)} x_n$.

The class of mappings considered in Theorem 1.1 includes nonexpansive mappings, as well as more general types of mappings; see the Appendix of Kondo [20] for further details. In Theorem 1.1, the sequence $\{x_n\}$ is defined with given sequences $\{y_n\}$, $\{z_n\}$, and $\{w_n\}$, and the required conditions for these sequences $\{y_n\}$, $\{z_n\}$, and $\{w_n\}$ are explicitly stated in (1.1) and (1.2). Consequently, many iterative schemes can be derived from this theorem. For example, consider the following iterative scheme:

$$
\begin{aligned}
z_n &= \lambda_n x_n + (1-\lambda_n) T x_n, \\
y_n &= \mu_n z_n + (1-\mu_n) S z_n, \\
x_{n+1} &= a_n y_n + b_n S y_n + c_n T y_n,
\end{aligned}
\tag{1.3}
$$

where an initial point $x_1 \in C$ is provided. The coefficients of the convex combinations λ_n and μ_n satisfy $\lambda_n \to 1$ and $\mu_n \to 1$, respectively; see Corollary 4.4 in Kondo [20]. It can be confirmed that y_n in (1.3) satisfies the conditions $\|y_n - q\| \leq \|x_n - q\|$ and $x_n - y_n \to 0$. Therefore, according to Theorem 1.1, the sequence $\{x_n\}$, defined by rule (1.3), converges weakly to a common fixed point of S and T. The iterative scheme in (1.3) is a three-step scheme; see Noor [27], Dashputre and Diwan [5], and Phuengrattana and Suantai [28]. By setting $\lambda_n = 1$ for all $n \in \mathbb{N}$ in (1.3), the two-step Ishikawa iterative scheme is obtained [10]. For more on the Ishikawa method, see Xu [32], Tan and Xu [31], and Berinde [2, 3]. This method, which generates infinitely many iterative schemes, is referred to as an *iterative scheme generating method (ISGM)*; see Kondo [17, 18, 19].

In 2003, Nakajo and Takahashi [26] proposed the CQ method and proved a strong convergence theorem for finding a fixed point of nonexpansive mapping. In 2006, Martinez-Yanes and Xu [24] extended the CQ method and proved the following theorem:

Theorem 1.2 ([24]). *Let* C *be a nonempty, closed, and convex subset of* H*. Let* $S : C \to C$ *be a nonexpansive mapping such that* $F(S) \neq \emptyset$*. Let* $\{\alpha_n\}$ *and* $\{\beta_n\}$ *be sequences of real numbers in the interval* $[0,1]$ *such that* $0 \leq \alpha_n \leq \alpha < 1$ *and* $\beta_n \to 1$*, where* $\alpha \in [0,1)$*. Define a sequence* $\{x_n\}$ *in* C *as follows:*

$$
\begin{aligned}
x_1 &= x \in C : \ \textit{given}, \\
y_n &= \beta_n x_n + (1-\beta_n) S x_n, \\
X_n &= \alpha_n x_n + (1-\alpha_n) S y_n, \\
C_n &= \{h \in C : \|X_n - h\|^2 \leq \|x_n - h\|^2 \\
&\quad + (1-\alpha_n)\left(\|y_n\|^2 - \|x_n\|^2 - 2\langle y_n - x_n,\ h\rangle\right)\}, \\
Q_n &= \{h \in C : \langle x - x_n,\ x_n - h\rangle \geq 0\} \\
x_{n+1} &= P_{C_n \cap Q_n} x
\end{aligned}
$$

for all $n \in \mathbb{N}$*. Then,* $\{x_n\}$ *converges strongly to an element* $\widehat{x}$ *in* $F(S)$*, where* $\widehat{x} = P_{F(S)} x$.

In 2008, Takahashi *et al.* [30] also developed the CQ method and established a strong convergence theorem utilizing metric projections on shrinking sets $\{C_n\}$, where $\{C_n\}$ satisfies the condition $C_n \subset C_{n-1} \subset \cdots \subset C_1 = C$. This approach is referred to as *the shrinking projection method*. For other related works, see Kimura and Nakajo [12], as well as Ibaraki and Saejung [9]. In 2023, Kondo [18] applied ISGMs with mean-valued sequences, such as

$$x_{n+1} = a_n y_n + b_n \frac{1}{n} \sum_{k=0}^{n-1} S^k z_n + c_n \frac{1}{n} \sum_{k=0}^{n-1} T^k w_n, \tag{1.4}$$

to the CQ and shrinking projection methods and obtained various strong convergence theorems. For iterative methods involving mean-valued sequences, refer to Shimizu and Takahashi [29], Atsushiba and Takahashi [1], and Kondo [23].

This study establishes ISGMs using the shrinking projection method incorporating the Martinez-Yanes and Xu method. Through these efforts, we derive numerous strong convergence theorems. These results enhance Theorem 1.1, which previously only provided weak convergence. Although ISGMs utilizing the shrinking projection method and the Martinez-Yanes and Xu method were also explored in Kondo [19], that study's findings depended on the properties of mean-valued sequences. In contrast, as demonstrated in Theorem 1.1, this study develops ISGMs without relying on mean-valued sequences. We explore a broader class of mappings, including nonexpansive mappings as specific instances.

The structure of this article is as follows: Section 2 provides essential preliminary information. In Section 3, we establish an ISGM using the shrinking projection method. Section 4 integrates the Martinez-Yanes and Xu iterative scheme with the shrinking projection method, further extending the ISGM. Section 5 offers a comparison between the present study and previous work [19], highlighting the unique contributions of this research. Finally, Section 6 presents an iterative scheme derived from the result in Section 3 to demonstrate the broad applicability of the main findings of this study.

