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Robust Control and CAPMs under a Quadratic Model with Inflation-Deflation Risk*

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Abstract

This study assumes homothetic robust Epstein-Zin utility and analyzes the consumption-investment problem and CAPMs under a quadratic security market model in which interest rates, the market price of risk, the variances and covariances of asset returns, and inflation rates are stochastic. First, we demonstrate that homothetic robust Epstein-Zin utility is interpreted as homothetic stochastic differential utility. Then, we show that robust investors determine the "worst-case probability" and the optimal consumption-investment. We clarify the theoretical structures of robust control. We derive a robust version of the two-factor CAPM and show that the CAPM can contribute to solving both the equity premium puzzle and the risk-free rate puzzle. Furthermore, we derive an approximate testable ICAPM.

Keywords Homothetic robust utility, Stochastic differential utility, Consumption-investment problem, CAPM, Stochastic volatility, Stochastic inflation

JEL classification C61, D81, G11, G12

1 Introduction

When studying consumption-investment problems and capital asset pricing models (CAPMs), it is important to address three key issues. The first issue is to incorporate into security market models the stylized facts that interest rates, the market price of risk, the variances and covariances of asset returns, and inflation rates are stochastic and mean-reverting. In particular,

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it is essential to incorporate stochastic inflation, because inflation-deflation risk causes asset prices to fluctuate directly and indirectly through monetary policies implemented to control inflation-deflation risk. It should be noted that the indirect effect has become much stronger than the direct effect because of the large-scale easing and tightening of monetary policy since the global financial crisis. The second issue is to consider the equity premium puzzle (Mehra and Prescott (1985)) and the risk-free rate puzzle. The third issue is to assume a utility that accounts for Knightian uncertainty, as recognized during the global financial crisis.

Regarding the first issue, Batbold, Kikuchi, and Kusuda (2022) examine the consumption-investment problem for long-term investors with constant relative risk aversion (CRRA) utility under a quadratic security market model that satisfies the above stylized facts, including inflation-deflation risk. The class of quadratic models, which is a generalization of the affine models presented by Duffie and Kan (1996), has been independently developed by Ahn, Dittmar, and Gallant (2002) and Leippold and Wu (2002).[1] Batbold *et al.* (2022) derive an optimal portfolio decomposed into myopic demand, intertemporal hedging demand, and inflation-deflation hedging demand, and show that all three types of demand are nonlinear functions of the state vector. Their numerical analysis presents the nonlinearity and significance of market timing effects. Such nonlinearity is attributed to the stochastic variances and covariances of asset returns, whereas such significance is attributed to inflation-deflation hedging demand in addition to myopic demand.

For the second issue, Mehra and Prescott (1985) show that the CAPM beased on the CRRA utility, which has a good property called homotheticity[2], cannot explain the high market price of risk observed in the securities market, that is, the equity premium puzzle. Given that CRRA utility does not separate relative risk aversion and elasticity of intertemporal substitution (EIS), Epstein and Zin (1989) propose Epstein-Zin utility that generalizes CRRA utility and separates these properties while retaining homotheticity. However, Weil (1989) shows that the CAPM based on the Epstein-Zin utility cannot explain either the high market price of risk or the low-risk free rate (*i.e.*, the risk-free rate puzzle) observed in the securities market.

For the thrid issue, Anderson, Hansen, and Sargent (2003) and Hansen

[1] Quadratic security market models are employed in studies of empirical analysis (Leippold and Wu (2007), Kim and Singleton (2012), and Kikuchi (2024)), security pricing (Chen, Filipović, and Poor (2004), Boyarchenko and Levendorskii (2007), and Filipović, Gourier, and Mancini (2016)), and optimal consumption-investment (Batbold *et al.* (2022), Kikuchi and Kusuda (2024a), and Kikuchi and Kusuda (2024b)), because the quadratic models are sufficiently general to incorporate the stylized facts in the securities markets mentioned above, while still being analytically tractable.

[2] A utility function U is homothetic if for any consumption plan c and $\tilde{c}$, and any scalar $\alpha > 0$, $U(\alpha\tilde{c}) \geq U(\alpha c) \Leftrightarrow U(\tilde{c}) \geq U(c)$.

and Sargent (2001) propose robust utility. Investors with robust utility regard the "base probability" as the most likely probability; however, they also consider other probabilities because the true probability is unknown. Given that robust utility lacks homotheticity, Maenhout (2004) proposes homothetic robust utility—characterized by relative risk aversion and relative ambiguity aversion—that represents the investor's degree of distrust of the base probability. Kikuchi and Kusuda (2024a) study the consumption-investment problem for long-term investors with homothetic robust utility under the quadratic security market model of Batbold *et al.* (2022). Because a nonlinear term appears in the partial differential equation (PDE) for the indirect utility function, Kikuchi and Kusuda (2024a) use a linear approximation method to derive an approximate optimal portfolio. Their numerical analysis confirms the nonlinearity and significance of market timing effects. Homothetic robust utility can be interpreted as homothetic robust CRRA utility in the sense that homothetic robust utility converges to CRRA utility as ambiguity aversion approaches zero. CRRA utility does not separate relative risk aversion and elasticity of intertemporal substitution (EIS). Epstein-Zin utility (Epstein and Zin (1989)) generalizes CRRA utility and separates these properties while retaining homotheticity. Maenhout (2004) also introduces homothetic robust Epstein-Zin (HREZ) utility to derive the CAPM. However, he does not show the properties of HREZ utility. Skiadas (2003) proves that robust utility is a stochastic differential utility (SDU), proposed by Duffie and Epstein (1992a). They demonstrate that SDU exhibits various desirable properties. Skiadas (2003) generalizes the result to a more general robust utility, including HREZ utility. However, this proof contains an error in the calculation process, as presented in Section 3.

Investors with homothetic robust utility first determine the "conditional worst-case probability" of minimizing utility for a given consumption and investment and then determine the optimal consumption and investment that maximize the utility under the conditional worst-case probability. These optimal consumption-investment decisions implicitly determine the worst-case probability. Homothetic robust utility is used in robust portfolio studies, such as Skiadas (2003), Maenhout (2006), Liu (2010), Branger, Larsen, and Munk (2013), Munk and Rubtsov (2014), Yi, Viens, Law, and Li (2015), Batbold, Kikuchi, and Kusuda (2019), and Kikuchi and Kusuda (2024a)[3]. These studies have done little to elucidate the theoretical structures of the i) budget constraint equation and market price of risk under the conditional worst-case probability and the worst-case probability; and ii) the two types of CAPMs under the base probability and the worst-case probability.

We assume HREZ utility and consider the consumption-investment problem and CAPMs under the quadratic security market model of Kikuchi and

[3]With the exception of Kikuchi and Kusuda (2024a), these ignore some stylized facts in the securities market.

Kusuda (2024a). When we derive CAPMs, we assume that the investor is the representative agent in equilibrium. As it is natural to assume that the representative agent is an infinitely lived investor, we consider the infinite-time consumption-investment problem for investors with HREZ utility. The purpose of this study is to show the properties of HREZ utility and present a theoretical analysis of the consumption-investment problem and CAPMs under the quadratic security market model. The main results of this study are summarized as follows. First, we prove that HREZ utility is SDU under certain integrability conditions by modifying the incorrect proof in Skiadas (2003). Then, following Duffie and Epstein (1992a), we show that HREZ utility is continuous, consistent, strictly increasing, risk averse, and homothetic.

Second, we consider the consumption-investment problem for the infinite-lived investor with HREZ utility and derive the conditional worst-case probability for a given consumption and "investment," which is the product of the volatility matrix of risky securities and the vector of the fractions of wealth invested in those risky securities. Comparing the budget constraint under the conditional worst-case probability with the budget constraint under the base probability, we find that the volatility of wealth is invariant, whereas the market price of risk in the return on wealth is replaced by the "investor price of risk under the conditional worst-case probability" discounted from the market price of risk. Given that the discount from the market price of risk is permanent, this implies that investors with HREZ utility assume long-term stagnation rather than increased volatility as the worst-case scenario.

Third, we derive the optimal consumption and investment, both of which depend on the unknown function that comprises the indirect utility function. The unknown function is a solution to a nonlinear PDE. We show that the optimal investment is the weighted average of the market price of risk and the "investor hedging value of intertemporal uncertainty." The weights are the relative risk tolerance and one minus the relative risk tolerance, respectively. We also show that the "investor price of risk under the worst-case probability" is the weighted average of the market price of risk and the investor hedging value of intertemporal uncertainty. The weights are the ratio of risk aversion to uncertainty aversion, and ambiguity aversion to uncertainty aversion, respectively. In addition, the optimal investment and investor price of risk under the worst-case probability are both the weighted averages of the market price of risk and the investor hedging value of intertemporal uncertainty.

Fourth, we derive robust versions of the intertemporal CAPM (ICAPM) based on Epstein-Zin utility and of the two-factor CAPM (Duffie and Epstein (1992b)). We show that the equilibrium market price of risk under the worst-case probability is consistent with the equilibrium market price of risk based on Epstein-Zin utility. Furthermore, we demonstrate that i) the equilibrium market price of risk based on HREZ utility is higher than that based on Epstein-Zin utility, and ii) the equilibrium risk-free rate based on HREZ utility is lower than that based on Epstein-Zin utility. Therefore, the robust CAPMs can contribute to solving both the equity premium puzzle and risk-free rate puzzle. Finally, we derive the exact solution of the nonlinear PDE for the unit EIS case and a loglinear approximate solution of the PDE for the general case. We then present the approximate optimal portfolio and approximate testable ICAPM based on a loglinear approximate solution.

The remainder of this paper is organized as follows. In Section 2, we review the quadratic security market model. In Section 3, we introduce HREZ utility and show its properties. In Section 4, we theoretically analyze the optimal robust consumption and investment problem. In Section 5, we derive robust CAPMs. In Section 6, we derive the optimal portfolio for the unit EIS case and an approximate optimal portfolio for the general case. In Section 7, future research directions are discussed.

2 Quadratic Security Market Model

In this section, we review the quadratic security market model and stochastic differential equations (SDEs) of no-arbitrage security price processes, based on the work of Kikuchi and Kusuda (2024a).

2.1 Quadratic Security Market Model

We consider frictionless US markets over the period $[0, \infty)$. Investors' common subjective probability and information structure are modeled by a complete filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathrm{P})$, where $\mathbb{F} = (\mathcal{F}_t)_{t \in [0,\infty)}$ is the natural filtration generated by an N-dimensional standard Brownian motion B_t. We denote the expectation operator under P by E and the conditional expectation operator given $\mathcal{F}_t$ by E_t.

There are markets for a consumption commodity and securities at every date $t \in [0, \infty)$, and the consumer price index p_t is observed. The traded securities are the instantaneously nominal risk-free security called the money market account and a continuum of zero-coupon bonds and zero-coupon inflation-indexed bonds whose maturity dates are $(t, t + \tau^*]$, where τ^* is the longest time to maturity of the bonds. Each zero-coupon bond has a 1 US dollar payoff at maturity, and each zero-coupon inflation-indexed bond has a p_T US dollar payoff at maturity T. Moreover, K-types of stocks or indices are traded.

At every date t, P_t, P_t^T, Q_t^T, and S_t^k denote the USD prices of the money market account, zero-coupon bond with maturity date T, zero-coupon inflation-indexed bond with maturity date T, and k-th index, respectively. Let A' and I denote the transpose of A and $N \times N$ identity matrix, respectively.

We assume the following quadratic security market model introduced by Kikuchi and Kusuda (2024a).

Assumption 1. *Let* $(\rho_0, \iota_0, \delta_{0k}, \sigma_{0k})$, $(\lambda, \rho, \iota, \sigma_p, \delta_k, \sigma_k)$, *and* $(\mathcal{R}, \Delta_k, \Sigma_k)$ *denote scalers, N-dimensional vectors, and* $N \times N$ *positive-definite symmetric matrices, respectively, where* $k \in \{1, \cdots, K\}$.

1. *State vector process* X_t *is N-dimensional and satisfies the following SDE:*
$$dX_t = -\mathcal{K}X_t\,dt + I\,dB_t, \tag{2.1}$$
where $\mathcal{K}$ *is an* $N \times N$ *lower triangular matrix.*

2. *The market price* λ_t *of risk and instantaneous nominal risk-free rate* r_t *are provided as*
$$\lambda_t = \lambda + \Lambda X_t, \tag{2.2}$$
$$r_t = \rho_0 + \rho' X_t + \frac{1}{2}X_t'\mathcal{R}X_t, \tag{2.3}$$
where Λ *is an* $N \times N$ *lower triangular matrix*

3. *The consumer price index* p_t *satisfies*
$$\frac{dp_t}{p_t} = \iota(X_t)\,dt + \sigma^p(X_t)'dB_t, \qquad p_0 = 1, \tag{2.4}$$
where $\mu^p(X_t)$ *and* $\sigma^p(X_t)$ *are given by*
$$\mu^p(X_t) = \iota_0 + \iota' X_t + \frac{1}{2}X_t'\mathcal{I}X_t, \tag{2.5}$$
$$\sigma^p(X_t) = \sigma_p + \Sigma_p X_t, \tag{2.6}$$
where $\mathcal{I}$ *is an* $N \times N$*positive-semidefinite symmetric matrix.*

4. *The dividend of the k-th stock or index is given by*
$$D_t^k = \left(\delta_{0k} + \delta_k' X_t + \frac{1}{2}X_t'\Delta_k X_t\right)\exp\left(\sigma_{0k}t + \sigma_k' X_t + \frac{1}{2}X_t'\Sigma_k X_t\right). \tag{2.7}$$

5. *The parameters introduced above and a matrix* $\bar{\mathcal{R}}$ *defined by*
$$\bar{\mathcal{R}} = \mathcal{R} - \mathcal{I} + \Sigma_p'\Lambda + \Lambda'\Sigma_p \tag{2.8}$$
satisfy the regularity conditions shown in Appendix A.1.

6. *Markets are complete and arbitrage-free.*

Batbold *et al.* (2022) show the SDEs of no-arbitrage security price processes.

Lemma 1. *Let $\tau = T - t$ denote the time to maturity of bond P_t^T or inflation-indexed bond Q_t^T. Under Assumption 1, the dynamics of security price processes satisfy the following:*

1. *The default-free bond with time τ to maturity:*

$$\frac{dP_t^T}{P_t^T} = \big(r_t + (\sigma(\tau) + \Sigma(\tau)X_t)'\lambda_t\big)\, dt + (\sigma(\tau) + \Sigma(\tau)X_t)'\, dB_t, \quad P_T^T = 1, \tag{2.9}$$

where $(\Sigma(\tau), \sigma(\tau))$ is a solution to the system of ODEs (A.3) and (A.4).

2. *The default-free inflation-indexed bond with time τ to maturity:*

$$\frac{dQ_t^T}{Q_t^T} = \Big(r_t + \Big(\sigma_q(\tau) + \Sigma_q(\tau)X_t\Big)'\lambda_t\Big)\, dt + \Big(\sigma_q(\tau) + \Sigma_q(\tau)X_t\Big)'\, dB_t, \tag{2.10}$$

where $(\Sigma_q(\tau), \sigma_q(\tau)) = (\bar{\Sigma}_q(\tau) + \Sigma_p, \bar{\varsigma}_q(\tau) + \sigma_p)$ and $(\bar{\Sigma}_q(\tau), \bar{\varsigma}_q(\tau))$ is a solution to the system of ODEs (A.5) and (A.6).

