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ITERATIVE SCHEME GENERATING METHOD BEYOND ISHIKAWA ITERATIVE METHOD

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Abstract. We propose an iterative scheme generating method to address common fixed point problems. Our approach yields diverse iterative schemes for finding common fixed points. The derivative results include the Ishikawa iterative method and its variations. An application to the variational inequality problem is provided to illustrate the usefulness of our method. The class of mappings we target is general. This category includes nonexpansive mappings and various other types, even those that lack continuity.

1. Introduction

Let H be denote a real Hilbert space with an inner product $\langle \cdot,\ \cdot \rangle$. The norm $\|\cdot\|$ in H is defined by $\|x\| = \sqrt{\langle x,\ x \rangle}$. Consider a mapping S from C into H, where C is a nonempty subset of H. Following convention, we denote a set of fixed points of S by

$$F(S) = \{x \in C : Sx = x\}.$$

As a fixed point represents a crucial point, such as a solution to a variational inequality problem under appropriate settings, numerous studies have investigated iterative methods to approximate fixed points of nonlinear mappings. Among others, in 1974, Ishikawa [11] introduced the following iterative scheme:

$$\begin{aligned} x_1 &\in C: \text{ given}, \\ z_n &= \lambda_n x_n + (1 - \lambda_n) S x_n, \\ x_{n+1} &= a_n x_n + (1 - a_n) S z_n \end{aligned} \tag{1.1}$$

for all $n \in \mathbb{N}$, where $S : C \to C$ is a nonlinear mapping, $a_n, \lambda_n \in [0, 1]$ are supposed to satisfy appropriate conditions such as $\lambda_n \to 1$. The iterative rule in (1.1) is a kind of two-step iterative methods, which coincides with the Mann type [26] when $\lambda_n = 1$ for all $n \in \mathbb{N}$.

Kondo proved the following theorem in a 2023 article:

Theorem 1.1 ([20]). *Let C be a nonempty, closed, and convex subset of H. Let $S, T : C \to C$ be quasi-nonexpansive and mean-demiclosed mappings*

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such that $F(S) \cap F(T) \neq \emptyset$. Let $P_{F(S)\cap F(T)}$ denote the metric projection from H onto $F(S) \cap F(T)$. Let $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ be sequences of real numbers in the interval $[0,1]$ such that $a_n + b_n + c_n = 1$ for all $n \in \mathbb{N}$, $\underline{\lim}_{n\to\infty} a_n b_n > 0$, and $\underline{\lim}_{n\to\infty} a_n c_n > 0$. Define a sequence $\{x_n\}$ in C as follows:

$$x_1 \in C: \text{ given}, \quad x_{n+1} = a_n x_n + b_n \frac{1}{n} \sum_{k=0}^{n-1} S^k z_n + c_n \frac{1}{n} \sum_{l=0}^{n-1} T^l w_n \tag{1.2}$$

for all $n \in \mathbb{N}$, where $\{z_n\}$ and $\{w_n\}$ are sequences in C that satisfy

$$\|z_n - q\| \leq \|x_n - q\| \text{ and } \|w_n - q\| \leq \|x_n - q\| \tag{1.3}$$

for all $q \in F(S) \cap F(T)$ and $n \in \mathbb{N}$. Then, the sequence $\{x_n\}$ converges weakly to a point $\widehat{x} \equiv \lim_{n\to\infty} P_{F(S)\cap F(T)} x_n \in F(S) \cap F(T)$.

Iterative schemes using mean-valued sequences have been studied since Baillon [3], Shimizu and Takahashi [30], and Atsushiba and Takahashi [2]. For a version such as (1.2), refer to Kondo and Takahashi [24]. In Theorem 1.1, a "mean-demiclosed mapping" means that any weak cluster point of a mean-valued sequence defined as (1.2) is a fixed point. This class of mappings contains nonexpansive mappings as special cases, with more general types of mappings falling under the purview of this theorem. For further details and recent advancements regarding mean-valued sequences, consult Kondo [18, 20, 22] and the articles cited therein.