2. Preliminaries

This section introduces preliminary concepts and results. Let H represent a real Hilbert space. For $x, y, z \in H$ and $a, b, c \in \mathbb{R}$ such that $a + b + c = 1$, the following holds:

$$\begin{aligned} &\|ax + by + cz\|^2 \\ &= a\|x\|^2 + b\|y\|^2 + c\|z\|^2 - ab\|x-y\|^2 - bc\|y-z\|^2 - ca\|z-x\|^2; \end{aligned} \tag{2.1}$$

see Maruyama *et al.* [25] and Zegeye and Shahzad [33]. In (2.1), the conditions $a, b, c \in [0, 1]$ are not strictly required. However, if $a, b, c \in [0, 1]$, the following inequality holds:

$$\|ax + by + cz\|^2 \leq a\|x\|^2 + b\|y\|^2 + c\|z\|^2. \tag{2.2}$$

Let F be a nonempty, closed, and convex subset of H. We define P_F as the *metric projection* from H onto F, meaning $\|u - P_F u\| \leq \|u - h\|$ for all $u \in H$ and $h \in F$. The metric projection P_F is nonexpansive and satisfies

$$\|u - P_F u\|^2 + \|P_F u - h\|^2 \leq \|u - h\|^2 \tag{2.3}$$

for all $u \in H$ and $h \in F$. Let C be a nonempty, closed, and convex subset of H with $x \in H$ and $d \in \mathbb{R}$. Then, a subset D of C defined by

$$D = \{h \in C : 0 \leq \langle x,\ h\rangle + d\} \tag{2.4}$$

is also closed and convex; refer to Martinez-Yanes and Xu [24].

A mapping $S : C \to H$ with $F(S) \neq \emptyset$ is called *quasi-nonexpansive* if

$$\|Sz - q\| \leq \|z - q\| \quad \text{for all } z \in C \text{ and } q \in F(S). \tag{2.5}$$

The set of all fixed points of a quasi-nonexpansive mapping is closed and convex; see Itoh and Takahashi [11]. Any nonexpansive mapping that has a fixed point is quasi-nonexpansive. Denote by $x_n \to x$ and $x_n \rightharpoonup x$ the strong and weak convergence to a point x, respectively. Let C be a nonempty, closed, and convex subset of H and let $S : C \to C$ with $F(S) \neq \emptyset$. The mapping S is referred to as *demiclosed* if

$$z_n - Sz_n \to 0 \text{ and } z_n \rightharpoonup u \Longrightarrow u \in F(S), \tag{2.6}$$

where $\{z_n\}$ represents a sequence in C. Note that it is often said that $I - S$ is demiclosed when (2.6) holds, where I denotes the identity mapping. The quasi-nonexpansive and demiclosed mappings include nonexpansive mappings and a broader category of mappings; for further details, see Appendix in Kondo [20].

In what follows, we assume the existence of a common fixed point for nonlinear mappings. The following is a simple version of classical results demonstrated in 1965 by Browder [4], Göhde [6], and Kirk [13] in certain classes of Banach spaces:

Theorem 2.1 ([4, 6, 13])**.** *Let C be a nonempty, closed, convex, and bounded subset of H. Let $S, T : C \to C$ be nonexpansive mappings such that $ST = TS$. Then, $F(S) \cap F(T)$ is not empty.*

For further developments on common fixed point theorems, see Hojo [7], Kondo [14, 16], and the articles cited therein.

3. Takahashi–Takeuchi–Kubota method

This section presents a strong convergence theorem approximating a common fixed point of two nonlinear mappings. We employ the shrinking projection method by Takahashi *et al.* [30]. To achieve this, we can relax a required assumption for mappings in comparison to (2.6). Let C be a nonempty, closed, and convex subset of a real Hilbert space H and let $S : C \to C$ with $F(S) \neq \emptyset$. Let $\{z_n\}$ be a sequence in C. Following Kondo [15], consider the following condition:

$$z_n - Sz_n \to 0 \text{ and } z_n \to u \Longrightarrow u \in F(S). \tag{3.1}$$

Demiclosed mappings (2.6) or continuous mappings satisfy the condition (3.1), and thus, broad classes of mappings, including nonexpansive mappings, satisfy this condition (3.1). In the remainder of this article, we will focus on quasi-nonexpansive mappings (2.5) that satisfy the condition (3.1). In the main theorems presented below, we assume the following setting:

($\star$) Let C be a nonempty, closed, and convex subset of a real Hilbert space H. Let $S, T : C \to C$ be quasi-nonexpansive mappings (2.5) that satisfy the condition (3.1). Suppose that $F(S) \cap F(T) \neq \emptyset$. Let $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ be sequences of real numbers in the interval $[0, 1]$ such that $a_n + b_n + c_n = 1$ for all $n \in \mathbb{N}$, $\underline{\lim}_{n\to\infty} a_n b_n > 0$, and $\underline{\lim}_{n\to\infty} a_n c_n > 0$. Let $\{u_n\}$ be a sequence in H such that $u_n \to u\,(\in H)$.

Then, we can prove the following theorem:

Theorem 3.1. *Assume the setting* $(\star)$. *Define a sequence* $\{x_n\}$ *in* C *as follows:*

$$\begin{aligned} x_1 &= x \in C: \ \textit{given}, \\ C_1 &= C, \\ X_n &= a_n y_n + b_n S z_n + c_n T w_n, \\ C_{n+1} &= \{h \in C_n : \|X_n - h\| \leq \|x_n - h\|\}, \\ x_{n+1} &= P_{C_{n+1}} u_{n+1} \end{aligned}$$

for all $n \in \mathbb{N}$, *where* $\{y_n\}$, $\{z_n\}$, *and* $\{w_n\}$ *are sequences in* C *that satisfy the following conditions:*

$$\|y_n - q\| \leq \|x_n - q\|, \quad \|z_n - q\| \leq \|x_n - q\|, \quad \|w_n - q\| \leq \|x_n - q\| \tag{3.2}$$

for all $q \in F(S) \cap F(T)$ *and* $n \in \mathbb{N}$ *and*

$$x_n - y_n \to 0, \quad x_n - z_n \to 0, \quad x_n - w_n \to 0, \tag{3.3}$$

as $n \to \infty$. *Then,* $\{x_n\}$ *converges strongly to an element* $\widehat{u}$ *in* $F(S) \cap F(T)$, *where* $\widehat{u} = P_{F(S)\cap F(T)} u$.

Proof. First, we verify the following: (a) C_n is closed and convex, (b) $F(S) \cap F(T) \subset C_n$ for all $n \in \mathbb{N}$, and (c) the sequences $\{x_n\}, \{y_n\}, \{z_n\}, \{w_n\}, \{X_n\}$ in C, and $\{C_n\}$ are properly defined. We start with the case $n = 1$.