3. *The k-th index:*

$$\frac{dS_t^k + D_t^k dt}{S_t^k} = \big(r_t + (\sigma_k + \Sigma_k X_t)'\lambda_t\big)\, dt + (\sigma_k + \Sigma_k X_t)'\, dB_t, \tag{2.11}$$

where Σ_k is a solution to Eq. (A.7) and σ_k is given by Eq. (A.8).

Proof. See Appendix A.1 in Kikuchi and Kusuda (2024a). □

3 HREZ Utility and Properties

We introduce HREZ utility and demonstrate its properties.

3.1 HREZ Utility

We begin with the following continuous-time version (Duffie and Epstein (1992a)) of Epstein-Zin utility.

$$\tilde{V}_t = \mathrm{E}_t\left[\int_t^{T^*} f(c_s, \tilde{V}_s)ds\right], \tag{3.1}$$

where f denotes the normalized aggregator of the form:

$$f(c, v) = \begin{cases} \dfrac{\beta}{1-\psi^{-1}}(1-\gamma)v\left(\Big(c\big((1-\gamma)v\big)^{-\frac{1}{1-\gamma}}\Big)^{1-\psi^{-1}} - 1\right), & \text{if} \quad \psi \neq 1, \\ \beta(1-\gamma)v\Big(\log c - \dfrac{1}{1-\gamma}\log\big((1-\gamma)v\big)\Big), & \text{if} \quad \psi = 1, \end{cases} \tag{3.2}$$

where $\beta > 0$ is the subjective discount rate, $\gamma \in (0,1) \cup (1,\infty)$ is the relative risk aversion, and $\psi > 0$ is the EIS.

Whereas an investor with robust utility regards probability P ("base probability") as the most likely probability, they also consider other probabilities because the true probability is unknown. Thus, the investor assumes set $\mathbb{P}$ of all equivalent probability measures[4] as alternative probabilities. According to Girsanov's theorem, any equivalent probability measure is characterized by a measurable process ξ_t with Novikov's integrability condition as the following Radon-Nikodym derivative:

$$\mathrm{E}_{T^*}\left[\frac{d\mathrm{P}^\xi}{d\mathrm{P}}\right] = \exp\left(\int_0^{T^*} \xi_t \, dB_t - \frac{1}{2}\int_0^{T^*} |\xi_t|^2 dt\right) \quad \forall T^* \in (0,\infty). \tag{3.3}$$

Therefore, the investor chooses the worst-case probability that minimizes the utility among $\mathbb{P}$ for every consumption plan. The investor rationally determines the worst-case probability by considering deviations from P.

Definition 1. *HREZ utility is defined by*

$$U(c) = \inf_{\mathrm{P}^\xi \in \mathbb{P}} \mathrm{E}^\xi \left[\int_0^{T^*} \left(f(c_t, V_t^\xi) + \frac{(1-\gamma)V_t^\xi}{2\theta} |\xi_t|^2 \right) dt \right], \tag{3.4}$$

where c is a consumption plan such that $c = (c_t)_{t \in [0,T^*)}$ is an adapted non-negative consumption-rate process, E^ξ is the expectation under P^ξ, $\theta > 0$ is relative ambiguity aversion, and V_t^ξ is the utility process, defined recursively as follows:

$$V_t^\xi = \mathrm{E}_t^\xi \left[\int_t^{T^*} \left(f(c_s, V_s^\xi) + \frac{(1-\gamma)V_s^\xi}{2\theta} |\xi_s|^2 \right) ds \right], \quad V_{T^*}^\xi = 0. \tag{3.5}$$

3.2 HREZ Utility as Homothetic SDU

For simplicity, we assume $\psi \neq 1$ in Eq. (3.4). First, we prove that HREZ utility is SDU under certain integrability conditions by modifying the incorrect proof in Skiadas (2003). Then, following Duffie and Epstein (1992a), we show that HREZ utility is continuous, consistent, strictly increasing, risk averse, and homothetic.

3.2.1 SDU Representation

Let $\beta^* = \dfrac{\beta(1-\gamma)}{1-\psi^{-1}}$. Then, Eq. (3.5) is rewritten as

$$V_t^\xi = \mathrm{E}_t^\xi \left[\int_t^{T^*} e^{-\beta^*(s-t)} \left(f^*(c_s, V_s^\xi) + \frac{(1-\gamma)V_s^\xi}{2\theta} |\xi_s|^2 \right) ds \right], \tag{3.6}$$

[4] A probability measure $\tilde{\mathrm{P}}$ is said to be an equivalent probability measure of P if and only if $\mathrm{P}(A) = 0 \Leftrightarrow \tilde{\mathrm{P}}(A) = 0$.

where

$$f^*(c,v) = \frac{\beta}{1-\psi^{-1}}(1-\gamma)v\Big(c\big((1-\gamma)v\big)^{-\frac{1}{1-\gamma}}\Big)^{1-\psi^{-1}}. \tag{3.7}$$

Suppose a progressively measurable pair (V^*, σ^*) satisfies

$$dV_t^* = -\left(f^*(c_t, V_t^*) - \beta^* V_t^* - \frac{\theta}{2(1-\gamma)V_t^*}|\sigma_t^*|^2\right)dt + (\sigma_t^*)' dB_t, \quad V_{T^*}^* = 0. \tag{3.8}$$

From the Girsanov theorem, as the standard Brownian motion under P^ξ is given by $B_t^\xi = B_t - \int_0^t \xi_s\, ds$, Eq. (3.8) is rewritten as

$$\begin{aligned} dV_t^* = -\left(f^*(c_t, V_t^*) - \beta^* V_t^* - \frac{\theta}{2(1-\gamma)V_t^*}|\sigma_t^*|^2 - (\sigma_t^*)'\xi_t\right) dt \\ + (\sigma_t^*)' dB_t^\xi, \quad V_{T^*}^* = 0. \end{aligned} \tag{3.9}$$

Then, the following equation holds:

$$\begin{aligned} V_t^\xi = V_t^* + \mathrm{E}_t^\xi\Bigg[\int_t^{T^*} e^{-\beta^*(s-t)}\Big(f^*(c_s, V_s^\xi) - f^*(c_s, V_s^*) \\ + \frac{1-\gamma}{2\theta}(V_s^\xi - V_s^*)|\xi_s|^2 + Q(\xi_s, \sigma_s^*, V_s)\Big)ds\Bigg], \end{aligned} \tag{3.10}$$

where

$$Q(\xi_s, \sigma_s^*, V_s) = \frac{(1-\gamma)V_s^*}{2\theta}\left|\xi_s + \frac{\theta}{(1-\gamma)V_s^*}\sigma_s^*\right|^2 \geq 0. \tag{3.11}$$

Remark 1. *Skiadas (2003) shows the following equation:*

$$V_t^\xi = V_t^* + \mathrm{E}_t^\xi\left[\int_t^{T^*} e^{-\beta^*(s-t)}\Big(f^*(c_s, V_s^\xi) - f^*(c_s, V_s^*) + Q^\xi(s, V_s^\xi)\Big)ds\right], \tag{3.12}$$

where

$$Q^\xi(s, V_s^\xi) = \frac{(1-\gamma)V_s^\xi}{2\theta}\left|\xi_s + \frac{\theta}{(1-\gamma)V_s^\xi}\sigma_s^*\right|^2. \tag{3.13}$$

It is evident that Eq. (3.12) does not hold. However, the subsequent proof in Skiadas (2003) can be applied in the same manner.

We define the function h^* as

$$h^*(\xi, c, v) = f^*(c, v) + \frac{1-\gamma}{2\theta}v|\xi|^2. \tag{3.14}$$

Then, Eq. (3.10) is rewritten as

$$V_t^\xi = V_t^* + \mathrm{E}_t^\xi\left[\int_t^{T^*} e^{-\beta^*(s-t)}\Big(h^*(\xi_s, c_s, V_s^\xi) - h^*(\xi_s, c_s, V_s^*) + Q(\xi_s, \sigma_s^*, V_s^*)\Big)ds\right], \tag{3.15}$$

and $h^*_{vv} = f^*_{vv}$ is calculated as

$$h^*_{vv}(\xi, c, v) = \beta(\gamma - \psi^{-1})c^{1-\psi^{-1}}\big((1-\gamma)v\big)^{-\frac{1-\psi^{-1}}{1-\gamma}-1}. \tag{3.16}$$

Thus, if $\gamma > \psi^{-1}$, then h^* is convex in its utility argument; conversely, if $\gamma < \psi^{-1}$, then h^* is concave in its utility argument. Therefore, we obtain

$$V^\xi_t - V^*_t \geq \begin{cases} \mathrm{E}^\xi_t\left[\int_t^{T^*} e^{-\beta^*(s-t)} h^*_v(\xi_s, c_s, V^*_s)(V^\xi_s - V^*_s)ds\right], & \text{if } \gamma > \psi^{-1}, \\ \mathrm{E}^\xi_t\left[\int_t^{T^*} e^{-\beta^*(s-t)}\big(-h^*_v(\xi_s, c_s, V^*_s)\big)(V^\xi_s - V^*_s)ds\right], & \text{if } \gamma < \psi^{-1}. \end{cases} \tag{3.17}$$

Under certain integrability conditions on h^*_v, in either case, "stochastic Gronwall-Bellman inequality"[5] implies that $V^\xi_t \geq V^*_t$ P-*a.s.* for all $t \in [0, T^*]$. Hence, the minimizer or worst-case probability ξ^* is given by

$$\xi^* = -\frac{\theta}{(1-\gamma)V^*}\sigma^*, \tag{3.18}$$

and $V^{\xi^*} = V^*$. Therefore, HREZ utility satisfying the above conditions is SDU of the unnormalized form (3.8).

Remark 2. *In Eq. (3.8), as $\theta \searrow 0$, V^* converges to Epstein-Zin utility. When $\psi^{-1} = \gamma$, Eq. (3.8) is simplified as*

$$dV^*_t = -\left(\frac{\beta}{1-\gamma}c_t^{1-\gamma} - \beta V^*_t - \frac{\theta}{2(1-\gamma)V^*_t}|\sigma^*_t|^2\right)dt + (\sigma^*_t)' dB_t, \quad V^*_{T^*} = 0, \tag{3.19}$$

that is, V^* becomes homothetic robust utility. Therefore, HREZ utility is a generalization of Epstein-Zin utility and homothetic robust utility.

Remark 3. *HREZ utility is SDU, and is continuous and consistent, as shown in Propositions 1 and 4 in Duffie and Epstein (1992a). Let HREZU$(\beta, \gamma, \psi, \theta)$ denote HREZ utility with $(\beta, \gamma, \psi, \theta)$ and the above conditions. From Proposition 6 in Duffie and Epstein (1992a), Eq. (3.8) shows that HREZU$(\beta, \gamma, \psi, \theta_1)$ is more risk averse than HREZU$(\beta, \gamma, \psi, \theta_2)$ if $\theta_1 > \theta_2$.*

3.2.2 Normalized Representation

Next, by using the ordinally equivalent utility presented by Duffie and Epstein (1992a), we show the normalized form of V^*. The ordinally equivalent utility $\bar{V}$ of V^* is defined by

$$\bar{V} = \frac{1}{1-(\gamma+\theta)}\big((1-\gamma)V^*\big)^{1-\frac{\theta}{1-\gamma}}. \tag{3.20}$$

[5] See Appendix B in Duffie and Epstein (1992a).

Then, from Ito's lemma, $\bar{V}$ satisfies

$$\bar{V}_t = \mathrm{E}_t \left[\int_t^{T^*} \bar{f}(c_s, \bar{V}_s) ds \right], \tag{3.21}$$

where

$$\bar{f}(c, v) = \frac{\beta}{1 - \psi^{-1}} \big(1 - (\gamma + \theta)\big) v \left(\left(c \Big(\big(1 - (\gamma + \theta)\big) v \Big)^{-\frac{1}{1-(\gamma+\theta)}} \right)^{1-\psi^{-1}} - 1 \right). \tag{3.22}$$

Remark 4. *From Proposition 3 in Duffie and Epstein (1992a), HREZ utility with the normalized aggregator $\bar{f}$ is strictly increasing because $\bar{f}$ is strictly increasing in consumption, as follows:*

$$\bar{f}_c(c, v) = \beta c^{-\psi^{-1}} \Big(\big(1 - (\gamma + \theta)\big) v \Big)^{1 - \frac{1-\psi^{-1}}{1-(\gamma+\theta)}} > 0. \tag{3.23}$$

In addition, from Proposition 7 in Duffie and Epstein (1992a), HREZ utility with the normalized aggregator $\bar{f}$ is risk averse because $\bar{f}$ is concave in consumption, as shown in the follows:

$$\bar{f}_{cc}(c, v) = -\beta \psi^{-1} c^{-\psi^{-1}-1} \Big(\big(1 - (\gamma + \theta)\big) v \Big)^{1 - \frac{1-\psi^{-1}}{1-(\gamma+\theta)}} < 0. \tag{3.24}$$

Furthermore, from Proposition A in Skiadas (1998), HREZ utility with the normalized aggregator $\bar{f}$ is information seeking if $\gamma + \theta > \psi^{-1}$ because $\bar{f}$ is convex in utility, as shown in the follows:

$$\bar{f}_{vv}(c, v) = \beta \big(\gamma + \theta - \psi^{-1}\big) c^{1-\psi^{-1}} \Big(\big(1 - (\gamma + \theta)\big) v \Big)^{-\frac{1-\psi^{-1}}{1-(\gamma+\theta)} - 1} > 0. \tag{3.25}$$

3.2.3 Observational Indistinguishability from Epstein-Zin Utility and Robustness Effect

By substituting $\psi^{-1} = \gamma$ into Eq. (3.22), we obtain the normalized aggregator of homothetic robust (HR) utility $\tilde{f}$:

$$\tilde{f}(c, v) = \frac{\beta}{1 - \gamma} \big(1 - (\gamma + \theta)\big) v \left(\left(c \Big(\big(1 - (\gamma + \theta)\big) v \Big)^{-\frac{1}{1-(\gamma+\theta)}} \right)^{1-\gamma} - 1 \right). \tag{3.26}$$

Assume that an investor who has Epstein-Zin utility with (β, γ, ψ) is information seeking, that is, $\gamma > \psi^{-1}$. We consider another investor who has HR utility with $(\beta, \hat{\gamma}, \theta)$. Let $\hat{\gamma} = \psi^{-1}$ and $\theta = \gamma - \psi^{-1}$. Then, Eqs. (3.2) and (3.26) show that HR utility with $(\beta, \hat{\gamma}, \theta)$ is observationally indistinguishable from Epstein-Zin utility with (β, γ, ψ). This is noted by Maenhout (2004), who also interprets the effect of homothetic robustness as follows. Given that the nonrobust agents have CRRA utility with $(\beta, \hat{\gamma})$, they are equally willing to substitute over time as across states because $\hat{\gamma}$ is the inverse of the EIS. Robustness makes the agent less willing to substitute across states as the relative risk aversion becomes $\hat{\gamma}+\theta > \hat{\gamma}$, without altering the willingness to substitute intertemporally, as the EIS remains $\hat{\gamma}^{-1}$.