For the sequences $\{z_n\}$ and $\{w_n\}$ in Theorem 1.1, only the conditions in (1.3) are required. Thus, setting $z_n = \lambda_n x_n + (1-\lambda_n) S x_n$ and $w_n = \mu_n x_n + (1-\mu_n) T x_n$, the following two-step iterative method is derived.

$$\begin{aligned} z_n &= \lambda_n x_n + (1-\lambda_n) S x_n, \\ w_n &= \mu_n x_n + (1-\mu_n) T x_n, \\ x_{n+1} &= a_n x_n + b_n \frac{1}{n} \sum_{k=0}^{n-1} S^k z_n + c_n \frac{1}{n} \sum_{l=0}^{n-1} T^l w_n, \end{aligned} \tag{1.4}$$

where an initial point $x_1 \in C$ is given and $\lambda_n, \mu_n \in [0,1]$ are coefficients of convex combinations without any restrictive conditions. It can be verified that z_n and w_n in (1.4) satisfy the conditions in (1.3). By interchanging the roles of S and T, we obtain the next iterative method:

$$\begin{aligned} z_n &= \lambda_n x_n + (1-\lambda_n) T x_n, \\ w_n &= \mu_n x_n + (1-\mu_n) S x_n, \\ x_{n+1} &= a_n x_n + b_n \frac{1}{n} \sum_{k=0}^{n-1} S^k z_n + c_n \frac{1}{n} \sum_{l=0}^{n-1} T^l w_n. \end{aligned} \tag{1.5}$$

Notice that z_n (respectively w_n) in (1.5) only depends on the mapping T (respectively S), at least directly. The sequences $\{z_n\}$ and $\{w_n\}$ in (1.5) also satisfy the conditions in (1.3). Furthermore, three-step iterative methods are derived from (1.2). For instance,

$$\begin{aligned} w_n &= \mu_n x_n + (1-\mu_n)\, T x_n, \\ z_n &= \lambda_n x_n + (1-\lambda_n)\, S w_n, \\ x_{n+1} &= a_n x_n + b_n \frac{1}{n}\sum_{k=0}^{n-1} S^k z_n + c_n \frac{1}{n}\sum_{l=0}^{n-1} T^l z_n. \end{aligned} \tag{1.6}$$

The sequences $\{z_n\}$ and $\{w_n\}$ in (1.6) satisfy the conditions in (1.3). For further elucidation on this point, refer to the proof of Corollary 4.5 in this article. Regarding three-step iterative methods; see Noor [28]. Four-step and more general iterative schemes are generated from Theorem 1.1. In this sense, this method may be called an *iterative scheme generating method using mean-valued sequences*. By integrating this method with projection methods, Kondo [22] obtained strong convergence theorems.

This study presents a novel iterative scheme generating method to address common fixed point problems without relying on mean-valued sequences. Our method yields various types of iterative schemes for locating common fixed points. The derived results encompass the Ishikawa iterative method and its variant (see Corollary 4.2 in this article). Additionally, we provide an application to the variational inequality problem to illustrate the utility of our method. We target a broad category of mappings characterized by quasi-nonexpansive and a condition regarding the demiclosedness. This class encompasses nonexpansive mappings and various other types, including mappings that are not continuous.

The structure of this article is as follows: Section 2 provides background information. In Section 3, the main theorem of this study is established. Section 4 introduces various iterative schemes derived from the main theorem. Section 5 applies to the variational inequality problem. As previously mentioned, the mappings targeted in this study are not limited to nonexpansive mappings. In Section 6, as an appendix, we present classes of mappings addressed in this study with an example.