(i) Given $x_1 \in C_1 \,(= C)$, we can select y_1, z_1, and $w_1 \in C$ that satisfy (3.2) and (3.3) for $n = 1$. For instance, if we set $y_1 = z_1 = w_1 = x_1$, then both (3.2) and (3.3) hold. With $x_1, y_1, z_1, w_1 \in C$, X_1 and C_2 are defined as follows:

$$\begin{aligned} X_1 &= a_1 y_1 + b_1 S z_1 + c_1 T w_1 \in C \ \text{ and} \\ C_2 &= \{h \in C_1 : \|X_1 - h\| \leq \|x_1 - h\|\}. \end{aligned}$$

As C_1 is closed and convex, C_2 is also closed and convex. Observe that $F(S) \cap F(T) \subset C_2$. Choose $q \in F(S) \cap F(T) \,(\subset C_1)$ arbitrarily. From (2.5) and (3.2), we have

$$\begin{aligned} \|X_1 - q\| &= \|a_1 y_1 + b_1 S z_1 + c_1 T w_1 - q\| \\ &\leq a_1 \|y_1 - q\| + b_1 \|S z_1 - q\| + c_1 \|T w_1 - q\| \\ &\leq a_1 \|y_1 - q\| + b_1 \|z_1 - q\| + c_1 \|w_1 - q\| \\ &\leq a_1 \|x_1 - q\| + b_1 \|x_1 - q\| + c_1 \|x_1 - q\| \\ &= \|x_1 - q\|. \end{aligned}$$

This indicates that $q \in C_2$. Therefore, $F(S) \cap F(T) \subset C_2$ as asserted. As $F(S) \cap F(T) \neq \emptyset$ is assumed, it follows that $C_2 \neq \emptyset$. Consequently, the metric projection P_{C_2} exists and $x_2 = P_{C_2} u_2$ is defined.

(ii) Given that $x_2 \in C_2$ (with $C_2 \subset C_1 = C$), we can choose y_2, z_2, and $w_2 \in C$ under the conditions provided in (3.2) and (3.3) for $n = 2$. Then, X_2 and C_3 are defined accordingly:

$$\begin{aligned} X_2 &= a_2 y_2 + b_2 S z_2 + c_2 T w_2 \in C \ \text{ and} \\ C_3 &= \{h \in C_2 : \|X_2 - h\| \leq \|x_2 - h\|\}. \end{aligned}$$

By the same reasoning as in case (i), we can confirm that C_3 is closed and convex and that $F(S) \cap F(T) \subset C_3$. As $F(S) \cap F(T) \neq \emptyset$ is supposed, we conclude that $C_3 \neq \emptyset$. Consequently, the metric projection P_{C_3} exists and $x_3 = P_{C_3} u_3$ is defined.

By repeating the same argument, we can establish (a), (b), and (c) as stated.

Define $\overline{u}_n = P_{C_n} u \,(\in C_n)$. The sequence $\{\overline{u}_n\}$ is contained in C, as $C_n \subset C_{n-1} \subset \cdots \subset C_1 = C$. From $\overline{u}_n = P_{C_n} u$ and $F(S) \cap F(T) \subset C_n$, it follows that

$$\|u - \overline{u}_n\| \leq \|u - q\| \tag{3.4}$$

for all $q \in F(S) \cap F(T)$ and $n \in \mathbb{N}$. This implies that the sequence $\{\overline{u}_n\}$ is bounded.

From $\overline{u}_n = P_{C_n} u$ and $\overline{u}_{n+1} = P_{C_{n+1}} u \in C_{n+1} \subset C_n$, it follows that

$$\|u - \overline{u}_n\| \leq \|u - \overline{u}_{n+1}\|$$

for all $n \in \mathbb{N}$. In other words, the sequence $\{\|u - \overline{u}_n\|\} \,(\subset \mathbb{R})$ is monotone increasing. As $\{\overline{u}_n\}$ is bounded, $\{\|u - \overline{u}_n\|\}$ is also bounded. Therefore, the sequence $\{\|u - \overline{u}_n\|\}$ of real numbers converges.

We now show that $\{\overline{u}_n\}$ converges in C, meaning that there exists $\overline{u} \in C$ such that

$$\overline{u}_n \to \overline{u}. \tag{3.5}$$

Choose $m, n \in \mathbb{N}$ such that $m \geq n$. As the sequence of sets $\{C_n\}$ is shrinking, it follows from $m \geq n$ that $C_m \subset C_n$. Given that $\overline{u}_n = P_{C_n} u$ and $\overline{u}_m = P_{C_m} u \in C_m \subset C_n$, we obtain from (2.3) that

$$\|u - \overline{u}_n\|^2 + \|\overline{u}_n - \overline{u}_m\|^2 \leq \|u - \overline{u}_m\|^2 .$$

As $\{\|u - \overline{u}_n\|\}$ converges, it holds that $\overline{u}_n - \overline{u}_m \to 0$ as $m, n \to \infty$, meaning that $\{\overline{u}_n\}$ is a Cauchy sequence in C. As C is closed in the real Hilbert space H, it is complete. Thus, there exists $\overline{u} \in C$ such that $\overline{u}_n \to \overline{u}$ as claimed.

We now prove that

$$x_n \to \overline{u}. \tag{3.6}$$

As the metric projection P_{C_n} is nonexpansive, it follows from the assumption $u_n \to u$ and (3.5) that

$$\begin{aligned} \|x_n - \overline{u}\| &\leq \|x_n - \overline{u}_n\| + \|\overline{u}_n - \overline{u}\| \\ &= \|P_{C_n} u_n - P_{C_n} u\| + \|\overline{u}_n - \overline{u}\| \\ &\leq \|u_n - u\| + \|\overline{u}_n - \overline{u}\| \to 0 \end{aligned}$$

as claimed. As $\{x_n\}$ converges, it is bounded. Moreover, from (3.3), we obtain

$$z_n \to \overline{u} \quad \text{and} \quad w_n \to \overline{u}. \tag{3.7}$$

Next, observe that

$$x_n - X_n \to 0. \tag{3.8}$$

Indeed, as $\{x_n\}$ is convergent, it holds that $x_n - x_{n+1} \to 0$ as $n \to \infty$. Given that $x_{n+1} = P_{C_{n+1}} u_{n+1} \in C_{n+1}$, it follows that $\|X_n - x_{n+1}\| \leq \|x_n - x_{n+1}\| \to 0$. Therefore, we have

$$\|x_n - X_n\| \leq \|x_n - x_{n+1}\| + \|x_{n+1} - X_n\| \to 0$$

as claimed. As $\{x_n\}$ is bounded, $\{X_n\}$ is also bounded, according to (3.8).