Remark 5. *Eqs. (3.2) and (3.22) show that the HREZ utility with $(\beta, \hat{\gamma}, \psi, \theta)$ is observationally indistinguishable from the Epstein-Zin utility with (β, γ, ψ) if $\hat{\gamma} + \theta = \gamma$. Following Maenhout (2004), we can interpret the effect of homothetic robustness as follows: Suppose that the nonrobust agent has the Epstein-Zin utility with (β, γ, ψ). What robustness does is to make the agent less willing to substitute across states, as the relative risk aversion becomes $\gamma + \theta > \gamma$, without altering the willingness to substitute intertemporally, as the EIS remains ψ.*

3.2.4 Homotheticity

Finally, we demonstrate that V^* is homothetic. The ordinally equivalent utility $\hat{V}$ of V^* is defined as

$$\hat{V} = |1-\gamma|\big((1-\gamma)V^*\big)^{\frac{1}{1-\gamma}}. \tag{3.27}$$

Then, from Ito's lemma, $\hat{V}$ satisfies

$$d\hat{V}_t = -\left(\hat{f}(c_s, \hat{V}_s) - \hat{\beta}\hat{V}_t - \frac{\gamma+\theta}{2\hat{V}_t}|\hat{\sigma}_t|^2\right) dt + \hat{\sigma}_t' dB_t, \quad \hat{V}_{T^*} = 0, \tag{3.28}$$

where $\hat{\beta} = \dfrac{\beta}{1-\psi^{-1}}$ and

$$\hat{f}(c,v) = \frac{\beta}{1-\psi^{-1}}(1-\gamma)c^{1-\psi^{-1}}\Big(\frac{v}{1-\gamma}\Big)^{\psi^{-1}}. \tag{3.29}$$

From Proposition 8 in Duffie and Epstein (1992a), SDE (3.28) and Eq. (3.29) show that $\hat{V}$ and its ordinally equivalent utility V^* are homothetic.

4 Theoretical Analysis of Robust Control

We introduce the robust consumption-investment problem and theoretically analyze the optimal robust control.

4.1 Robust Consumption-Investment Problem

We consider the infinite-time consumption-investment problem of the infinite-lived investor.

Assumption 2. *The investor's utility is HREZ utility of the form:*

$$U(c) = \inf_{\mathrm{P}^{\xi} \in \mathbb{P}} \mathrm{E}^{\xi} \left[\int_0^{\infty} \left(f(c_t, V_t^{\xi}) + \frac{(1-\gamma)V_t^{\xi}}{2\theta} |\xi_t|^2 \right) dt \right], \tag{4.1}$$

where $\gamma > 1$, $\psi > 1$, *and*

$$V_t^{\xi} = \mathrm{E}_t^{\xi} \left[\int_t^{\infty} \left(f(c_s, V_s^{\xi}) + \frac{(1-\gamma)V_s^{\xi}}{2\theta} |\xi_s|^2 \right) ds \right]. \tag{4.2}$$

Remark 6. *Note that the HREZ utility in Assumption 2 is information seeking, because* $\gamma + \theta > 1 > \psi^{-1}$.

4.1.1 Portfolio

Let $P_t(\tau) = P_t^T$ and $Q_t(\tau) = Q_t^T$ where $\tau = T - t$.

Assumption 3. *The investor invests in* $P_t(\tau_1), \cdots, P_t(\tau_{I_P}), Q_t(\tau_1^q), \cdots, Q_t(\tau_{I_Q}^q)$, *and* $S_t^1, \cdots, S_t^K$ *at time* t *where* $I_P + I_Q + K = N$. *Let* $\Phi^P(\tau)$ *and* $\Phi^Q(\tau^q)$ *denote the portfolio weights of a default-free bond with* τ*-time to maturity and a default-free inflation-indexed bond with* τ^q*-time to maturity, respectively. Let* Φ^k *denote the portfolio weight of the k-th index. Let* Φ_t *and* $\Sigma(X_t)$ *denote the portfolio weight and volatility matrix at time t, respectively.* Φ_t *and* $\Sigma(X_t)$ *are expressed as follows:*

$$\Phi_t = \begin{pmatrix} \Phi_t^P(\tau_1) \\ \vdots \\ \Phi_t^P(\tau_{I_P}) \\ \Phi_t^Q(\tau_1^q) \\ \vdots \\ \Phi_t^Q(\tau_{I_Q}^q) \\ \Phi_t^1 \\ \vdots \\ \Phi_t^K \end{pmatrix}, \quad \Sigma(X_t) = \begin{pmatrix} \big(\sigma(\tau_1) + \Sigma(\tau_1)X_t\big)' \\ \vdots \\ \big(\sigma(\tau_{I_P}) + \Sigma(\tau_{I_P})X_t\big)' \\ \big(\sigma_q(\tau_1^q) + \Sigma_q(\tau_1^q)X_t\big)' \\ \vdots \\ \big(\sigma_q(\tau_{I_Q}^q) + \Sigma_q(\tau_{I_Q}^q)X_t\big)' \\ (\sigma_1 + \Sigma_1 X_t)' \\ \vdots \\ (\sigma_K + \Sigma_K X_t)' \end{pmatrix}. \tag{4.3}$$

4.1.2 Real Budget Constraint

Given that the inflation-deflation risk is introduced in the quadratic security market model of Kikuchi and Kusuda (2024a), we derive the real budget constraint equation. Therefore, we define the real market price of risk $\bar{\lambda}(X_t)$ and real instantaneous interest rate $\bar{r}(X_t)$ as

$$\bar{\lambda}(X_t) = \lambda_t - \sigma^p(X_t), \tag{4.4}$$

$$\bar{r}(X_t) = r_t - \mu^p(X_t) + \lambda_t' \sigma^p(X_t). \tag{4.5}$$

Note that $\bar{\lambda}(X_t)$ is an affine function of X_t, and $\bar{r}(X_t)$ is a quadratic function of X_t.

$$\begin{aligned} \bar{\lambda}(X_t) &= \bar{\lambda} + \bar{\Lambda} X_t, && (4.6) \\ \bar{r}(X_t) &= \bar{\rho}_0 + \bar{\rho}' X_t + \frac{1}{2} X_t' \bar{\mathcal{R}} X_t, && (4.7) \end{aligned}$$

where $\bar{\mathcal{R}}$ is given by Eq. (2.8) and

$$\begin{aligned} (\bar{\lambda}, \bar{\Lambda}) &= (\lambda - \sigma_p, \Lambda - \Sigma_p), && (4.8) \\ \bar{\rho}_0 &= \rho_0 - \iota_0 + \lambda' \sigma^p, && (4.9) \\ \bar{\rho} &= \rho - \iota + \Lambda' \sigma_p + \Sigma_p' \lambda. && (4.10) \end{aligned}$$

$\mathit{\Phi}$ is assumed to be an adapted process. Let $\bar{W}$ denote the real wealth process and $\bar{W}_0 > 0$.

Lemma 2. *Under Assumptions 1–3, given an initial state $(\bar{W}_0, X_0)$, consumption plan c, and self-financing portfolio weight $\mathit{\Phi}$, the real budget constraint equation is*

$$\frac{d\bar{W}_t}{\bar{W}_t} = \left(\bar{r}(X_t) + \bar{\varsigma}_t' \bar{\lambda}(X_t) - \frac{c_t}{\bar{W}_t} \right) dt + \bar{\varsigma}_t' \, dB_t, \quad \bar{W}_t > 0 \quad \forall t \in (0, \infty), \tag{4.11}$$

where

$$\bar{\varsigma}_t = \Sigma(X_t)' \mathit{\Phi}_t - \sigma^p(X_t). \tag{4.12}$$

Proof. See Appendix A.1 in Kikuchi and Kusuda (2024b). □

Remark 7. *The real budget constraint represents the instantaneous real rate of return on wealth. Eq. (4.11) shows that increasing the investment in the measure of $\bar{\varsigma}_t$ increases the wealth volatility, whereas the real expected excess return on wealth increases in proportion to $\bar{\varsigma}_t$. Thus, the (real) market price $\bar{\lambda}(X_t)$ of risk is interpreted as the price per unit of investment for all investors.*

4.1.3 Robust Control Problem

The real budget constraint (4.11) indicates that $(c, \bar{\varsigma})$ is the control in the optimal consumption-investment problem. $\mathbf{X} = (\bar{W}, X')'$. We call $\bar{\varsigma}$ the investment control. We say that a control $(c, \bar{\varsigma})$ is admissible if it satisfies the real budget constraint equation (4.11) with initial state $\mathbf{X}_0$ and there are measurable functions $\hat{c}(\mathbf{x})$ and $\hat{\varsigma}(\mathbf{x})$ such that $c_t = \hat{c}(\mathbf{X}_t)$ and $\bar{\varsigma}_t = \hat{\varsigma}(\mathbf{X}_t)$ for every $t \in [0, \infty)$. Let $\mathcal{B}(\mathbf{X}_0)$ denote the set of admissible controls. Furthermore, we call ξ in Eq. (3.3) the probability control. We say that probability control ξ is admissible if it satisfies Novikov's condition and there is a measurable function $\hat{\xi}(\mathbf{x})$ such that $\hat{\xi}(\mathbf{X}_t) = \xi_t$ for every $t \in [0, \infty)$. Let $\hat{\mathbb{P}}(\mathbf{X}_0)$ denote the set of admissible probability controls.

Given $\mathcal{F}_t$ and $\mathbf{X}_t$, the investor's robust consumption-investment problem and value function are recusively defined as

$$V_t = \sup_{(c,\varsigma)\in\mathcal{B}(\mathbf{X}_t)} \inf_{\mathrm{P}^\xi\in\hat{\mathbb{P}}(\mathbf{X}_t)} \mathrm{E}^\xi \left[\int_t^\infty \left(f(c_s, V_s) + \frac{(1-\gamma)V_s}{2\theta} |\xi_s|^2 \right) ds \right]. \quad (4.13)$$

The recursive definition of the above value function is justified by the fact that HREZ utility is consistent. Given $\mathcal{F}_t$ and $\mathbf{X}_t = \mathbf{x}$, the indirect utility function is defined as $J(\mathbf{x}) = V_t$.

4.2 Conditional Worst-case Probability

First, the conditional worst-case probability for a given control is presented. As the standard Brownian motion under P^ξ is given by $B_t^\xi = B_t - \int_0^t \xi_s \, ds$, the SDE for $\mathbf{X}_t$ under P^ξ is rewritten as

$$d\mathbf{X}_t = \left(\begin{pmatrix} \bar{W}_t(\bar{r}_t + \varsigma_t'\bar{\lambda}_t) - c_t \\ -\mathcal{K}X_t \end{pmatrix} + \begin{pmatrix} \bar{W}_t\varsigma_t' \\ I \end{pmatrix} \xi_t \right) dt + \begin{pmatrix} \bar{W}_t\varsigma_t' \\ I \end{pmatrix} dB_t^\xi. \quad (4.14)$$

The Hamilton-Jacobi-Bellman (HJB) equation for problem (4.13) is then expressed as

$$0 = \sup_{(\hat{c},\hat{\varsigma})\in\mathbb{R}_+\times\mathbb{R}^N} \inf_{\hat{\xi}\in\mathbb{R}^N} \left\{ \begin{pmatrix} w\left(\bar{r}(x) + \hat{\varsigma}'\bar{\lambda}(x)\right) - \hat{c} \\ -\mathcal{K}x \end{pmatrix}' \begin{pmatrix} J_w \\ J_x \end{pmatrix} + \hat{\xi}' \begin{pmatrix} w\hat{\varsigma}' \\ I \end{pmatrix}' \begin{pmatrix} J_w \\ J_x \end{pmatrix} \right.$$
$$\left. + \frac{1}{2} \operatorname{tr} \left[\begin{pmatrix} w\hat{\varsigma}' \\ I \end{pmatrix} \begin{pmatrix} w\hat{\varsigma}' \\ I \end{pmatrix}' \begin{pmatrix} J_{ww} & J_{wx} \\ J_{xw} & J_{xx} \end{pmatrix} \right] + f(\hat{c}, J) + \frac{(1-\gamma)J}{2\theta} |\hat{\xi}|^2 \right\}. \quad (4.15)$$

The conditional worst-case probability $\hat{\xi}^{\hat{c},\hat{\varsigma}}$ for given control $(\hat{c}, \hat{\varsigma})$ satisfies

$$\hat{\xi}^{\hat{c},\hat{\varsigma}}(\mathbf{x}) = -\frac{\theta}{(1-\gamma)J} \begin{pmatrix} w\hat{\varsigma}' \\ I \end{pmatrix}' \begin{pmatrix} J_w \\ J_x \end{pmatrix}. \quad (4.16)$$

The real budget constraint (4.11) under the conditional worst-case probability $\mathrm{P}^{\hat{\xi}^*}$ for the given control $(\hat{c}, \hat{\varsigma})$ is rewritten as

$$\frac{d\bar{W}_t}{\bar{W}_t} = \left\{ \bar{r}(X_t) + \hat{\varsigma}(\mathbf{X}_t)'\hat{\lambda}(\mathbf{X}_t) - \frac{\hat{c}(\mathbf{X}_t)}{\bar{W}_t} \right\} dt + \hat{\varsigma}(\mathbf{X}_t)' \, dB_t^{\hat{\xi}}, \quad (4.17)$$

where

$$\hat{\lambda}(\mathbf{x}) = \bar{\lambda}(x) + \hat{\xi}^{\hat{c},\hat{\varsigma}}(\mathbf{x}) = \bar{\lambda}(x) - \frac{\theta}{(1-\gamma)J} \begin{pmatrix} w\hat{\varsigma}(\mathbf{x})' \\ I \end{pmatrix}' \begin{pmatrix} J_w \\ J_x \end{pmatrix}. \quad (4.18)$$

Remark 8. *In Eq. (4.17), the real market price $\bar{\lambda}(X_t)$ of risk in the real budget constraint Eq. (4.11) is replaced with $\hat{\lambda}(\mathbf{X}_t)$, which is the investor price per unit of investment under the conditional worst-case probability for a given control. As shown by Remark 7, when ambiguity is not considered, the price per unit of investment risk is the real market price $\bar{\lambda}(X_t)$ of risk, which is common to all investors. By contrast, $\hat{\lambda}(\mathbf{X}_t)$ varies across investors. Eq. (4.17) shows that ambiguity averse investors value the price per unit of investment below the real market price of risk under the conditional worst-case probability.*

Henceforth, we refer to the real market price of risk simply as the market price of risk.