2. Preliminaries

In this section, we summarize preliminary information. Let $\{x_n\}$ be a sequence in a real Hilbert space H and let $x \in H$. Strong and weak convergence of $\{x_n\}$ to x is denoted by $x_n \to x$ and $x_n \rightharpoonup x$, respectively. We know that $x_n \rightharpoonup x$ if and only if for any subsequence $\{x_{n_i}\}$ of $\{x_n\}$, there exists a subsequence $\{x_{n_j}\}$ of $\{x_{n_i}\}$ such that $x_{n_j} \rightharpoonup x$. It holds that if $x_n \rightharpoonup x$ and $y_n \to y$, then $\langle x_n,\ y_n\rangle \to \langle x,\ y\rangle$.

Let C be a nonempty, closed, and convex subset of H. A mapping $S : C \to H$ is called *nonexpansive* if $\|Sx - Sy\| \le \|x - y\|$ for all $x, y \in C$. A mapping S with $F(S) \neq \emptyset$ is called *quasi-nonexpansive* if

$$\|Sx - q\| \le \|x - q\| \quad \text{for all } x \in C \text{ and } q \in F(S). \tag{2.1}$$

A set of fixed points of a quasi-nonexpansive mapping is closed and convex; see Itoh and Takahashi [12]. A mapping $I - S : C \to H$ is called *demiclosed* if

$$x_n - Sx_n \to 0 \text{ and } x_n \rightharpoonup p \Longrightarrow p \in F(S), \tag{2.2}$$

where I represents the identity mapping. In this article, we shed light on this type of mappings, in other words, quasi-nonexpansive mappings that satisfy the condition (2.2). This class of mappings encompasses nonexpansive mappings with fixed points:

Proposition 2.1. *Let $S : C \to H$ be a nonexpansive mapping with a fixed point, where C is a nonempty, closed, and convex subset of H. Then, S is quasi-nonexpansive and $I - S$ is demiclosed.*

A proof of Proposition 2.1 is found in Takahashi [31]. This class of mappings includes more general types of mappings than nonexpansive mappings. To demonstrate the breadth of results obtained in this study, we introduce various classes of mappings that are quasi-nonexpansive with the condition (2.2) in Section 6.

Let F be a nonempty, closed, and convex subset of H. For any $x \in H$, there exists a unique element $p \in F$ that satisfies $\|x - p\| \leq \|x - q\|$ for all $q \in F$. This mapping $x \longmapsto p$ is called a *metric projection* from H onto F, denoted by P_F. A metric projection P_F is nonexpansive and satisfies the inequality

$$\langle x - P_F x,\ P_F x - q \rangle \geq 0 \tag{2.3}$$

for all $x \in H$ and $q \in F$. Conversely, if $p \in F$ satisfies $\langle x - p,\ p - q \rangle \geq 0$ for all $q \in F$, then $p = P_F x$.

The following lemmas are utilized in the proof of the main theorem:

Lemma 2.1 ([33])**.** *Let F be a nonempty, closed, and convex subset of H, let P_F be the metric projection from H onto F, and let $\{x_n\}$ be a sequence in H. If $\|x_{n+1} - q\| \leq \|x_n - q\|$ for all $q \in F$ and $n \in \mathbb{N}$, then $\{P_F x_n\}$ is convergent in F.*

Lemma 2.2 ([27, 37])**.** *Let $x, y, z \in H$ and let $a, b, c \in \mathbb{R}$ such that $a+b+c = 1$. Then, the following equation holds:*

$$\begin{aligned} \|ax + by + cz\|^2 &= a\|x\|^2 + b\|y\|^2 + c\|z\|^2 \\ &\quad -ab\|x - y\|^2 - bc\|y - z\|^2 - ca\|z - x\|^2 . \end{aligned}$$

Although Lemma 2.2 addresses the case of 3, its findings extend to more general scenarios. For investigation concerning the n case, refer to Lemma 1.1 in Zegeye and Shahzad [37]. It is noteworthy that in Lemma 2.2, the conditions $a, b, c \in [0, 1]$ are not necessary.