We show that

$$y_n - Sz_n \to 0 \quad \text{and} \quad y_n - Tw_n \to 0. \tag{3.9}$$

Select $q \in F(S) \cap F(T)$ arbitrarily. As S and T are quasi-nonexpansive (2.5), from (2.1) and (3.2), the following holds:

$$\begin{aligned}
& \|X_n - q\|^2 \\
= \ & \|a_n (y_n - q) + b_n (Sz_n - q) + c_n (Tw_n - q)\|^2 \\
= \ & a_n \|y_n - q\|^2 + b_n \|Sz_n - q\|^2 + c_n \|Tw_n - q\|^2 \\
& -a_n b_n \|y_n - Sz_n\|^2 - b_n c_n \|Sz_n - Tw_n\|^2 - c_n a_n \|Tw_n - y_n\|^2 \\
\\
\leq \ & a_n \|y_n - q\|^2 + b_n \|z_n - q\|^2 + c_n \|w_n - q\|^2 \\
& -a_n b_n \|y_n - Sz_n\|^2 - b_n c_n \|Sz_n - Tw_n\|^2 - c_n a_n \|Tw_n - y_n\|^2 \\
\leq \ & a_n \|x_n - q\|^2 + b_n \|x_n - q\|^2 + c_n \|x_n - q\|^2 \\
& -a_n b_n \|y_n - Sz_n\|^2 - b_n c_n \|Sz_n - Tw_n\|^2 - c_n a_n \|Tw_n - y_n\|^2 \\
= \ & \|x_n - q\|^2 \\
& -a_n b_n \|y_n - Sz_n\|^2 - b_n c_n \|Sz_n - Tw_n\|^2 - c_n a_n \|Tw_n - y_n\|^2 .
\end{aligned}$$

As $b_n c_n \|Sz_n - Tw_n\|^2 \geq 0$, we have

$$\begin{aligned}
& a_n b_n \|y_n - Sz_n\|^2 + a_n c_n \|y_n - Tw_n\|^2 \\
& \leq \|x_n - q\|^2 - \|X_n - q\|^2 \\
& \leq (\|x_n - q\| + \|X_n - q\|) \left| \|x_n - q\| - \|X_n - q\| \right| \\
& \leq (\|x_n - q\| + \|X_n - q\|) \|x_n - X_n\| .
\end{aligned}$$

As both $\{x_n\}$ and $\{X_n\}$ are bounded, and from (3.8), along with the assumptions $\underline{\lim}_{n\to\infty} a_n b_n > 0$ and $\underline{\lim}_{n\to\infty} a_n c_n > 0$, we obtain (3.9) as asserted.

Next, we aim to demonstrate that

$$z_n - Sz_n \to 0 \quad \text{and} \quad w_n - Tw_n \to 0. \tag{3.10}$$

Using (3.3) and (3.9), we have

$$\|z_n - Sz_n\| \leq \|z_n - x_n\| + \|x_n - y_n\| + \|y_n - Sz_n\| \to 0.$$

The second part of (3.10) can be verified in a similar way. As S and T satisfy the condition (3.1), according to (3.7) and (3.10), it holds that $\overline{u} \in F(S) \cap F(T)$.

Finally, we verify that

$$\overline{u} \left(= \lim_{n\to\infty} \overline{u}_n = \lim_{n\to\infty} x_n \right) = \widehat{u} \left(= P_{F(S)\cap F(T)} u \right) .$$

As $\overline{u} \in F(S) \cap F(T)$ and $\widehat{u} = P_{F(S)\cap F(T)} u$, it is sufficient to prove that $\|u - \overline{u}\| \leq \|u - \widehat{u}\|$. From $\widehat{u} \in F(S) \cap F(T)$ and (3.4), it holds that $\|u - \overline{u}_n\| \leq \|u - \widehat{u}\|$. From (3.5), we obtain $\|u - \overline{u}\| \leq \|u - \widehat{u}\|$. Therefore, $\overline{u} = \widehat{u}$. Given (3.6), we can conclude that $x_n \to \widehat{u} \, (= \overline{u})$. This completes the proof. □

For the convergent sequence $\{u_n\} \, (\subset H)$ in Theorem 3.1, see Theorem 4.1 and 5.2 in Hojo *et al.* [8]. Setting $y_n = z_n = w_n = x_n$ in Theorem 3.1 yields the following corollary, which corresponds to Theorem 4.1 in Kondo [15]:

Corollary 3.1 ([15]). *Assume the setting* $(\star)$. *Define a sequence* $\{x_n\}$ *in* C *as follows:*

$$\begin{aligned} x_1 &= x \in C: \text{ given}, \\ C_1 &= C, \\ X_n &= a_n x_n + b_n S x_n + c_n T x_n, \\ C_{n+1} &= \{h \in C_n : \|X_n - h\| \le \|x_n - h\|\}, \\ x_{n+1} &= P_{C_{n+1}} u_{n+1} \end{aligned}$$

for all $n \in \mathbb{N}$. *Then,* $\{x_n\}$ *converges strongly to an element* $\widehat{u}$ *in* $F(S) \cap F(T)$, *where* $\widehat{u} = P_{F(S)\cap F(T)} u$.

4. Martinez-Yanes and Xu Method

In this section, we incorporate the method of Martinez-Yanes and Xu [24] into the usual shrinking projection method presented in the previous section (Section 3). We prove the following theorem:

Theorem 4.1. *Assume the setting* $(\star)$. *Define a sequence* $\{x_n\}$ *in* C *as follows:*

$$\begin{aligned} x_1 &= x \in C: \text{ given}, \\ C_1 &= C, \\ X_n &= a_n y_n + b_n S z_n + c_n T w_n, \\ C_{n+1} &= \left\{h \in C_n : \|X_n - h\|^2 \le a_n \|y_n - h\|^2 + b_n \|z_n - h\|^2 + c_n \|w_n - h\|^2\right\}, \\ x_{n+1} &= P_{C_{n+1}} u_{n+1} \end{aligned}$$

for all $n \in \mathbb{N}$, *where* $\{y_n\}$, $\{z_n\}$, *and* $\{w_n\}$ *are sequences in* C *that satisfy the following conditions:*

$$\|y_n - q\| \le \|x_n - q\|, \quad \|z_n - q\| \le \|x_n - q\|, \quad \|w_n - q\| \le \|x_n - q\| \tag{4.1}$$

for all $q \in F(S) \cap F(T)$ *and* $n \in \mathbb{N}$ *and*

$$x_n - y_n \to 0, \quad x_n - z_n \to 0, \quad x_n - w_n \to 0 \tag{4.2}$$

as $n \to \infty$. *Then,* $\{x_n\}$ *converges strongly to an element* $\widehat{u}$ *in* $F(S) \cap F(T)$, *where* $\widehat{u} = P_{F(S)\cap F(T)} u$.