Remark 9. *In Eq. (4.17), under the conditional worst-case probability assumed by investors with HREZ utility, the investment control $\hat{\varsigma}(\mathbf{X}_t)$, which is the volatility of the wealth process, is as assumed under the base probability; however, its price $\hat{\lambda}(\mathbf{X}_t)$ is permanently discounted from the market price of risk. This implies that investors with HREZ utility assume long-term stagnation rather than increased volatility as the worst-case scenario.*

Substituting $\hat{\xi}^*$ into the HJB Eq. (4.15) yields

$$0 = \sup_{(\hat{c},\hat{\varsigma})\in\mathbb{R}_+\times\mathbb{R}^N} \left[J_t + \begin{pmatrix} w\big(\bar{r}(x) + \hat{\varsigma}'\bar{\lambda}(x)\big) - \hat{c} \\ -\mathcal{K}x \end{pmatrix}' \begin{pmatrix} J_w \\ J_x \end{pmatrix} \right.$$
$$\left. + \frac{1}{2}\,\mathrm{tr}\left[\begin{pmatrix} w\hat{\varsigma}' \\ I \end{pmatrix} \begin{pmatrix} w\hat{\varsigma}' \\ I \end{pmatrix}' \begin{pmatrix} J_{ww} & J_{wx} \\ J_{xw} & J_{xx} \end{pmatrix} \right] + f(\hat{c}, J) - \frac{\theta}{2(1-\gamma)J} \left| \begin{pmatrix} w\hat{\varsigma}' \\ I \end{pmatrix}' \begin{pmatrix} J_w \\ J_x \end{pmatrix} \right|^2 \right]. \tag{4.19}$$

Let $\bar{W}^*$ denote the optimal real wealth. Let $\mathbf{x}^* = (w^*, x')'$. Moreover, let $\hat{c}^*(\mathbf{x}^*)$ and $\hat{\varsigma}^*(\mathbf{x}^*)$ denote the optimal consumption and investment controls, respectively. Define the "worst-case probability" and "investor price of risk under the worst-case probability" as

$$\hat{\xi}^*(\mathbf{x}^*) = -\frac{\theta}{(1-\gamma)J} \begin{pmatrix} w^*\hat{\varsigma}^*(\mathbf{x})' \\ I \end{pmatrix}' \begin{pmatrix} J_w \\ J_x \end{pmatrix}, \tag{4.20}$$

$$\hat{\lambda}^*(\mathbf{x}^*) = \bar{\lambda}(x) - \theta \left(\frac{w^* J_w}{(1-\gamma)J}\hat{\varsigma}^*(\mathbf{x}) + \frac{J_x}{(1-\gamma)J} \right). \tag{4.21}$$

Remark 10. *In Eq. (4.20), the term $w^*\hat{\varsigma}^*(\mathbf{x})J_w + J_x$ on the indirect utility process corresponds to the term σ^* on the utility process V^* in the SDE (3.8). Thus, the worst-case probability $\hat{\xi}^*(\mathbf{x}^*)$ in the Markovian consumption-investment problem is consistent with the worst-case probability ξ^* in the HREZ utility functional in Eq. (3.18).*

4.3 A First Expression of the Optimal Robust Consumption-Investment

Let

$$\mathcal{U} = -\frac{w^* J_{ww}}{J_w} + \theta \frac{w^* J_w}{(1-\gamma)J}. \tag{4.22}$$

We obtain the following lemma.

Lemma 3. *Under Assumptions 1–3, the optimal control is given by*

$$\hat{c}^*(\mathbf{X}_t^*) = \begin{cases} \beta J_w^{-1}(1-\gamma)J, & \text{if} \quad \psi = 1, \\ \beta^{\psi} J_w^{-\psi}\big((1-\gamma)J\big)^{\frac{\gamma\psi-1}{\gamma-1}}, & \text{if} \quad \psi \neq 1, \end{cases} \tag{4.23}$$

$$\hat{\varsigma}^*(\mathbf{X}_t^*) = \frac{1}{\mathcal{U}}\left(\bar{\lambda}(X_t) + \frac{J_{xw}}{J_w} - \frac{\theta J_x}{(1-\gamma)J}\right), \tag{4.24}$$

where J is a solution of the following PDE:

$$\begin{aligned} 0 = &\frac{1}{2}\mathrm{tr}\left[J_{xx}\right] - \frac{\theta}{2(1-\gamma)J}|J_x|^2 \\ &- \frac{1}{2}\left(w^{*2}J_{ww} - \frac{\theta w^2 J_w^2}{(1-\gamma)J}\right)^{-1}|\pi(x)|^2 + \bar{r}(x)w^* J_w - (\mathcal{K}x)' J_x \\ &+ \begin{cases} \beta\big\{(1-\gamma)(\log \hat{c}^* - 1) - \log\big((1-\gamma)\big)\big\}J, & \text{if} \quad \psi = 1, \\ \dfrac{1}{\psi - 1}\hat{c}^* J_w - \dfrac{\beta(1-\gamma)}{1-\psi^{-1}}J, & \text{if} \quad \psi \neq 1, \end{cases} \end{aligned} \tag{4.25}$$

where

$$\pi(x) = -w^* J_w\left(\bar{\lambda}(x) + \frac{J_{xw}}{J_w} - \frac{\theta J_x}{(1-\gamma)J}\right). \tag{4.26}$$

Proof. See Appendix C. □

Remark 11. *Strictly speaking, $(\hat{c}^*(\mathbf{X}_t^*), \hat{\varsigma}^*(\mathbf{X}_t))$ is only a candidate for optimal control, because we do not provide a verification theorem. However, we tentatively call this optimal control in this study.*

It follows from Eqs. (4.12) and (4.24), optimal robust portfolio Φ_t^* satisfies

$$\Sigma(X_t)'\Phi_t^* - \sigma^p(X_t) = \frac{1}{\mathcal{U}}\left(\bar{\lambda}_t + \frac{J_{xw}}{J_w} - \theta\frac{J_x}{(1-\gamma)J}\right). \tag{4.27}$$

Thus, from Eq. (4.4), we decompose the optimal robust portfolio into the following four terms:

$$\begin{aligned} \Phi_t^* = &\frac{1}{\mathcal{U}}\Sigma(X_t)'^{-1}\lambda_t + \frac{1}{\mathcal{U}}\Sigma(X_t)'^{-1}\frac{J_{xw}}{J_w} \\ &- \frac{1}{\mathcal{U}}\Sigma(X_t)'^{-1}\frac{\theta J_x}{(1-\gamma)J} + \left(1 - \frac{1}{\mathcal{U}}\right)\Sigma(X_t)'^{-1}\sigma^p(X_t). \end{aligned} \tag{4.28}$$

The first term is the myopic demand. The fourth term insures inflation-deflation risk. Following Kikuchi and Kusuda (2024a), we call the fourth term the "inflation-deflation hedging demand," as presented by Brennan and Xia (2002), Sangvinatsos and Wachter (2005), Batbold *et al.* (2022), and Kikuchi and Kusuda (2024a).

Remark 12. *Note that the indirect utility function depends on both relative risk aversion and ambiguity aversion because the PDE (4.25) depends not only on relative risk aversion but also on relative ambiguity aversion. Thus, the second and third terms in Eq. (4.28) are related to the intertemporal uncertainty on marginal indirect utility and indirect utility, respectively. The second term hedges against intertemporal uncertainty in marginal indirect utility due to state changes, whereas the third term hedges against intertemporal uncertainty in indirect utility due to state changes. Therefore, we call the second term the "intertemporal marginal indirect utility hedging demand" and the third term the "intertemporal indirect utility hedging demand." Note that intertemporal indirect utility hedging demand disappears when* $\theta = 0$.

From the PDE (4.25), we infer that the indirect utility function takes the form in Eq. (4.29):

$$J(\mathbf{x}) = \begin{cases} \dfrac{w^{1-\gamma}}{1-\gamma} G(x), & \text{if} \quad \psi = 1, \\ \dfrac{w^{1-\gamma}}{1-\gamma} \big(G(x)\big)^{\frac{1-\gamma}{\psi-1}}, & \text{if} \quad \psi \neq 1. \end{cases} \tag{4.29}$$

Thus, the partial derivatives of J with respect to w are given by Eq. (B.1) in Appendix B.

Remark 13. *Eq. (4.18) is rewritten as*

$$\hat{\lambda}(\mathbf{x}^*) = \bar{\lambda}(x) - \theta \left(\frac{w^* J_w}{(1-\gamma)J} \hat{\varsigma}^*(\mathbf{x}^*) + \frac{J_x}{(1-\gamma)J} \right). \tag{4.30}$$

The second term represents the discount from the market price of risk to $\hat{\lambda}(\mathbf{x}^*)$*. Eq. (4.30) shows that the discount is proportional to the investor's ambiguity aversion. Substituting Eq. (B.1) into Eq. (4.30) yields*

$$\hat{\lambda}(\mathbf{x}^*) = \bar{\lambda}(x) - \theta \left(\hat{\varsigma}^*(\mathbf{x}^*) + \frac{J_x}{(1-\gamma)J} \right). \tag{4.31}$$

Eq. (4.31) shows that the discount from the market price of risk to $\hat{\lambda}(\mathbf{x}^*)$ *increases with investment* $\hat{\varsigma}(\mathbf{x}^*)$*. As the discount from the market price of risk increases with relative ambiguity aversion and investment, these combined effects suppress the ambiguity averse investor's optimal investment.*

Substituting Eq. (B.1) into Eq. (4.22), we obtain

$$\mathcal{U} = \gamma + \theta. \tag{4.32}$$

Batbold *et al.* (2022) call the sum of relative risk aversion and relative ambiguity aversion "relative uncertainty aversion." Given that the reciprocal of relative risk aversion is called relative risk tolerance, they also call the reciprocal of relative uncertainty aversion "relative uncertainty tolerance."

Remark 14. *Eq. (4.32) indicates that $\mathcal{U}^{-1}$ is the relative uncertainty tolerance; Eq. (4.24) shows that the optimal investment is proportional to relative uncertainty tolerance.*

The partial derivatives of J with respect to x are given by Eq. (B.2) in Appendix B. Substituting the derivatives of J into Eq. (4.21), the investor price of risk under the worst-case probability is given by

$$\hat{\lambda}^*(\mathbf{x}^*) = \frac{\gamma}{\gamma+\theta}\bar{\lambda}(x) + \frac{\theta}{\gamma+\theta}\eta^*(x), \tag{4.33}$$

where

$$\eta^*(x) = \begin{cases} \dfrac{1}{\gamma-1}\dfrac{G_x(x)}{G(x)}, & \text{if} \quad \psi = 1, \\ -\dfrac{1}{\psi-1}\dfrac{G_x(x)}{G(x)}, & \text{if} \quad \psi \neq 1. \end{cases} \tag{4.34}$$

We refer to $\eta^*(X_t)$ as the "investor hedging value of intertemporal uncertainty."

Remark 15. *Eq. (4.33) shows that the investor price $\hat{\lambda}^*(\mathbf{x}^*)$ of risk under the worst-case probability depends only on x and not on w^*. This result follows from the homotheticity of HREZ utility. Eq. (4.33) also shows that $\hat{\lambda}^*(\mathbf{x}^*)$ is the weighted average of the market price of risk and the investor hedging value of intertemporal uncertainty. The weights are the ratios of risk aversion to uncertainty aversion, and ambiguity aversion to uncertainty aversion, respectively. The optimal investor price of uncertainty converges to the market price of risk in the case of Epstein-Zin utility ($\theta \searrow 0$), and to the investor hedging value of intertemporal uncertainty as relative ambiguity aversion diverges to infinity.*

4.4 A Second Expression of the Optimal Robust Control

We obtain the following proposition.

Proposition 1. *Under Assumptions 1–3, the optimal wealth, consumption, and investment for the problem (4.13) satisfy Eqs. (4.35), (4.36), and (4.37),* *respectively.*

$$\bar{W}_t^* = \bar{W}_0 \exp\biggl(\int_0^t \Bigl(\bar{r}(X_s) + \hat{\varsigma}^*(X_s)'\bar{\lambda}_s - \frac{1}{2}|\hat{\varsigma}^*(X_s)|^2 - \check{c}(X_s)\Bigr)\, ds + \int_0^t \hat{\varsigma}^*(X_s)'\, dB_s\biggr), \quad (4.35)$$

$$\hat{c}^*(\mathbf{X}_t^*) = \begin{cases} \beta \bar{W}_t^*, & \text{if} \quad \psi = 1, \\ \dfrac{\beta^\psi}{G(X_t)} \bar{W}_t^*, & \text{if} \quad \psi \neq 1, \end{cases} \quad (4.36)$$

$$\hat{\varsigma}^*(\mathbf{X}_t^*) = \frac{1}{\gamma+\theta}\bar{\lambda}(X_t) + \left(1 - \frac{1}{\gamma+\theta}\right)\eta^*(X_t), \quad (4.37)$$

where G is a solution of the following PDE:

1. *The unit EIS case:*

$$\frac{1}{2}\mathrm{tr}\left[\frac{G_{xx}}{G}\right] + \frac{\theta}{2(\gamma-1)(\gamma+\theta)}\left|\frac{G_x}{G}\right|^2 - \left(\mathcal{K}x + \frac{\gamma+\theta-1}{\gamma+\theta}\bar{\lambda}(x)\right)'\frac{G_x}{G} - \beta \log G - \left(\frac{\gamma-1}{2(\gamma+\theta)}|\bar{\lambda}(x)|^2 + (\gamma-1)\bar{r}(x) + \beta(\log\beta - 1)(\gamma-1)\right) = 0. \quad (4.38)$$

2. *The general case:*

$$\frac{1}{2}\,\mathrm{tr}\left[\frac{G_{xx}}{G}\right] - \frac{\psi-(\gamma+\theta)^{-1}}{2(\psi-1)}\left|\frac{G_x}{G}\right|^2 - \Bigl(\mathcal{K}x + \bigl(1-(\gamma+\theta)^{-1}\bigr)\bar{\lambda}(x)\Bigr)'\frac{G_x}{G} + \frac{\beta^\psi}{G} + \frac{\psi-1}{2(\gamma+\theta)}|\bar{\lambda}(x)|^2 + (\psi-1)\,\bar{r}(x) - \beta\psi = 0. \quad (4.39)$$

Proof. See Appendix D. □

Remark 16. *Eq. (4.37) shows that the optimal investment is the weighted average of the market price of risk and the investor hedging value of intertemporal uncertainty. The weights are the relative risk tolerance and one minus the relative risk tolerance, respectively. Interestingly, the optimal investment and investor price of risk under the worst-case probability are both the weighted averages of the market price of risk and the investor hedging value of intertemporal uncertainty.*

We obtain the optimal robust portfolio using Eq. (4.12). The optimal robust portfolio is

$$\Phi_t^* = \frac{1}{\gamma+\theta}\Sigma(X_t)'^{-1}\lambda_t + \left(1-\frac{1}{\gamma+\theta}\right)\Sigma(X_t)'^{-1}\eta^*(X_t) + \left(1-\frac{1}{\gamma+\theta}\right)\Sigma(X_t)'^{-1}\sigma^p(X_t). \quad (4.40)$$

Remark 17. *The intertemporal marginal indirect utility hedging demand and intertemporal indirect utility hedging demand in Eq. (4.28) are integrated into the second term in Eq. (4.40). Hereafter, we refer to the second term in Eq. (4.40) as the "intertemporal uncertainty hedging demand."*

5 Robust CAPMs

In this section, we assume the general case, that is, $\psi \neq 1$ and derive the robust CAPMs and equilibrium risk-free rate.

5.1 Basic Setting

We make the following assumption.

Assumption 4. *In equilibrium, the following holds:*

1. *The representative agent's utility is the HREZ utility given by Eq. (4.1).*
2. *The representative agent invests in $S^1, \cdots, S^N$. Then, Φ_t and $\Sigma(X_t)$ are expressed as follows:*

$$\Phi_t = \begin{pmatrix} \Phi_t^1 \\ \vdots \\ \Phi_t^N \end{pmatrix}, \quad \Sigma(X_t) = \begin{pmatrix} (\sigma_1 + \Sigma_1 X_t)' \\ \vdots \\ (\sigma_N + \Sigma_N X_t)' \end{pmatrix}. \tag{5.1}$$

3. *The total dividends D^M of the market portfolio are given by*

$$D_t^M = \left(\delta_{0M} + \delta_M' X_t + \frac{1}{2} X_t' \Delta_M X_t\right) \exp\left(\sigma_{0M} t + \sigma_M' X_t + \frac{1}{2} X_t' \Sigma_M X_t\right). \tag{5.2}$$

5.2 Robust CAPMs

Let W and S^M denote the nominal wealth and market capitalization of the market portfolio, respectively. The representative agent's optimal portfolio becomes the market portfolio,that is, $W^* = S^M$ and the representative agent's optimal nominal consumption becomes the aggregate nominal consumption. This is equal to the total dividends of the market portfolio, that is, $p_t \hat{c}^*(\mathbf{X}_t^*) = D_t^M$. Furthermore, the investor price of risk under the worst-case probability becomes the "market price of risk under the worst-case probability." Let $\bar{\mu}_t^c$ and $\bar{\sigma}_t^c$ denote the expected growth rate and volatility of aggregate consumption, respectively. Let $\bar{S}^k = \dfrac{S^k}{p}$, $\bar{D}^k = \dfrac{D^k}{p}$, $\bar{\sigma}^M = \sigma^M - \sigma^p$ and $\bar{\sigma}^k = \sigma^k - \sigma^p$, where $k = 1, \cdots, N$.