Remark 4.1. *See the definition of* C_{n+1}. *It follows that*

$$\|X_n - h\|^2 \le a_n \|y_n - h\|^2 + b_n \|z_n - h\|^2 + c_n \|w_n - h\|^2$$

$$\begin{aligned} &(4.3) \quad \iff \quad 0 \le a_n \|y_n\|^2 + b_n \|z_n\|^2 + c_n \|w_n\|^2 - \|X_n\|^2 \\ &\qquad\qquad\qquad -2 \langle a y_n + b z_n + c w_n - X_n, \ h \rangle \\ &(4.4) \quad \iff \quad \|X_n - h\|^2 \le \|y_n - h\|^2 + b_n \left(\|z_n\|^2 - \|y_n\|^2 - 2 \langle z_n - y_n, \ h\rangle\right) \\ &\qquad\qquad\qquad +c_n \left(\|w_n\|^2 - \|y_n\|^2 - 2 \langle w_n - y_n, \ h\rangle\right). \end{aligned}$$

From (4.4), *Theorem* 4.1 *corresponds to the Martinez-Yanes and Xu type; see Theorem* 1.2 *in Section* 1. *Suppose that* $X_n, y_n, z_n, w_n \in C$ *and* $a_n, b_n, c_n \in \mathbb{R}$ *are given. From* (2.4) *and* (4.3), *the set* C_{n+1} *is closed and convex if* C_n *is closed and convex.*

Proof. At the outset, observe that (a) C_n is closed and convex, (b) $F(S) \cap F(T) \subset C_n$ for all $n \in \mathbb{N}$, and (c) the sequences $\{x_n\}$, $\{y_n\}$, $\{z_n\}$, $\{w_n\}$, $\{X_n\}\,(\subset C)$, and $\{C_n\}$ are defined properly. We begin with the case of $n = 1$.

(i) Given $x_1 \in C_1\,(= C)$, we can select y_1, z_1, and $w_1 \in C$ to satisfy (4.1) and (4.2) for $n = 1$. For instance, by letting $y_1 = z_1 = w_1 = x_1$, those conditions are fulfilled. With $x_1, y_1, z_1, w_1 \in C$, X_1 and C_2 are defined as follows:

$$X_1 = a_1 y_1 + b_1 S z_1 + c_1 T w_1 \in C \quad \text{and}$$
$$C_2 = \left\{ h \in C_1 : \|X_1 - h\|^2 \leq a_1 \|y_1 - h\|^2 + b_1 \|z_1 - h\|^2 + c_1 \|w_1 - h\|^2 \right\}.$$

From (2.4) and (4.3), C_2 is closed and convex as $C_1\,(= C)$ is closed and convex. We verify that $F(S) \cap F(T) \subset C_2$. Let $q \in F(S) \cap F(T)\,(\subset C_1)$. As S and T are quasi-nonexpansive (2.5), from (2.2), it follows that

$$\begin{aligned}
\|X_1 - q\|^2 &= \|a_1 y_1 + b_1 S z_1 + c_1 T w_1 - q\|^2 \\
&= \|a_1 (y_1 - q) + b_1 (S z_1 - q) + c_1 (T w_1 - q)\|^2 \\
&\leq a_1 \|y_1 - q\|^2 + b_1 \|S z_1 - q\|^2 + c_1 \|T w_1 - q\|^2 \\
&\leq a_1 \|y_1 - q\|^2 + b_1 \|z_1 - q\|^2 + c_1 \|w_1 - q\|^2,
\end{aligned}$$

which implies that $q \in C_2$. Therefore, $F(S) \cap F(T) \subset C_2$ as asserted. Given the assumption that $F(S) \cap F(T) \neq \emptyset$, C_2 is nonempty. Thus, the metric projection P_{C_2} exists and $x_2 = P_{C_2} u_2$ is defined.

(ii) Given $x_2 \in C_2\,(\subset C_1 = C)$, we can select y_2, z_2, and $w_2 \in C$ such that (4.1) and (4.2) are satisfied for $n = 2$. With these elements, X_2 and C_3 are defined as follows:

$$X_2 = a_2 y_2 + b_2 S z_2 + c_2 T w_2 \in C \quad \text{and}$$
$$C_3 = \left\{ h \in C_2 : \|X_2 - h\|^2 \leq a_2 \|y_2 - h\|^2 + b_2 \|z_2 - h\|^2 + c_2 \|w_2 - h\|^2 \right\}.$$

Using the same argument as in case (i), we can demonstrate that C_3 is closed and convex and that $F(S) \cap F(T) \subset C_3$. From the assumption $F(S) \cap F(T) \neq \emptyset$, we conclude that $C_3 \neq \emptyset$. Consequently, the metric projection P_{C_3} exists and $x_3 = P_{C_3} u_3$ is defined.

Repeating the same analysis guarantees that (a), (b), and (c) are true.

Define $\overline{u}_n = P_{C_n} u \in C_n$. As $C_n \subset C_{n-1} \subset \cdots \subset C_1 = C$, $\{\overline{u}_n\}$ is a sequence contained in C. We claim that

$$\|u - \overline{u}_n\| \leq \|u - q\| \tag{4.5}$$

for all $q \in F(S) \cap F(T)$ and $n \in \mathbb{N}$, this follows from the definition $\overline{u}_n = P_{C_n} u$ and the fact that $q \in F(S) \cap F(T) \subset C_n$. Thus, we can conclude from (4.5) that $\{\overline{u}_n\}$ is bounded.

Note that

$$\|u - \overline{u}_n\| \leq \|u - \overline{u}_{n+1}\| \tag{4.6}$$

for all $n \in \mathbb{N}$. As $\overline{u}_n = P_{C_n} u$ and $\overline{u}_{n+1} = P_{C_{n+1}} u \in C_{n+1} \subset C_n$, the inequality (4.6) follows. This implies that $\{\|u - \overline{u}_n\|\}$ is monotone increasing. As $\{\|u - \overline{u}_n\|\}$ is bounded, it is convergent.