In this section, the process $\bar{\sigma}^W(X_t)$, *etc.*, expressed as a function of the state, is abbreviated as $\bar{\sigma}_t^W$, *etc.* Further, $\hat{\varsigma}^*$ is referred to as $\bar{\sigma}^W$ because the optimal investment control is the volatility of $\bar{W}^*$. Note that from Eq. (4.36), the following equation holds.

$$\frac{G_x}{G} = \bar{\sigma}^W - \bar{\sigma}^c. \tag{5.3}$$

We obtain the following proposition.

Proposition 2. *Under Assumptions 1 and 4, the equilibrium real return rate on the k-th stock satisfies the following:*

Under the probability P*:*

$$\mathrm{E}_t\left[\frac{d\bar{S}_t^k + \bar{D}_t^k dt}{\bar{S}_t^k}\right] = \bar{r}_t dt + (\bar{\sigma}^k)'\bar{\lambda}_t dt, \tag{5.4}$$

where $\bar{\lambda}$ *and* $\bar{r}$ *are the equilibrium real market price of risk (MPR) and the equilibrium real risk-free rate, respectively, and* $(\bar{\lambda}, \bar{r})$ *is given by the following two models:*

1. *Robust ICAPM:*

$$\bar{\lambda}_t = (\gamma+\theta)\bar{\sigma}_t^W + \big(1-(\gamma+\theta)\big)\left(-\frac{1}{\psi-1}\frac{G_x}{G}\right), \tag{5.5}$$

$$\begin{aligned} \bar{r}_t = \frac{1}{1-\psi}\bigg\{ & -\beta\psi + \frac{\psi-1}{2(\gamma+\theta)}|\bar{\lambda}_t|^2 - \big(1-(\gamma+\theta)^{-1}\big)\bar{\lambda}_t'\frac{G_x}{G} \\ & + \frac{1}{2}\,\mathrm{tr}\left[\frac{G_{xx}}{G}\right] - \frac{\psi-(\gamma+\theta)^{-1}}{2(\psi-1)}\left|\frac{G_x}{G}\right|^2 - (\mathcal{K}X_t)'\frac{G_x}{G} + \frac{\beta^\psi}{G}\bigg\}. \end{aligned} \tag{5.6}$$

2. *Robust two-factor CAPM:*

$$\bar{\lambda}_t = -\frac{\gamma+\theta-1}{\psi-1}\bar{\sigma}_t^c + \frac{(\gamma+\theta)\psi-1}{\psi-1}\bar{\sigma}_t^M, \tag{5.7}$$

$$\bar{r}_t = \beta + \psi^{-1}\bar{\mu}_t^c + \frac{1}{2}\frac{(\gamma+\theta)\psi^{-1}-1}{\psi-1}|\bar{\sigma}_t^c|^2 - \frac{1}{2}\frac{(\gamma+\theta)\psi-1}{\psi-1}|\bar{\sigma}_t^M|^2. \tag{5.8}$$

Under the worst-case probability $\mathrm{P}^{\hat{\xi}^*}$*:*

$$\mathrm{E}_t^{\hat{\xi}^*}\left[\frac{d\bar{S}_t^k + \bar{D}_t^k dt}{\bar{S}_t^k}\right] = \bar{r}_t dt + (\bar{\sigma}_t^k)'\bar{\lambda}_t^* dt, \tag{5.9}$$

where $\bar{\lambda}_t^*$ *is the equilibrium real MPR under the worst-case probability, given by the following two models:*

1. *Robust ICAPM:*

$$\bar{\lambda}_t^* = \gamma\bar{\sigma}_t^W + (1-\gamma)\left(-\frac{1}{\psi-1}\frac{G_x}{G}\right). \tag{5.10}$$

2. Robust two-factor CAPM:

$$\bar{\lambda}_t^* = -\frac{\gamma - 1}{\psi - 1}\bar{\sigma}_t^c + \frac{\gamma\psi - 1}{\psi - 1}\bar{\sigma}_t^M. \tag{5.11}$$

Proof. See Appendix E.1. □

Remark 18. *Maenhout (2004) derives the CAPM based on HREZ utility. However, there is no state process in his security market model and we obtain* $\bar{\sigma}^c = \hat{\varsigma}^* = \bar{\sigma}^M$ *from Eq. (5.3) because* $G_x = 0$*. Thus, the two-factor CAPM (5.7) and the equilibrium risk-free rate (5.8) are simplified to the following single-factor CAPM.*[6] *as shown by Maenhout (2004).*

$$\bar{\lambda} = (\gamma + \theta)\bar{\sigma}^c, \tag{5.12}$$

$$\bar{r} = \beta + \psi^{-1}\bar{\mu}^c - \frac{1}{2}(\gamma + \theta)(\psi^{-1} + 1)|\bar{\sigma}^c|^2. \tag{5.13}$$

This demonstrates that the incorporation of the state process into the security market model, along with the generalization of utility, is essential not only in the consumption-investment problem, but also in the capital asset pricing problem.

5.3 Equity Premium Puzzle and Risk-free Rate Puzzle

We consider the the CAPM in the case of Epstein-Zin utility. Let $\bar{\lambda}^{\mathrm{EZ}}$ and $\bar{r}^{\mathrm{EZ}}$ denote the equilibrium MPR and the equilibrium real risk free rate in the case of Epstein-Zin utility (*i.e.*, $\theta = 0$). Then, Eqs. (5.7) and (5.8) are expressed as

$$\bar{\lambda}_t^{\mathrm{EZ}} = -\frac{\gamma - 1}{\psi - 1}\bar{\sigma}_t^c + \frac{\gamma\psi - 1}{\psi - 1}\bar{\sigma}_t^M, \tag{5.14}$$

$$\bar{r}_t^{\mathrm{EZ}} = \beta + \psi^{-1}\bar{\mu}_t^c + \frac{1}{2}\frac{\gamma\psi^{-1} - 1}{\psi - 1}|\bar{\sigma}_t^c|^2 - \frac{1}{2}\frac{\gamma\psi - 1}{\psi - 1}|\bar{\sigma}_t^M|^2. \tag{5.15}$$

Eq. (5.14) is the two-factor CAPM shown by Duffie and Epstein (1992b), which is a linear combination of the consumption-based CAPM and market portfolio-based CAPM. Note $\bar{\lambda}_t^{\mathrm{EZ}} = \bar{\lambda}_t^*$, that is, the equilibrium real MPR in the case of Epstein-Zin utility is equal to that under the worst-case probability in the case of HREZ utility. Weil (1989) shows that the CAPM beased on the Epstein-Zin utility cannot explain either the high market price of risk or the low-risk free rate observed in securities markets. From Eqs. (5.7), (5.14), (5.8), and (5.15), we obtain

$$\bar{\lambda}_t - \bar{\lambda}_t^{\mathrm{EZ}} = \frac{\theta}{\psi - 1}\big(\psi\bar{\sigma}_t^M - \bar{\sigma}_t^c\big), \tag{5.16}$$

$$\bar{r}_t - \bar{r}_t^{\mathrm{EZ}} = -\frac{\theta}{2(\psi - 1)}\big(\psi|\bar{\sigma}_t^M|^2 - \psi^{-1}|\bar{\sigma}_t^c|^2\big). \tag{5.17}$$

[6] Note that Maenhout (2004) does not consider inflation, so his CAPM equates nominal and real prices.

Given that the market portfolio return rate is more volatile than the growth rate of aggregate consumption, we assume $\bar{\sigma}^M > \bar{\sigma}^c$ and $|\bar{\sigma}^M|^2 > |\bar{\sigma}^c|^2$. Then, as $\psi > 1$, $\bar{\lambda}_t > \bar{\lambda}_t^{\mathrm{EZ}}$, and $\bar{r}_t < \bar{r}_t^{\mathrm{EZ}}$. Therefore, the robust CAPMs can contribute to solving both the equity premium puzzle (Mehra and Prescott (1985)) and the risk-free rate puzzle (Weil (1989)).

5.4 Robust CAPMs based on Nominal Prices

Typically, empirical analyses of CAPMs use data on nominal prices rather than real prices. Therefore, we demonstrate robust CAPMs based on nominal prices. Let $\sigma^W = \bar{\sigma}^W + \sigma^p$ and $\sigma^c = \bar{\sigma}^c + \sigma^p$. Let μ_t^c denote the nominal expected growth rate of aggreagte consumption. Then, note that the following equation holds.

$$\bar{\mu}^c = \mu^c - \mu^p - (\bar{\sigma}^c)'\sigma^p = \mu^c - \iota - (\sigma^c)'\sigma^p + |\sigma^p|^2. \tag{5.18}$$

Proposition 3. *Under Assumptions 1 and 4, the equilibrium nominal return rate on the k-th stock satisfies the following:*

Under the probability P*:*

$$\mathrm{E}_t\left[\frac{dS_t^k + D_t^k dt}{S_t^k}\right] = r_t dt + (\sigma_t^k)'\lambda_t dt, \tag{5.19}$$

where λ_t and r_t are the equilibrium nominal market price of risk and the equilibrium nominal risk-free rate, respectively, and (λ, r) is given by the following two models:

1. *Robust ICAPM with inflation-deflation factor:*

$$\lambda_t = (\gamma+\theta)\sigma_t^W + \big(1-(\gamma+\theta)\big)\left(-\frac{1}{\psi-1}\frac{G_x}{G}\right) + \big(1-(\gamma+\theta)\big)\sigma_t^p, \tag{5.20}$$

$$\begin{aligned} r_t = \mu_t^p - \lambda_t'\sigma_t^p + \frac{1}{1-\psi}\Bigg\{ &-\beta\psi + \frac{\psi-1}{2(\gamma+\theta)}|\lambda_t - \sigma_t^p|^2 \\ &- \big(1-(\gamma+\theta)^{-1}\big)(\lambda_t - \sigma_t^p)'\frac{G_x}{G} + \frac{1}{2}\mathrm{tr}\left[\frac{G_{xx}}{G}\right] \\ &- \frac{\psi-(\gamma+\theta)^{-1}}{2(\psi-1)}\left|\frac{G_x}{G}\right|^2 - (\mathcal{K}X_t)'\frac{G_x}{G} + \frac{\beta^\psi}{G}\Bigg\}. \end{aligned} \tag{5.21}$$

2. *Robust three-factor CAPM:*

$$\lambda_t = -\frac{\gamma+\theta-1}{\psi-1}\sigma_t^c + \frac{(\gamma+\theta)\psi-1}{\psi-1}\sigma_t^M + \big(1-(\gamma+\theta)\big)\sigma_t^p, \tag{5.22}$$

$$\begin{aligned} r_t = \beta + \psi^{-1}\mu_t^c &+ \frac{1}{2}\frac{(\gamma+\theta)\psi^{-1}-1}{\psi-1}|\sigma_t^c|^2 - \frac{1}{2}\frac{(\gamma+\theta)\psi-1}{\psi-1}|\sigma_t^M|^2 \\ + (1-\psi^{-1})\mu_t^p &- (\gamma+\theta-1)\psi^{-1}(\sigma_t^c)'\sigma_t^p + \frac{1}{2}(\gamma+\theta-2)(1-\psi^{-1})|\sigma_t^p|^2. \end{aligned} \tag{5.23}$$

Under the worst-case probability $\mathrm{P}^{\hat{\xi}^*}$*:*

$$\mathrm{E}_t^{\hat{\xi}^*}\left[\frac{dS_t^k + D_t^k dt}{S_t^k}\right] = r_t dt + (\sigma_t^k)' \lambda_t^* dt, \tag{5.24}$$

where λ_t^* *is the equilibrium nominal market price of risk under the worst-case probability, given by by the following two models:*

1. *Robust ICAPM with inflation-deflation factor:*

$$\lambda_t^* = \gamma \sigma_t^W + (1-\gamma)\left(-\frac{1}{\psi - 1}\frac{G_x}{G}\right) + (1-\gamma)\sigma_t^p. \tag{5.25}$$

2. *Robust three-factor CAPM:*

$$\lambda_t^* = -\frac{\gamma - 1}{\psi - 1}\sigma_t^c + \frac{\gamma\psi - 1}{\psi - 1}\sigma_t^M + (1-\gamma)\sigma_t^p. \tag{5.26}$$

Proof. See Appendix E.2. □

The equilibrium real risk-free rate (5.23) is decomposed into the sum of seven terms:the subjective discount rate, expected growth rate of aggregate consumption, variance of the growth rate of aggregate consumption, variance of the market portfolio return rate, expected inflation rate, covariance between the growth rate of aggregate consumption and the inflation rate, and variance of the inflation rate.

6 Analytical Expression of the Optimal Robust Consumption-Investment

First, for the unit EIS case, that is, $\psi = 1$, we derive the optimal solution. Second, for the general case, we derive an approximate optimal solution. Finally, we derive a testable ICAPM.

6.1 Optimal Solution for the Unit EIS Case

An analytical solution of the PDE (4.38) is expressed as:

$$G(x) = \exp\left(a_0 + a'x + \frac{1}{2}x'Ax\right), \tag{6.1}$$

where A is a symmetric matrix.

We obtain the following theorem.

Theorem 1. *Under Assumptions 1–3, the indirect utility function, optimal consumption, and optimal investment for problem (4.13) satisfy Eqs. (6.2), (4.36), and (6.3), respectively:*

$$J(\mathbf{X}_t^*) = \frac{\bar{W}_t^{*1-\gamma}}{1-\gamma} \exp\left(a_0 + a'X_t + \frac{1}{2}X_t'AX_t\right), \tag{6.2}$$

$$\hat{\varsigma}^*(X_t) = \frac{1}{\gamma+\theta}(\bar{\lambda} + \bar{\Lambda}X_t) + \left(1 - \frac{1}{\gamma+\theta}\right)\left(\frac{1}{\gamma-1}(a + AX_t)\right), \tag{6.3}$$

where (A, a, a_0) *is a solution of the simultaneous Eqs. (6.4)–(6.6):*

$$\frac{\gamma(\gamma+\theta-1)}{(\gamma-1)(\gamma+\theta)}A^2 - \left(\mathcal{K} + \frac{\gamma+\theta-1}{\gamma+\theta}\bar{\Lambda}\right)' A - A\left(\mathcal{K} + \frac{\gamma+\theta-1}{\gamma+\theta}\bar{\Lambda}\right) - (\gamma-1)\left(\frac{1}{\gamma+\theta}\bar{\Lambda}'\bar{\Lambda} + \bar{\mathcal{R}}\right) = 0, \tag{6.4}$$

$$\left(\frac{\gamma(\gamma+\theta-1)}{(\gamma-1)(\gamma+\theta)}A - \left(\mathcal{K} + \frac{\gamma+\theta-1}{\gamma+\theta}\bar{\Lambda}\right)\right)' a - \left(\frac{\gamma+\theta-1}{\gamma+\theta}A\bar{\lambda} + \frac{\gamma-1}{\gamma+\theta}\bar{\Lambda}'\bar{\lambda} + (\gamma-1)\bar{\rho}\right) = 0, \tag{6.5}$$

$$\beta a_0 = \frac{1}{2}\,\mathrm{tr}[A] + \frac{\gamma(\gamma+\theta-1)}{2(\gamma-1)(\gamma+\theta)}|a|^2 - \frac{\gamma+\theta-1}{\gamma+\theta}\bar{\lambda}'a - \frac{\gamma-1}{2\gamma(\gamma+\theta)}|\bar{\lambda}|^2 - \frac{\gamma-1}{\gamma}\bar{\rho}_0 - \frac{\beta(\gamma-1)}{\gamma}(\log\beta - 1). \tag{6.6}$$

Proof. See Appendix F.1. □

Eq. (4.36) indicates that the optimal consumption-wealth ratio is constant and independent of the state process, which is unrealistic. However, this result implies that the optimal consumption-wealth ratio is stable if the EIS does not deviate from one and the state process does not deviate significantly from zero. In the next section, we consider the general case in which a nonhomogeneous term appears in the PDE for indirect utility. Then, we use a loglinear approximation based on the stability of the optimal consumption-wealth ratio to derive an approximate solution.

6.2 Approximate Optimal Solution for the General Case

Next, for the general case, that is, $\psi \neq 1$, we derive an approximate optimal solution by applying the loglinear approximation method presented by Campbell and Viceira (2002) to our quadratic security market model. In the PDE (4.39), both nonlinear and nonhomogeneous terms appear. From Eq. (4.36), the nonhomogeneous term $\frac{\beta^\psi}{G}$ is expressed as $\frac{\beta^\psi}{G} = \frac{\hat{c}^*}{w}$. Considering that the optimal consumption-wealth ratio is stable, Campbell and

Viceira (2002) make a loglinear approximation of the nonhomogeneous term and derive an approximate analytical solution. We apply the loglinear approximation to the nonhomogeneous term as follows:

$$\frac{1}{G(x)} \approx g_0 - g_1 \log G(x), \tag{6.7}$$

where

$$\begin{aligned} g_0 &= g_1(1 - \log g_1), & (6.8)\\ g_1 &= \exp\Big(-\mathrm{E}\Big[\lim_{t\to\infty} \log G(X_t)\Big]\Big). & (6.9) \end{aligned}$$

In the PDE (4.39), approximating the nonhomogeneous term by Eq. (6.7) yields the following approximate PDE:

$$\frac{1}{2}\,\mathrm{tr}\left[\frac{G_{xx}}{G}\right] - \frac{\psi-(\gamma+\theta)^{-1}}{2(\psi-1)}\left|\frac{G_x}{G}\right|^2 - \Big(\mathcal{K}x + \big(1-(\gamma+\theta)^{-1}\big)\big(\bar{\lambda}+\bar{\Lambda}x\big)\Big)'\frac{G_x}{G}$$
$$-\beta^{\psi} g_1 \log G + \beta^{\psi} g_0 + \frac{\psi-1}{2(\gamma+\theta)}\left|\bar{\lambda}+\bar{\Lambda}x\right|^2 + (\psi-1)\left(\bar{\rho}_0 + \bar{\rho}'x + \frac{1}{2}x'\bar{\mathcal{R}}x\right) - \beta\psi = 0. \tag{6.10}$$

The optimal control based on the approximate PDE (6.10) is called the approximate optimal control and is denoted by $(\tilde{c}^*(\mathbf{X}_t^*), \tilde{\varsigma}^*(X_t))$. We obtain the following proposition.

Theorem 2. *Under Assumptions 1–3, the approximate optimal consumption and investment for problem (4.13) satisfy Eqs. (6.11) and (6.12), respectively:*

$$\tilde{c}^*(\mathbf{X}_t^*) = \tilde{W}_t^* \exp\left[-\left(a_0 + a'X_t + \frac{1}{2}X_t'AX_t\right)\right], \tag{6.11}$$

$$\tilde{\varsigma}^*(X_t) = \frac{1}{\gamma+\theta}\left(\bar{\lambda}+\bar{\Lambda}X_t\right) + \left(1-\frac{1}{\gamma+\theta}\right)\left(-\frac{1}{\psi-1}(a+AX_t)\right), \tag{6.12}$$

where (A, a, a_0) is a solution of the simultaneous Eqs. (6.13)–(6.15):

$$-\frac{1-(\gamma+\theta)^{-1}}{\psi-1}A^2 - \big(\mathcal{K}+\big(1-(\gamma+\theta)^{-1}\big)\bar{\Lambda}\big)'A - A\left(\mathcal{K}+\big(1-(\gamma+\theta)^{-1}\big)\bar{\Lambda}\right)$$
$$-\beta^{\psi} g_1 A + (\psi-1)\Big((\gamma+\theta)^{-1}\bar{\Lambda}'\bar{\Lambda} + \bar{\mathcal{R}}\Big) = 0, \tag{6.13}$$

$$-\frac{1-(\gamma+\theta)^{-1}}{\psi-1}Aa - \mathcal{K}'a - \big(1-(\gamma+\theta)^{-1}\big)(A\bar{\lambda}+\bar{\Lambda}'a)$$
$$-\beta^{\psi} g_1 a + (\psi-1)\Big((\gamma+\theta)^{-1}\bar{\Lambda}'\bar{\lambda} + \bar{\rho}\Big) = 0, \tag{6.14}$$

$$\frac{1}{2}\operatorname{tr}[A] - \frac{1-(\gamma+\theta)^{-1}}{2(\psi-1)}|a|^2 - \big(1-(\gamma+\theta)^{-1}\big)\bar{\lambda}'a$$

$$+ \beta^{\psi} g_1(1-a_0-\log g_1) + (\psi-1)\left(\frac{(\gamma+\theta)^{-1}}{2}|\bar{\lambda}|^2 + \bar{\rho}_0\right) - \beta\psi = 0, \quad (6.15)$$

where g_1 is expressed by .

$$g_1 = \exp\left(-a_0 - \frac{1}{2}\operatorname{tr}\left[\left(\mathcal{K}+\mathcal{K}'\right)^{-1}A\right]\right). \quad (6.16)$$

Furthermore, the approximate optimal portfolio $\tilde{\Phi}^*_t$ is given by

$$\tilde{\Phi}^*_t = \frac{1}{\gamma+\theta}\Sigma(X_t)'^{-1}\Big(\lambda+\Lambda X_t\Big)+\left(1-\frac{1}{\gamma+\theta}\right)\Sigma(X_t)'^{-1}\left(-\frac{1}{\psi-1}\right)\Big(a+AX_t\Big)$$
$$+\left(1-\frac{1}{\gamma+\theta}\right)\Sigma(X_t)'^{-1}\Big(\sigma_p+\Sigma_p X_t\Big). \quad (6.17)$$

Proof. See Appendix F.2. □

6.3 A Testable Robust Three-Factor ICAPM

The robust three-factor ICAPM (5.20) is untestable because it contains the unknown function G. Finally, from Eqs. (5.20) and (6.12), we obtain a testable robust three-factor ICAPM:

$$\lambda(X_t) = (\gamma+\theta)\sigma^M(X_t)+\big(1-(\gamma+\theta)\big)\Big(-\frac{1}{\psi-1}(a+AX_t)\Big)+\big(1-(\gamma+\theta)\big)\sigma^p(X_t). \quad (6.18)$$

7 Conclusion

We studied the consumption-investment problem based on HREZ utility under the quadratic security market model that satisfies the stylized facts. First, we proved that HREZ utility is SDU under certain integrability conditions by modifying the incorrect proof in Skiadas (2003). Then, following Duffie and Epstein (1992a), we showed that HREZ utility is continuous, consistent, strictly increasing, risk averse, and homothetic. Second, for the infinite-time consumption-investment problem for the infnite-lived investor with HREZ utility, we derived the conditional worst-case probability for a given consumption and investment. We compared the budget constraint under the conditional worst-case probability with that under the base probability. We found that the volatility of wealth is invariant, whereas the market price of risk in the return on wealth is replaced by the investor price of risk under the conditional worst-case probability discounted from the market price of risk. Given that the discount from the market price of risk is permanent, the implication is that investors with HREZ utility assume long-term stagnation rather than increased volatility as the worst-case scenario.

Third, we derived the optimal consumption and investment, both of which depend on the unknown function that comprises the indirect utility function. We demonstrated that the optimal investment is the weighted average of the market price of risk and the investor hedging value of intertemporal uncertainty. The weights are the relative risk tolerance and one minus the relative risk tolerance, respectively. We also demonstrated that the investor price of risk under the worst-case probability is the weighted average of the market price of risk and the investor hedging value of intertemporal uncertainty. The weights are the ratios of risk aversion to uncertainty aversion, and ambiguity aversion to uncertainty aversion, respectively. In addition, the optimal investment and investor price of risk under the worst-case probability are both the weighted averages of the market price of risk and the investor hedging value of intertemporal uncertainty.

Fourth, we derived robust versions of the intertemporal CAPM (ICAPM) based on Epstein-Zin utility and of the two-factor CAPM. We showed that the equilibrium market price of risk under the worst-case probability is consistent with the equilibrium market price of risk based on Epstein-Zin utility. Furthermore, we demonstrated that i) the equilibrium market price of risk based on HREZ utility is higher than that based on Epstein-Zin utility, and ii) the equilibrium risk-free rate based on HREZ utility is lower than that based on Epstein-Zin utility. Therefore, the robust CAPMs can contribute to solving both the equity premium puzzle and risk-free rate puzzles. Finally, we derived the exact solution of the nonlinear PDE for the unit EIS case and a loglinear approximate solution of the PDE for the general case. We then presented the approximate optimal portfolio and approximate testable ICAPM based on a loglinear approximate solution.

Cochrane (2001) presents two approaches to solving the equity premium and risk-free rate puzzles. One is to model uninsured idiosyncratic risk and market frictions, and the other is to modify the representative agent's utility. We believe that our quadratic security market model and HREZ utility are promising, because the quadratic model explains the stylized facts in the securities markets and HREZ utility incorporates Knightean uncertainty. To address both puzzles, we intend to incorporate the following into our model: i) uninsured idiosyncratic risk; ii) "near-singularity risk" in the variance-covariance matrix of asse returns that accounts for transaction costs; iii) nontradable human capital in wealth; iv) "homothetic generalized robust utility" in which relative ambiguity aversion depends on the source of risk. i), ii), and iii) are the first approach, and iv) is the second. Proposing such robust CAPMs that can solve both the equity premium puzzle and the risk-free rate puzzle is our future research.

A Quadratic Security Market Model

A.1 Regularity Conditions on Parameters

- $\mathcal{K} + \mathcal{K}'$ and $\bar{\mathcal{R}}$ are positive-definite.
- $(\rho_0, \rho, \mathcal{R})$ and $(\delta_{0j}, \delta_j, \mathit{\Delta}_j)$ satisfy[7]

$$\rho_0 = \frac{1}{2}\rho'\mathcal{R}^{-1}\rho, \tag{A.1}$$

$$\delta_{0k} = \frac{1}{2}\delta_k'\mathit{\Delta}_j^{-1}k\delta_j. \tag{A.2}$$

A.2 Parameters on Return Rates of Securities

1. The default-free bond with time τ to maturity: $(\Sigma(\tau), \sigma(\tau))$ in Eq. (2.9) is a solution to the following system of ODEs.

$$\frac{d\Sigma(\tau)}{d\tau} = \Sigma(\tau)^2 - (\mathcal{K} + \Lambda)'\Sigma(\tau) - \Sigma(\tau)(\mathcal{K} + \Lambda) - \mathcal{R} \tag{A.3}$$

$$\frac{d\sigma(\tau)}{d\tau} = -(\mathcal{K} + \Lambda - \Sigma(\tau))'\sigma(\tau) - (\Sigma(\tau)\lambda + \rho), \tag{A.4}$$

   with $(\Sigma, \sigma)(0) = (0, 0)$.

2. The default-free inflation-indexed bond with time τ to maturity: $(\Sigma(\tau), \sigma(\tau))$ in Eq. (2.10) is a solution to the following system of ODEs.

$$\frac{d\bar{\Sigma}_q(\tau)}{d\tau} = \bar{\Sigma}_q(\tau)^2 - (\mathcal{K} + \bar{\Lambda})'\bar{\Sigma}_q(\tau) - \bar{\Sigma}_q(\tau)(\mathcal{K} + \bar{\Lambda}) - \bar{\mathcal{R}}, \tag{A.5}$$

$$\frac{d\bar{\sigma}_q(\tau)}{d\tau} = -(\mathcal{K} + \bar{\Lambda} - \bar{\Sigma}_q(\tau))'\bar{\sigma}_q(\tau) - (\bar{\Sigma}_q(\tau)\bar{\lambda} + \bar{\rho}), \tag{A.6}$$

   with $(\bar{\Sigma}_q, \bar{\sigma}_q)(0) = (0, 0)$.

3. The k-th index and the market portfolio: In Eq. (2.11), Σ_k is a solution to Eq. (A.7) and σ_k is given by Eq. (A.8).

$$0 = \Sigma_k^2 - (\mathcal{K} + \Lambda)'\Sigma_k - \Sigma_k(\mathcal{K} + \Lambda) + \Delta_k - \mathcal{R}, \tag{A.7}$$

$$\sigma_k = (\mathcal{K} + \Lambda - \Sigma_k)'^{-1}(\delta_k - \rho - \Sigma_k\lambda), \tag{A.8}$$

B Derivatives of the Indirect Utility Function

The partial derivatives of J with respect to w are given by

$$\begin{aligned} wJ_w &= (1 - \gamma)J, \\ w^2 J_{ww} &= -\gamma(1 - \gamma)J. \end{aligned} \tag{B.1}$$

[7]Conditions (A.1) and (A.2) ensure that the nominal risk-free rate and divided are non-negative, respectively.