We now prove that $\{\overline{u}_n\}$ converges in C; that is, there exists $\overline{u} \in C$ such that

$$\overline{u}_n \to \overline{u}. \tag{4.7}$$

Let $m, n \in \mathbb{N}$ with $m \geq n$. As $\overline{u}_n = P_{C_n} u$ and $\overline{u}_m = P_{C_m} u \in C_m \subset C_n$, from (2.3), it holds that

$$\|u - \overline{u}_n\|^2 + \|\overline{u}_n - \overline{u}_m\|^2 \leq \|u - \overline{u}_m\|^2 .$$

As $\{\|u - \overline{u}_n\|\}$ is convergent, it follows that $\overline{u}_n - \overline{u}_m \to 0$ as $m, n \to \infty$. Thus, $\{\overline{u}_n\}$ is a Cauchy sequence in C. As C is complete, there exists $\overline{u} \in C$ such that $\overline{u}_n \to \overline{u}$ as claimed.

Our next aim is to demonstrate that $\{x_n\}$ has the same limit point, that is,

$$x_n \to \overline{u}. \tag{4.8}$$

As the metric projection is nonexpansive, using (4.7) and the hypothesis $u_n \to u$, we obtain

$$\begin{aligned} \|x_n - \overline{u}\| &\leq \|x_n - \overline{u}_n\| + \|\overline{u}_n - \overline{u}\| \\ &= \|P_{C_n} u_n - P_{C_n} u\| + \|\overline{u}_n - \overline{u}\| \\ &\leq \|u_n - u\| + \|\overline{u}_n - \overline{u}\| \to 0 \end{aligned}$$

as n tends to infinity. This shows that (4.8) holds true as claimed. Consequently, $\{x_n\}$ is bounded. From (4.1), $\{y_n\}$, $\{z_n\}$, and $\{w_n\}$ are also bounded. Furthermore, according to (4.2) and (4.8), we have

$$z_n \to \overline{u} \quad \text{and} \quad w_n \to \overline{u}. \tag{4.9}$$

As $\{x_n\}$ converges, it holds that

$$x_n - x_{n+1} \to 0. \tag{4.10}$$

Next, let us show that

$$X_n - x_{n+1} \to 0. \tag{4.11}$$

As $x_{n+1} = P_{C_{n+1}} u_{n+1} \in C_{n+1}$, it follows from the definition of C_{n+1} that

$$\begin{aligned} & \|X_n - x_{n+1}\|^2 \\ \leq\ & a_n \|y_n - x_{n+1}\|^2 + b_n \|z_n - x_{n+1}\|^2 + c_n \|w_n - x_{n+1}\|^2 \\ \leq\ & a_n \left(\|y_n - x_n\| + \|x_n - x_{n+1}\|\right)^2 + b_n \left(\|z_n - x_n\| + \|x_n - x_{n+1}\|\right)^2 \\ & + c_n \left(\|w_n - x_n\| + \|x_n - x_{n+1}\|\right)^2 . \end{aligned} \tag{4.12}$$

From (4.2) and (4.10), we can conclude that $X_n - x_{n+1} \to 0$ as stated. From (4.10) and (4.11), we have $x_n - X_n \to 0$. As $\{x_n\}$ is bounded, $\{X_n\}$ is also bounded.

Observe that

$$y_n - S z_n \to 0 \quad \text{and} \quad y_n - T w_n \to 0. \tag{4.13}$$

Choose $q \in F(S) \cap F(T)$ arbitrarily. Using (2.1), (2.5), and (4.1) yields

$$\begin{aligned} & \|X_n - q\|^2 \\ =\ & \|a_n (y_n - q) + b_n (S z_n - q) + c_n (T w_n - q)\|^2 \\ =\ & a_n \|y_n - q\|^2 + b_n \|S z_n - q\|^2 + c_n \|T w_n - q\|^2 \\ & - a_n b_n \|y_n - S z_n\|^2 - b_n c_n \|S z_n - T w_n\|^2 - c_n a_n \|T w_n - y_n\|^2 \\ \leq\ & a_n \|y_n - q\|^2 + b_n \|z_n - q\|^2 + c_n \|w_n - q\|^2 \\ & - a_n b_n \|y_n - S z_n\|^2 - b_n c_n \|S z_n - T w_n\|^2 - c_n a_n \|T w_n - y_n\|^2 \end{aligned}$$

$$\begin{aligned} &\leq a_n \|x_n - q\|^2 + b_n \|x_n - q\|^2 + c_n \|x_n - q\|^2 \\ &\quad - a_n b_n \|xy_n - Sz_n\|^2 - b_n c_n \|Sz_n - Tw_n\|^2 - c_n a_n \|Tw_n - y_n\|^2 \\ &= \|x_n - q\|^2 \\ &\quad - a_n b_n \|y_n - Sz_n\|^2 - b_n c_n \|Sz_n - Tw_n\|^2 - c_n a_n \|Tw_n - y_n\|^2 . \end{aligned}$$

As $b_n c_n \|Sz_n - Tw_n\|^2 \geq 0$, we have

$$\begin{aligned} &a_n b_n \|y_n - Sz_n\|^2 + a_n c_n \|y_n - Tw_n\|^2 \\ &\leq \|x_n - q\|^2 - \|X_n - q\|^2 \\ &\leq (\|x_n - q\| + \|X_n - q\|) \left| \|x_n - q\| - \|X_n - q\| \right| \\ &\leq (\|x_n - q\| + \|X_n - q\|) \|x_n - X_n\| . \end{aligned}$$

Recall that $\{x_n\}$ and $\{X_n\}$ are bounded and $x_n - X_n \to 0$. Thus, we obtain $y_n - Sz_n \to 0$ and $y_n - Tw_n \to 0$ as asserted.

From (4.2) and (4.13), it follows that

$$z_n - Sz_n \to 0 \quad \text{and} \quad w_n - Tw_n \to 0. \tag{4.14}$$

As S and T satisfy the condition (3.1), from (4.9) and (4.14), we obtain $\overline{u} \in F(S) \cap F(T)$.