The partial derivatives of J with respect to x are given by

$$J_x = \begin{cases} J\dfrac{G_x}{G}, & \text{if} \quad \psi = 1, \\ \dfrac{1-\gamma}{\psi - 1} J \dfrac{G_x}{G}, & \text{if} \quad \psi \neq 1, \end{cases}$$

$$wJ_{xw} = \begin{cases} (1-\gamma) J\dfrac{G_x}{G}, & \text{if} \quad \psi = 1, \\ \dfrac{(1-\gamma)^2}{\psi - 1} J \dfrac{G_x}{G}, & \text{if} \quad \psi \neq 1, \end{cases} \tag{B.2}$$

$$J_{xx} = \begin{cases} J\left(\dfrac{G_x}{G}\dfrac{G_x'}{G} + \dfrac{G_{xx}}{G}\right), & \text{if} \quad \psi = 1, \\ \dfrac{1-\gamma}{\psi - 1} J \left(\dfrac{2-\gamma-\psi}{\psi-1}\dfrac{G_x}{G}\dfrac{G_x'}{G} + \dfrac{G_{xx}}{G}\right), & \text{if} \quad \psi \neq 1. \end{cases}$$

C Proof of Lemma 3

C.1 Proof for the Unit EIS Case

Assume $\psi = 1$. Substituting $f(c_t, J) = \beta(1-\gamma)J\log c_t - \beta v \log\big((1-\gamma)J\big)$ into the HJB Eq. (4.19) yields

$$\begin{aligned} \sup_{(\hat{c},\hat{\varsigma})\in\mathbb{R}_+\times\mathbb{R}^N} \Bigg[& J_t + \begin{pmatrix} w\left(\bar{r}(x) + \hat{\varsigma}'\bar{\lambda}(x)\right) - \hat{c} \\ -\mathcal{K}x \end{pmatrix}' \begin{pmatrix} J_w \\ J_x \end{pmatrix} \\ & + \frac{1}{2}\operatorname{tr}\left[\begin{pmatrix} w\hat{\varsigma}' \\ I \end{pmatrix}\begin{pmatrix} w\hat{\varsigma}' \\ I \end{pmatrix}'\begin{pmatrix} J_{ww} & J_{wx} \\ J_{xw} & J_{xx} \end{pmatrix}\right] - \frac{\theta}{2(1-\gamma)J}\left|\begin{pmatrix} w\hat{\varsigma}' \\ I \end{pmatrix}'\begin{pmatrix} J_w \\ J_x \end{pmatrix}\right|^2 \\ & + \beta(1-\gamma)J\log\hat{c} - \beta J\log\big((1-\gamma)J\big)\Bigg] = 0. \end{aligned} \tag{C.1}$$

It is evident that the optimal control $(\hat{c}^*, \hat{\varsigma}^*)$ in the HJB Eq. (C.1) satisfies Eqs. (4.23) and (4.24). The consumption-related terms in the HJB Eq. (C.1) are computed as

$$-\hat{c}^* J_w + \beta(1-\gamma)J\log\hat{c}^* - \beta J\log\big((1-\gamma)J\big) = \beta J\big\{(1-\gamma)(\log\hat{c}^* - 1) - \log\big((1-\gamma)J\big)\big\}. \tag{C.2}$$

The investment-related terms in the HJB Eq. (C.1) are computed as

$$w J_w \bar{\lambda}(x)' \hat{\varsigma}^* + \frac{1}{2} \operatorname{tr}\left[\begin{pmatrix} w(\hat{\varsigma}^*)' \\ I \end{pmatrix} \begin{pmatrix} w(\hat{\varsigma}^*)' \\ I \end{pmatrix}' \begin{pmatrix} J_{ww} & J_{wx} \\ J_{xw} & J_{xx} \end{pmatrix} \right] - \frac{\theta}{2(1-\gamma)J} \left| \begin{pmatrix} w(\hat{\varsigma}^*)' \\ I \end{pmatrix}' \begin{pmatrix} J_w \\ J_x \end{pmatrix} \right|^2$$

$$= \frac{1}{2} \operatorname{tr}[J_{xx}] - \frac{\theta}{2(1-\gamma)J} |J_x|^2 - \frac{|\pi(x)|^2}{2w^2 \left(J_{ww} - \dfrac{\theta J_w^2}{(1-\gamma)J} \right)}, \quad \text{(C.3)}$$

where $\pi(x)$ is given by Eq. (4.26).

Substituting the optimal control (4.23) and (4.24) into the HJB Eq. (C.1) and using Eqs. (C.2) and (C.3), we obtain the PDE (4.25).

C.2 Proof for the General Case

Assume $\psi \neq 1$. Substituting $f(\hat{c}, J) = \frac{\beta}{1-\psi^{-1}} \hat{c}_t^{1-\psi^{-1}} \left((1-\gamma)J\right)^{1-\frac{1-\psi^{-1}}{1-\gamma}} - \frac{\beta(1-\gamma)}{1-\psi^{-1}} J$ into the HJB Eq. (4.19) yields

$$\sup_{(\hat{c},\hat{\varsigma}) \in \mathbb{R}_+ \times \mathbb{R}^N} \left[\begin{pmatrix} w\left(\bar{r}_t + \hat{\varsigma}' \bar{\lambda}_t\right) - \hat{c} \\ -\mathcal{K} x \end{pmatrix}' \begin{pmatrix} J_w \\ J_x \end{pmatrix} + \frac{1}{2} \operatorname{tr}\left[\begin{pmatrix} w\hat{\varsigma}' \\ I \end{pmatrix} \begin{pmatrix} w\hat{\varsigma}' \\ I \end{pmatrix}' \begin{pmatrix} J_{ww} & J_{wx} \\ J_{xw} & J_{xx} \end{pmatrix} \right] - \frac{\theta}{2(1-\gamma)J} \left| \begin{pmatrix} w\hat{\varsigma}' \\ I \end{pmatrix}' \begin{pmatrix} J_w \\ J_x \end{pmatrix} \right|^2 + \frac{\beta}{1-\psi^{-1}} \hat{c}^{1-\psi^{-1}} \left((1-\gamma)J\right)^{1-\frac{1-\psi^{-1}}{1-\gamma}} - \frac{\beta(1-\gamma)}{1-\psi^{-1}} J \right] = 0. \quad \text{(C.4)}$$

The optimal control $(\hat{c}^*, \hat{\varsigma}^*)$ in the HJB Eq. (C.4) satisfies Eqs. (4.23) and (4.24). The consumption-related terms in the HJB Eq. (C.4) are computed as

$$-\hat{c}^* J_w + f(\hat{c}^*, J) = \hat{c}^* \left(-J_w + \frac{1}{1-\psi^{-1}} J_w \right) - \frac{\beta(1-\gamma)}{1-\psi^{-1}} J = \frac{1}{\psi - 1} \hat{c}^* J_w - \frac{\beta(1-\gamma)}{1-\psi^{-1}} J. \quad \text{(C.5)}$$

The investment-related terms in the HJB Eq. (C.4) are computed as

$$
\begin{aligned}
&wJ_w\bar{\lambda}(x)'\hat{\varsigma}^* + \frac{1}{2}\operatorname{tr}\left[\begin{pmatrix} w(\hat{\varsigma}^*)' \\ I \end{pmatrix}\begin{pmatrix} w(\hat{\varsigma}^*)' \\ I \end{pmatrix}'\begin{pmatrix} J_{ww} & J_{wx} \\ J_{xw} & J_{xx} \end{pmatrix}\right] \\
&\qquad\qquad - \frac{\theta}{2(1-\gamma)J}\left|\begin{pmatrix} w(\hat{\varsigma}^*)' \\ I \end{pmatrix}'\begin{pmatrix} J_w \\ J_x \end{pmatrix}\right|^2 \\
&= \frac{1}{2}\operatorname{tr}[J_{xx}] - \frac{\theta}{2(1-\gamma)J}|J_x|^2 - \left(w^2J_{ww} - \frac{\theta(wJ_w)^2}{(1-\gamma)J}\right)^{-1}|\pi(x)|^2, \quad \text{(C.6)}
\end{aligned}
$$

where $\pi(x)$ is given by Eq. (4.26).

Substituting the optimal control (4.23) and (4.24) into the HJB Eq. (C.4) and using Eqs. (C.5) and (C.6), we obtain the PDE (4.25).

D Proof of Proposition 1

D.1 Proof for the Unit EIS Case

Assume $\psi \neq 1$. The optimal consumption (4.36) immediately follows from Eq. (4.23). Eq. (4.26) is rewritten as

$$
\pi(x) = J\left((\gamma-1)\bar{\lambda}(x) + (\gamma+\theta-1)\frac{G_x(x)}{G(x)}\right). \quad \text{(D.1)}
$$

Inserting Eqs. (4.32) and the derivatives of J into Eq. (4.24), we obtain the optimal investment (4.37). From Eq. (D.1) and the derivatives of J, the first to third terms in the PDE (4.25) are calculated as

$$
\begin{aligned}
&\frac{1}{2}\operatorname{tr}[J_{xx}] - \frac{\theta}{2(1-\gamma)J}|J_x|^2 - \frac{|\pi(x)|^2}{2w^2\left(J_{ww} - \dfrac{\theta J_w^2}{(1-\gamma)J}\right)} \\
=&\frac{1}{2}J\left(\operatorname{tr}\left[\frac{G_x}{G}\frac{G_x'}{G} + \frac{G_{xx}}{G}\right] + \frac{\theta}{\gamma-1}\left|\frac{G_x}{G}\right|^2 - \frac{1}{(\gamma-1)(\gamma+\theta)}\left|(\gamma-1)\bar{\lambda}(x) + (\gamma+\theta-1)\frac{G_x}{G}\right|^2\right) \\
=&\frac{1}{2}J\left(\operatorname{tr}\left[\frac{G_{xx}}{G}\right] - \frac{\gamma-1}{\gamma+\theta}|\bar{\lambda}(x)|^2 - \frac{2(\gamma+\theta-1)}{\gamma+\theta}\bar{\lambda}(x)'\frac{G_x}{G} + \left(1 + \frac{\theta}{\gamma-1} - \frac{(\gamma+\theta-1)^2}{(\gamma-1)(\gamma+\theta)}\right)\left|\frac{G_x}{G}\right|^2\right) \\
=&J\left(\frac{1}{2}\operatorname{tr}\left[\frac{G_{xx}}{G}\right] - \frac{\gamma-1}{2(\gamma+\theta)}|\bar{\lambda}(x)|^2 - \frac{\gamma+\theta-1}{\gamma+\theta}\bar{\lambda}(x)'\frac{G_x}{G} + \frac{\theta}{2(\gamma-1)(\gamma+\theta)}\left|\frac{G_x}{G}\right|^2\right).
\end{aligned}
\quad \text{(D.2)}
$$

The fourth and fifth terms in the PDE (4.25) are computed as

$$
\bar{r}(x)wJ_w - (\mathcal{K}x)'J_x = J\left(-(\gamma-1)\bar{r}(x) - (\mathcal{K}x)'\frac{G_x}{G}\right). \quad \text{(D.3)}
$$

The sixth term in the PDE (4.25) is calculated from Eq. (4.36) as

$$\begin{aligned}
&\beta J\big\{(1-\gamma)(\log \hat{c}^* - 1) - \log\big((1-\gamma)J\big)\big\} \\
=&\beta J\big\{(1-\gamma)(\log\beta + \log w - 1) - \big((1-\gamma)\log w + \log G\big)\big\} \\
=&\beta J\big((1-\gamma)(\log\beta - 1) - \log G\big).
\end{aligned} \tag{D.4}$$

Substituting Eqs. (D.2)–(D.4) into the PDE (4.25) and dividing by J yields the PDE (4.38).

D.2 Proof for the General Case

Assume $\psi \neq 1$. From Eq. (4.23), the optimal consumption (4.36) is calculated as

$$\hat{c}^* = \beta^\psi \left(\frac{(1-\gamma)J}{w}\right)^{-\psi} \big((1-\gamma)J\big)^{\frac{\gamma\psi-1}{\gamma-1}} = \beta^\psi w^\psi \left(w^{1-\gamma} G^{\frac{1-\gamma}{\psi-1}}\right)^{\frac{\psi-1}{\gamma-1}} = \beta^\psi \frac{w}{G}. \tag{D.5}$$

Eq. (4.26) is rewritten as

$$\pi(x) = (\gamma-1)J\left(\bar{\lambda}(x) + \frac{\gamma+\theta-1}{1-\psi}\frac{G_x(x)}{G(x)}\right). \tag{D.6}$$

Inserting Eq. (4.32) and the derivatives of J into Eq. (4.24), we obtain the optimal investment (4.37). From Eq. (D.6) and the derivatives of J, the first to third terms in the PDE (4.25) are calculated as

$$\begin{aligned}
&\frac{1}{2}\operatorname{tr}[J_{xx}] - \frac{\theta}{2(1-\gamma)J}|J_x|^2 - \frac{1}{2}\left(w^2 J_{ww} - \frac{\theta(wJ_w)^2}{(1-\gamma)J}\right)^{-1}|\pi(x)|^2 \\
=&J\Bigg\{\frac{1-\gamma}{2(\psi-1)}\operatorname{tr}\left[\frac{2-\gamma-\psi}{\psi-1}\frac{G_x}{G}\frac{G_x'}{G} + \frac{G_{xx}}{G}\right] - \frac{(1-\gamma)\theta}{2(\psi-1)^2}\left|\frac{G_x}{G}\right|^2 \\
&+ \frac{1-\gamma}{2(\psi-1)^2(\gamma+\theta)}\left|(\psi-1)\bar{\lambda}(x) - (\gamma+\theta-1)\frac{G_x}{G}\right|^2\Bigg\} \\
=&\frac{1-\gamma}{\psi-1}J\Bigg\{\frac{1}{2}\operatorname{tr}\left[\frac{2-\gamma-\psi}{\psi-1}\frac{G_x}{G}\frac{G_x'}{G} + \frac{G_{xx}}{G}\right] - \frac{\theta}{2(\psi-1)}\left|\frac{G_x}{G}\right|^2 \\
&+ \frac{1}{2(\psi-1)(\gamma+\theta)}\left|(\psi-1)\bar{\lambda}(x) - (\gamma+\theta-1)\frac{G_x}{G}\right|^2\Bigg\} \\
=&\frac{1-\gamma}{\psi-1}J\Bigg\{\frac{1}{2}\operatorname{tr}\left[\frac{G_{xx}}{G}\right] + \frac{\psi-1}{2(\gamma+\theta)}|\bar{\lambda}(x)|^2 - \big(1-(\gamma+\theta)^{-1}\big)\bar{\lambda}(x)'\frac{G_x}{G} \\
&- \frac{1}{2(\psi-1)}\Big(\gamma+\psi-2+\theta - \big(1-(\gamma+\theta)^{-1}\big)(\gamma+\theta-1)\Big)\left|\frac{G_x}{G}\right|^2\Bigg\} \\
=&\frac{1-\gamma}{\psi-1}J\Bigg\{\frac{1}{2}\operatorname{tr}\left[\frac{G_{xx}}{G}\right] + \frac{\psi-1}{2}(\gamma+\theta)^{-1}|\bar{\lambda}(x)|^2 - \big(1-(\gamma+\theta)^{-1}\big)\bar{\lambda}(x)'\frac{G_x}{G} \\
&- \frac{1}{2(\psi-1)}\big(\psi-(\gamma+\theta)^{-1}\big)\left|\frac{G_x}{G}\right|^2\Bigg\}.
\end{aligned} \tag{D.7}$$

The fourth and fifth terms in the PDE (4.25) are computed as follows:

$$\bar{r}(x)wJ_w - (\mathcal{K}x)'J_x = \frac{1-\gamma}{\psi-1}J\left(-\frac{G_\tau}{G} + (\psi-1)\bar{r}(x) - (\mathcal{K}w)'\frac{G_x}{G}\right). \quad \text{(D.8)}$$

The sixth and seventh terms in the PDE (4.25) are calculated from Eq. (4.36) as follows:

$$\frac{1}{\psi-1}\hat{c}^* J_w - \frac{\beta(1-\gamma)}{1-\psi^{-1}}J = \frac{1}{\psi-1}\left(\beta^\psi \frac{w}{G}\frac{(1-\gamma)J}{w} + \beta(\gamma-1)\psi J\right) = \frac{1-\gamma}{\psi-1}J\left(\frac{\beta^\psi}{G} - \beta\psi\right). \quad \text{(D.9)}$$

Substituting Eqs. (D.7)–(D.9) into the PDE (4.25) and dividing by $\frac{1-\gamma}{\psi-1}J$ yields the PDE (4.39).