Our objective is to demonstrate that $x_n \to \widehat{u}$. From (4.8), it is sufficient to show that

$$\overline{u} \left(= \lim_{n\to\infty} \overline{u}_n = \lim_{n\to\infty} x_n \right) = \widehat{u} \left(= P_{F(S)\cap F(T)} u \right) .$$

Applying (4.5) for $q = \widehat{u} \in F(S) \cap F(T)$, we have $\|u - \overline{u}_n\| \leq \|u - \widehat{u}\|$ for all $n \in \mathbb{N}$. From (4.7), it holds that $\|u - \overline{u}\| \leq \|u - \widehat{u}\|$. As $\overline{u} \in F(S) \cap F(T)$ and $\widehat{u} = P_{F(S)\cap F(T)} u$, we obtain $\overline{u} = \widehat{u}$. This concludes the proof. □

Setting $y_n = z_n = w_n = x_n$ in Theorem 4.1, we again obtain Corollary 3.1.

5. Remarks

This section provides brief notes regarding the main theorems of this study in comparison with previous results. Let $S : C \to C$ with $F(S) \neq \emptyset$ and let $\{z_n\}$ be a bounded sequence in C, where C is a nonempty, closed, and convex subset of a real Hilbert space H. Define $Z_n = \frac{1}{n} \sum_{k=0}^{n-1} S^k z_n \,(\in C)$. We call a mapping $S : C \to C$ *mean-demiclosed* if

$$Z_{n_j} \rightharpoonup u \text{ (weak convergence)} \implies u \in F(S) . \tag{5.1}$$

According to Kondo and Takahashi [21], a nonexpansive mapping is mean-demiclosed; see also Claim 1 in Kondo [18] or Proposition 2.1 in Kondo [19]. Furthermore, consider the following condition:

$$Z_{n_j} \to u \text{ (strong convergence)} \implies u \in F(S) . \tag{5.2}$$

A mean-demiclosed mapping (5.1) satisfies the condition (5.2) and therefore, broad classes of mappings, including nonexpansive mappings, satisfy this condition (5.2); see Appendix in Kondo [18]. Consider the following setting:

($\star\star$) Let C be a nonempty, closed, and convex subset of a real Hilbert space H. Let $S, T : C \to C$ be quasi-nonexpansive mappings (2.5) that satisfy the condition (5.2). Suppose that $F(S) \cap F(T) \neq \emptyset$. Let $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ be sequences of real numbers in the interval $[0, 1]$ such that $a_n + b_n + c_n = 1$ for all $n \in \mathbb{N}$,

$\underline{\lim}_{n\to\infty} a_n b_n > 0$, and $\underline{\lim}_{n\to\infty} a_n c_n > 0$. Let $\{u_n\}$ be a sequence in H such that $u_n \to u\,(\in H)$.

The only difference between the settings $(\star)$ and $(\star\star)$ is with regard to the mapping conditions (3.1) and (5.2). The following two theorems are contained in Kondo [19]:

Theorem 5.1 ([19])**.** *Assume the setting $(\star\star)$. Define a sequence $\{x_n\}$ in C as follows:*

$$\begin{aligned} x_1 &= x \in C: \text{ given},\\ C_1 &= C,\\ X_n &= a_n y_n + b_n \frac{1}{n}\sum_{l=0}^{n-1} S^l z_n + c_n \frac{1}{n}\sum_{l=0}^{n-1} T^l w_n,\\ C_{n+1} &= \{h \in C_n : \|X_n - h\| \le \|x_n - h\|\},\\ x_{n+1} &= P_{C_{n+1}} u_{n+1} \end{aligned}$$

for all $n \in \mathbb{N}$, where $\{y_n\}$, $\{z_n\}$, and $\{w_n\}$ are sequences in C that satisfy the following conditions:

$$\|y_n - q\| \le \|x_n - q\|, \quad \|z_n - q\| \le \|x_n - q\|, \quad \|w_n - q\| \le \|x_n - q\|$$

for all $q \in F(S) \cap F(T)$ and $n \in \mathbb{N}$ and

$$x_n - y_n \to 0 \tag{5.3}$$

as $n \to \infty$. Then, $\{x_n\}$ converges strongly to an element $\widehat{u} \in F(S) \cap F(T)$, where $\widehat{u} = P_{F(S)\cap F(T)} u$.

Theorem 5.2 ([19])**.** *Assume the setting $(\star\star)$. Define a sequence $\{x_n\}$ in C as follows:*

$$\begin{aligned} x_1 &= x \in C: \text{ given},\\ C_1 &= C,\\ X_n &= a_n y_n + b_n \frac{1}{n}\sum_{l=0}^{n-1} S^l z_n + c_n \frac{1}{n}\sum_{l=0}^{n-1} T^l w_n,\\ C_{n+1} &= \left\{h \in C_n : \|X_n - h\|^2 \le a_n \|y_n - h\|^2 + b_n \|z_n - h\|^2 + c_n \|w_n - h\|^2\right\},\\ x_{n+1} &= P_{C_{n+1}} u_{n+1} \end{aligned}$$

for all $n \in \mathbb{N}$, where $\{y_n\}$, $\{z_n\}$, and $\{w_n\}$ are sequences in C that satisfy the following conditions:

$$\|y_n - q\| \le \|x_n - q\|, \quad \|z_n - q\| \le \|x_n - q\|, \quad \|w_n - q\| \le \|x_n - q\|$$

for all $q \in F(S) \cap F(T)$ and $n \in \mathbb{N}$ and

$$x_n - y_n \to 0, \quad x_n - z_n \to 0, \quad x_n - w_n \to 0 \tag{5.4}$$

as $n \to \infty$. Then, $\{x_n\}$ converges strongly to an element $\widehat{u} \in F(S) \cap F(T)$, where $\widehat{u} = P_{F(S)\cap F(T)} u$.

First, we compare Theorem 3.1 with 5.1. As can be seen in (3.3) and (5.3), Theorem 3.1 requires additional assumptions $x_n - z_n \to 0$ and $x_n - w_n \to 0$, although it can be established without relying on mean-valued sequences. Furthermore, the conditions for mappings S and T in Theorem 3.1 differ from those in Theorem 5.1.

For quasi-nonexpansive mappings with (3.1), see Appendix in Kondo [20] and for quasi-nonexpansive mappings with (5.2), see Appendix in Kondo [18].