E Proof of Propositions 2 and 3

E.1 Proof of Proposition 2

Eq. (5.5) immediately follows from Eq. (4.37). From Lemma 1, we obtain

$$dS_t^M + D_t^M dt = S_t^M\big(r_t + (\sigma_t^M)'\lambda_t\big)\,dt + S_t^M(\sigma_t^M)'\,dB_t, \quad \text{(E.1)}$$

where $\sigma_t^M = \sigma_M + \Sigma_M X_t$, and Σ_M is a solution to Eq. (E.2) and σ_M is given by Eq. (E.3).

$$0 = \Sigma_M^2 - (\mathcal{K}+\Lambda)'\Sigma_M - \Sigma_M(\mathcal{K}+\Lambda) + \Delta_M - \mathcal{R}, \quad \text{(E.2)}$$

$$\sigma_M = (\mathcal{K}+\Lambda-\Sigma_M)'^{-1}(\delta_M - \rho - \Sigma_M\lambda). \quad \text{(E.3)}$$

The representative agent's nominal budget constraint is

$$dW_t^* = W_t^*\left(r_t + \big(\hat{\varsigma}_t + \sigma_t^p\big)'\lambda_t - \frac{p_t\hat{c}^*(\mathbf{X}_t^*)}{W_t^*}\right)dt + W_t^*\big(\hat{\varsigma}_t + \sigma_t^p\big)'\,dB_t. \quad \text{(E.4)}$$

Given that $W_t^* = S_t^M$ and $p_t\hat{c}^*(\mathbf{X}_t^*) = D_t^M$, from Eq. (E.1), the representative agent's nominal budget constraint is also expressed as

$$\begin{aligned} dW_t^* &= dS_t^M + \big(D_t^M - p_t c_t^*\big)dt \\ &= W_t^*\left(r_t + \big(\bar{\sigma}_t^M + \sigma_t^p\big)'\lambda_t\right)dt + W_t^*\big(\bar{\sigma}_t^M + \sigma_t^p\big)'\,dB_t. \end{aligned} \quad \text{(E.5)}$$

Eqs. (E.4) and (E.5) show that $\hat{\varsigma}^*(X_t) = \bar{\sigma}_t^M$. Thus, $\bar{\sigma}_t^W = \bar{\sigma}_t^M$. Inserting $\bar{\sigma}_t^W = \bar{\sigma}_t^M$ into Eq. (5.3) yields

$$\frac{G_x}{G} = \bar{\sigma}^M - \bar{\sigma}^c. \quad \text{(E.6)}$$

Substituting Eq. (E.6) into Eq. (5.5), we obtain Eq. (5.7). Inserting Eq. (5.5) into Eq. (4.33) yields Eq. (5.10). Substituting Eq. (E.6) and $\bar{\sigma}_t^W = \bar{\sigma}_t^M$ into Eq. (5.10) yields Eq (5.11). Applying Ito's lemma to $\bar{W}^* = \beta^{-\psi} c^* G$ yields

$$\frac{d\bar{W}^*}{\bar{W}^*} = \frac{dc^*}{c^*} + \left(\frac{G_x}{G}\right)'(-\mathcal{K}X_t dt + dB_t) + \frac{1}{2}\mathrm{tr}\left[\frac{G_{xx}}{G}\right]dt + \left(\frac{dc^*}{c^*}\right)'\frac{dG}{G}. \quad \text{(E.7)}$$

Thus, from the real budget constraint equation (4.11) and Eq. (E.6), we obtain

$$\bar{r}_t + \bar{\lambda}_t' \bar{\sigma}_t^M - \frac{\beta^\psi}{G} = \bar{\mu}_t^c + \frac{1}{2}\mathrm{tr}\left[\frac{G_{xx}}{G}\right] - \left(\frac{G_x}{G}\right)' \mathcal{K}x + (\bar{\sigma}_r^c)'(\bar{\sigma}_t^M - \bar{\sigma}_t^c). \quad \text{(E.8)}$$

Using the PDE (4.39), we obtain

$$\begin{aligned} \bar{r}_t &= \bar{\mu}_t^c + (\bar{\sigma}_t^c)'(\bar{\sigma}_t^M - \bar{\sigma}_t^c) - \bar{\lambda}_t'\bar{\sigma}_t^M + \frac{1}{2}\mathrm{tr}\left[\frac{G_{xx}}{G}\right] - \left(\frac{G_x}{G}\right)'\mathcal{K}x + \frac{\beta^\psi}{G} \\ &= \bar{\mu}_t^c + (\bar{\sigma}_t^c)'(\bar{\sigma}_t^M - \bar{\sigma}_t^c) - \bar{\lambda}_t'\bar{\sigma}_t^M + \frac{\psi - (\gamma+\theta)^{-1}}{2(\psi-1)}\left|\frac{G_x}{G}\right|^2 \\ &\quad + \big(1 - (\gamma+\theta)^{-1}\big)\bar{\lambda}_t'\frac{G_x}{G} - \frac{\psi-1}{2(\gamma+\theta)}|\bar{\lambda}_t|^2 - (\psi-1)\bar{r}_t + \beta\psi. \end{aligned} \quad \text{(E.9)}$$

The above equation is rewritten as

$$\begin{aligned} \psi\bar{r}_t = \beta\psi + \bar{\mu}_t^c + (\bar{\sigma}_t^c)'(\bar{\sigma}_t^M - \bar{\sigma}_t^c) - \bar{\lambda}_t'\bar{\sigma}_t^M + \frac{(\gamma+\theta)\psi - 1}{2(\gamma+\theta)(\psi-1)}\left|\frac{G_x}{G}\right|^2 \\ + \frac{\gamma+\theta-1}{\gamma+\theta}\bar{\lambda}_t'\frac{G_x}{G} - \frac{\psi-1}{2(\gamma+\theta)}|\bar{\lambda}_t|^2 \end{aligned} \quad \text{(E.10)}$$

Substituting Eqs. (E.6) and (5.7) into the above equation yields

$$\bar{r}_t = \beta + \psi^{-1}\bar{\mu}_t^c + \frac{\psi^{-1}}{2(\gamma+\theta)(\psi-1)}\bar{h}(\bar{\sigma}_t^c, \bar{\sigma}_t^M), \quad \text{(E.11)}$$

where

$$\begin{aligned} \bar{h}(\bar{\sigma}_t^c, \bar{\sigma}_t^M) &= 2(\gamma+\theta)\Big\{(\psi-1)(\bar{\sigma}_t^c)'(\bar{\sigma}_t^M - \bar{\sigma}_t^c) - \Big(\big((\gamma+\theta)\psi-1\big)\bar{\sigma}_t^M - (\gamma+\theta-1)\bar{\sigma}_t^c\Big)'\bar{\sigma}_t^M\Big\} \\ &+ \big((\gamma+\theta)\psi-1\big)|\bar{\sigma}_t^M - \bar{\sigma}_t^c|^2 + 2(\gamma+\theta-1)\Big(\big((\gamma+\theta)\psi-1\big)\bar{\sigma}_t^M - (\gamma+\theta-1)\bar{\sigma}_t^c\Big)'(\bar{\sigma}_t^M - \bar{\sigma}_t^c) \\ &- \Big|\big((\gamma+\theta)\psi-1\big)\bar{\sigma}_t^M - (\gamma+\theta-1)\bar{\sigma}_t^c\Big|^2. \end{aligned} \quad \text{(E.12)}$$

Then, $\bar{h}(\bar{\sigma}_t^c, \bar{\sigma}_t^M)$ is calculated as

$$\bar{h}(\bar{\sigma}_t^c, \bar{\sigma}_t^M) = (\gamma+\theta)\Big\{(\gamma+\theta-\psi)|\bar{\sigma}_t^c|^2 - \big((\gamma+\theta)\psi-1\big)|\bar{\sigma}_t^M|^2\Big\}. \quad \text{(E.13)}$$

Substituting the above equation into Eq. (E.11), we obtain Eq. (5.8).

E.2 Proof of Proposition 3

Substituting Eq. (5.5) and $\bar{\sigma}^W = \sigma^W - \sigma^p$ into $\lambda = \bar{\lambda} + \sigma^p$ yields Eq. (5.20). From Eq. (E.6), the following equation holds.

$$\frac{G_x}{G} = \sigma^M - \sigma^c. \tag{E.14}$$

Inserting Eq. (E.14) and $\sigma^W = \sigma^M$ yields Eqs. (5.22). In the same way, we obtain Eqs. (5.25) and (5.26). For the equilibrium nominal risk-free rate, substituting Eqs. (5.18), $\bar{\sigma}^c = \sigma^c - \sigma^p$, and $\bar{\sigma}^M = \sigma^M - \sigma^p$ into Eq. (4.5) yields

$$\begin{aligned} r_t = \beta + \psi^{-1}\big(\mu_t^c - \mu_t^p - (\sigma_t^c)'\sigma_t^p + |\sigma_t^p|^2\big) + \frac{1}{2}\frac{(\gamma+\theta)\psi^{-1} - 1}{\psi - 1}|\sigma_t^c - \sigma_t^p|^2 \\ - \frac{1}{2}\frac{\gamma + \theta - \psi^{-1}}{1 - \psi^{-1}}|\sigma_t^M - \sigma_t^p|^2 + \mu_t^p - \lambda_t'\sigma_t^p. \end{aligned} \tag{E.15}$$

Subtituting Eq. (5.22) into the above equation, we obtain

$$r_t = \beta + \psi^{-1}\mu_t^c + (1 - \psi^{-1})\mu_t^p + \frac{1}{\psi - 1}h(\sigma_t^c, \sigma_t^M, \sigma_t^p), \tag{E.16}$$

where

$$\begin{aligned} h(\sigma_t^c, \sigma_t^M, \sigma_t^p) = (\psi^{-1} - 1)\big((\sigma_t^c)'\sigma_t^p - |\sigma_t^p|^2\big) \\ + \frac{1}{2}\big((\gamma+\theta)\psi^{-1} - 1\big)|\sigma_t^c - \sigma_t^p|^2 - \frac{1}{2}\big((\gamma+\theta)\psi - 1\big)|\sigma_t^M - \sigma_t^p|^2 \\ + (\gamma+\theta-1)(\sigma_t^c)'\sigma_t^p - \big((\gamma+\theta)\psi - 1\big)(\sigma^M)'\sigma_t^p + (\gamma+\theta-1)(\psi-1)|\sigma_t^p|^2. \end{aligned} \tag{E.17}$$

Then, $h(\sigma_t^c, \sigma_t^M, \sigma_t^p)$ is calculated as

$$\begin{aligned} h(\sigma_t^c, \sigma_t^M, \sigma_t^p) = \frac{1}{2}\big((\gamma+\theta)\psi^{-1} - 1\big)|\sigma_t^c|^2 - \frac{1}{2}\big((\gamma+\theta)\psi - 1\big)|\sigma_t^M|^2 \\ - (\gamma+\theta-1)\psi^{-1}(\psi-1)(\sigma_t^c)'\sigma_t^p + \frac{1}{2}(\gamma+\theta-2)(1-\psi^{-1})(\psi-1)|\sigma_t^p|^2. \end{aligned} \tag{E.18}$$

Substituting the above equation into Eq. (E.16), we obtain Eq. (5.23).

F Proof of Theorems

F.1 Proof of Theorem 1

Inserting $G_x = (a + Ax)G$ into Eq. (4.34) yields

$$\eta^*(x) = -\frac{1}{\psi - 1}(a + Ax). \tag{F.1}$$

By substituting Eq. (F.1) into (4.37), we obtain the optimal investment (6.3). Substituting Eqs. (4.4), (4.5), (6.1) and derivatives of G into the PDE (4.38) and noting $A' = A$ and $x'\big(\mathcal{K} + \big(1-(\gamma+\theta)^{-1}\big)\bar{\Lambda}\big)' Ax = x'A\big(\mathcal{K} + \big(1-(\gamma+\theta)^{-1}\big)\bar{\Lambda}\big)x$, we obtain

$$\begin{aligned}
&\frac{1}{2}\mathrm{tr}[A] + \frac{1}{2}\left(1 + \frac{\theta}{(\gamma-1)(\gamma+\theta)}\right)\left(|a|^2 + 2a'Ax + x'A^2x\right) \\
&\quad - \left\{\frac{\gamma+\theta-1}{\gamma+\theta}\bar{\lambda} + \left(\mathcal{K} + \frac{\gamma+\theta-1}{\gamma+\theta}\bar{\Lambda}\right)x\right\}' a - \frac{\gamma+\theta-1}{\gamma+\theta}\bar{\lambda}' Ax \\
&\quad - \frac{1}{2}x'\left(\mathcal{K} + \frac{\gamma+\theta-1}{\gamma+\theta}\bar{\Lambda}\right)' Ax - \frac{1}{2}x'A\left(\mathcal{K} + \frac{\gamma+\theta-1}{\gamma+\theta}\bar{\Lambda}\right)x \\
&\quad - \beta\left(a_0 + a'x + \frac{1}{2}x'Ax\right) - \frac{\gamma-1}{2(\gamma+\theta)}\left(|\bar{\lambda}|^2 + 2\bar{\lambda}'\bar{\Lambda}x + x'\bar{\Lambda}'\bar{\Lambda}x\right) \\
&\qquad - (\gamma-1)\left(\bar{\rho}_0 + \bar{\rho}'x + \frac{1}{2}x'\bar{\mathcal{R}}x\right) - \beta(\log\beta - 1)(\gamma-1) = 0. \qquad \text{(F.2)}
\end{aligned}$$

As Eq. (F.2) is identical on x, we obtain Eqs. (6.4)–(6.6).

F.2 Proof of Theorem 2

An analytical solution of the PDE (6.10) is expressed as Eq. (6.1). Substituting Eq. (6.1) into Eq. (4.36) yields the optimal consumption (6.11). By substituting Eq. (F.1) into (4.37), we obtain the optimal investment (6.12). Substituting Eq. (6.1) into Eq. (6.9) yields

$$g_1 = \exp\left(\left[-a_0 - a'\mathrm{E}[\lim_{t\to\infty} X_t] - \frac{1}{2}\mathrm{E}[\lim_{t\to\infty} X_t' A X_t]\right]\right). \qquad \text{(F.3)}$$

As Eq. (2.1) is transformed into $d\big(e^{t\mathcal{K}}X_t\big) = e^{t\mathcal{K}}\,dB_t$, X_t is solved as $X_t = e^{-t\mathcal{K}}X_0 + \int_0^t e^{(s-t)\mathcal{K}}\,dB_s$. Hence, the stationary distribution of the state vector is $N(0, (\mathcal{K}+\mathcal{K}')^{-1})$. Thus, g_1 is calculated as Eq. (6.16). Substituting

G and its derivatives into the PDE (6.10) yields

$$\frac{1}{2}\operatorname{tr}\left[aa' + A + ax'A + Axa' + Axx'A\right] - \frac{\psi - (\gamma+\theta)^{-1}}{2(\psi-1)}\left(a' + x'A\right)(a + Ax)$$
$$- \left\{\left(1-(\gamma+\theta)^{-1}\right)\bar{\lambda} + \left(\mathcal{K} + \left(1-(\gamma+\theta)^{-1}\right)\bar{\Lambda}\right)x\right\}' a - \left(1-(\gamma+\theta)^{-1}\right)\bar{\lambda}'Ax$$
$$- \frac{1}{2}x'\left(\mathcal{K} + \left(1-(\gamma+\theta)^{-1}\right)\bar{\Lambda}\right)' Ax - \frac{1}{2}x'A\left(\mathcal{K} + \left(1-(\gamma+\theta)^{-1}\right)\bar{\Lambda}\right)X_t$$
$$- \beta^{\psi} g_1 \left(\log g_1 - 1 + a_0 + a'x + \frac{1}{2}x'Ax\right)$$
$$+ \frac{(\psi-1)(\gamma+\theta)^{-1}}{2}\left(|\bar{\lambda}|^2 + 2\bar{\lambda}'\bar{\Lambda}x + x'\bar{\Lambda}'\bar{\Lambda}x\right)$$
$$+ (\psi - 1)\left(\bar{\rho}_0 + \bar{\rho}'x + \frac{1}{2}x'\bar{\mathcal{R}}x\right) - \beta\psi = 0. \quad \text{(F.4)}$$

Therefore, we obtain Eqs. (6.13)–(6.15).

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Statements and Declarations

Competing Interests

The authors declare no competing interests relevant to the contents of this article.

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