Next, we compare Theorem 4.1 with 5.2. In these two theorems, the required conditions on the sequences $\{y_n\}$, $\{z_n\}$, and $\{w_n\}$ are the same as those in Theorem 3.1. For this point, see Remark 5.2 in Kondo [19]. In other words, Theorem 4.1 can be proved without using mean-valued sequences and without any additional conditions on the sequences $\{y_n\}$, $\{z_n\}$, and $\{w_n\}$. However, the conditions on the mappings differ, as in the cases of Theorems 3.1 and 5.1.

6. Corollary

In this section, we provide a convergence result deduced from Theorem 3.1 to demonstrate the applicability and effectiveness of the main theorems of this study.

Corollary 6.1. *Assume the setting* $(\star)$. *Let* $\{\lambda_n\}$, $\{\mu_n\}$, *and* $\{\nu_n\}$ *be sequences of real numbers in the interval* $[0,1]$ *such that* $\lambda_n + \mu_n + \nu_n = 1$ *for all* $n \in \mathbb{N}$ *and* $\lambda_n \to 1$. *Define a sequence* $\{x_n\}$ *in* C *as follows:*

$$
\begin{aligned}
x_1 &= x \in C: \text{ given}, \\
C_1 &= C, \\
y_n &= \lambda_n x_n + \mu_n S x_n + \nu_n T x_n, \\
X_n &= a_n y_n + b_n S y_n + c_n T y_n, \\
C_{n+1} &= \{h \in C_n : \|X_n - h\| \leq \|x_n - h\|\}, \\
x_{n+1} &= P_{C_{n+1}} u_{n+1}
\end{aligned} \tag{6.1}
$$

for all $n \in \mathbb{N}$. *Then, the sequence* $\{x_n\}$ *converges strongly to an element* $\widehat{u} \in F(S) \cap F(T)$, *where* $\widehat{u} = P_{F(S)\cap F(T)} u$.

Proof. From Theorem 3.1, it is sufficient to demonstrate that

$$
\begin{aligned}
&\|y_n - q\| \leq \|x_n - q\| \quad \text{for all } q \in F(S) \cap F(T) \text{ and } n \in \mathbb{N}, \\
&x_n - y_n \to 0 \quad \text{as } n \to \infty.
\end{aligned}
$$

Before that, we shall verify that (a) C_n is closed and convex, (b) $F(S) \cap F(T) \subset C_n$ for all $n \in \mathbb{N}$, and (c) the sequences $\{x_n\}, \{z_n\}, \{y_n\}, \{X_n\}$, and $\{C_n\}$ are properly defined. These parts can be shown in a similar manner to the proof of Theorem 3.1 and thus, we omit them here.

Observe that $\|y_n - q\| \leq \|x_n - q\|$. Let $q \in F(S) \cap F(T)$ and $n \in \mathbb{N}$. As S and T are quasi-nonexpansive (2.5), the following holds:

$$
\begin{aligned}
\|y_n - q\| &= \|\lambda_n (x_n - q) + \mu_n (S x_n - q) + \nu_n (T x_n - q)\| \\
&\leq \lambda_n \|x_n - q\| + \mu_n \|S x_n - q\| + \nu_n \|T x_n - q\| \\
&\leq \|x_n - q\|
\end{aligned}
$$

as asserted.

Define $\overline{u}_n = P_{C_n} u \in C_n$. In a similar manner to the proof of Theorem 3.1, we can show that there exists $\overline{u} \in C$ such that $\overline{u}_n \to \overline{u}$ and $x_n \to \overline{u}$. As $\{x_n\}$ is convergent, it is bounded. Moreover, as S and T are quasi-nonexpansive, $\{S x_n\}$ and $\{T x_n\}$ are also bounded. In fact, for $q \in F(S) \cap F(T)$, it holds that

$$
\begin{aligned}
\|S x_n\| &\leq \|S x_n - q\| + \|q\| \\
&\leq \|x_n - q\| + \|q\|
\end{aligned}
$$

for all $n \in \mathbb{N}$. As $\{x_n\}$ is bounded, $\{Sx_n\}$ is also bounded. Similarly, we can verify that $\{Tx_n\}$ is also bounded as claimed.

We show that $x_n - y_n \to 0$. As $\lambda_n \to 1$, it follows that $\mu_n \to 0$ and $\nu_n \to 0$. Thus, we have

$$\begin{aligned}\|x_n - y_n\| &= \|x_n - (\lambda_n x_n + \mu_n Sx_n + \nu_n Tx_n)\| \\ &= \|(1-\lambda_n)\, x_n - \mu_n Sx_n - \nu_n Tx_n\| \\ &\leq (1-\lambda_n)\,\|x_n\| + \mu_n \|Sx_n\| + \nu_n \|Tx_n\| \to 0\end{aligned}$$

as asserted. The desired result follows from Theorem 3.1. □

Furthermore, the iterative scheme (6.1) can be replaced by

$$\begin{aligned}y_n &= \lambda_n x_n + \mu_n Sx_n + \nu_n Tx_n + \xi_n T^2 x_n, \\ X_n &= a_n y_n + b_n Sy_n + c_n Ty_n, \\ C_{n+1} &= \{h \in C_n : \|X_n - h\| \leq \|x_n - h\|\}, \\ x_{n+1} &= P_{C_{n+1}} u_{n+1},\end{aligned} \tag{6.2}$$

where $x_1 = x \in C$ is given and $C_1 = C$. Furthermore, in (6.2), the parameters $\lambda_n, \mu_n, \nu_n, \xi_n \in [0,1]$ are required to satisfy $\lambda_n \to 1$. This type of iterative scheme, which includes the term $T^2 x_n$, was utilized by Maruyama *et al.* [25] to address more general class of mappings than nonexpansive mappings; see also Kondo [15, 22] and the articles cited therein. Hence, it is effective for nonexpansive mappings and the class of mappings discussed in this study.

Apart from the iterative schemes (6.1) in Corollary 6.1 and (6.2), infinitely many iterative methods to locate common fixed points of nonlinear mappings are generated from Theorems 3.1 and 4.1; see also Kondo [17, 18, 19, 20]. As a final remark, we can prove similar results using the CQ method by Nakajo and Takahashi [26].

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(Atsumasa Kondo) Department of Economics, Shiga University, Banba 1-1-1, Hikone, Shiga 522-0069, Japan
E-mail address: a-kondo@biwako.shiga-u.ac.jp