In this paper's subsequent sections, we assume that there exists a common fixed point for nonlinear mappings. A simplified version of the common fixed point theorem for nonexpansive mappings can be articulated in the context of a real Hilbert space as follows:

Theorem 2.1 ([4, 8, 13])**.** *Let C be a nonempty, closed, convex, and bounded subset of H. Let $S, T : C \to C$ be nonexpansive mappings such that $ST = TS$. Then, S and T possess a common fixed point.*

Key assumptions are the commutativity $ST = TS$ of the mappings and the boundedness of the domain C. For common fixed point theorems about more general types of mappings, refer to Kondo [17, 19], and articles cited therein. Additionally, note that setting T as the identity mapping I in Theorem 2.1, we derive a fixed point theorem for a single nonexpansive mapping S as $SI = IS$ holds true.

3. Main Result

In this section, we present an iterative scheme generation method. This method yields various iterative schemes that weakly approximate common fixed points. For example, the Ishikawa iterative scheme and its variant are derived from this method, as discussed in the next section. The fundamental components of the proof have been developed and refined in numerous prior studies; refer to articles cited in Kondo [16].

Theorem 3.1. *Let C be a nonempty, closed, and convex subset of a real Hilbert space H, let $S, T : C \to C$ be quasi-nonexpansive mappings such that $I - S$ and $I - T$ are demiclosed, where I is the identity mapping. Suppose that $F(S) \cap F(T) \neq \emptyset$. Let $P_{F(S)\cap F(T)}$ denote the metric projection from H onto $F(S) \cap F(T)$. Let $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ be sequences of real numbers in the interval $[0, 1]$ such that $a_n + b_n + c_n = 1$ for all $n \in \mathbb{N}$, $\underline{\lim}_{n\to\infty} a_n b_n > 0$, and $\underline{\lim}_{n\to\infty} a_n c_n > 0$. Define a sequence $\{x_n\}$ in C as follows:*

$$x_1 \in C : \text{ given,}$$
$$x_{n+1} = a_n y_n + b_n S z_n + c_n T w_n$$

for all $n \in \mathbb{N}$, where $\{y_n\}$, $\{z_n\}$, and $\{w_n\}$ are sequences in C that satisfy

$$\|y_n - q\| \le \|x_n - q\|\,,\ \|z_n - q\| \le \|x_n - q\|\,,\ \text{and}\ \|w_n - q\| \le \|x_n - q\| \tag{3.1}$$

for all $q \in F$ and $n \in \mathbb{N}$ and

$$x_n - y_n \to 0\,,\ x_n - z_n \to 0\,,\ \text{and}\ x_n - w_n \to 0\,. \tag{3.2}$$

Then, the sequence $\{x_n\}$ converges weakly to a point $\widehat{x} \in F(S) \cap F(T)$, where $\widehat{x} \equiv \lim_{n\to\infty} P_{F(S)\cap F(T)} x_n$.

Proof. First, we show that

$$\|x_{n+1} - q\| \le \|x_n - q\| \tag{3.3}$$

for any $q \in F(S) \cap F(T)$ and $n \in \mathbb{N}$. To achieve this, let us arbitrarily select $q \in F(S) \cap F(T)$ and $n \in \mathbb{N}$. As S and T are quasi-nonexpansive (2.1), by employing (3.1), we have

$$\begin{aligned}\|x_{n+1}-q\| &= \|a_n y_n + b_n S z_n + c_n T w_n - q\| \\ &= \|a_n (y_n - q) + b_n (S z_n - q) + c_n (T w_n - q)\| \\ &\le a_n \|y_n - q\| + b_n \|S z_n - q\| + c_n \|T w_n - q\| \\ &\le a_n \|y_n - q\| + b_n \|z_n - q\| + c_n \|w_n - q\| \\ &\le a_n \|x_n - q\| + b_n \|x_n - q\| + c_n \|x_n - q\| \\ &= \|x_n - q\|,\end{aligned}$$

as claimed. This indicates three facts: First, $\{\|x_n - q\|\}$ converges in $\mathbb{R}$ for all $q \in F(S) \cap F(T)$. Second, $\{x_n\}$ is bounded. Third, according to Lemma 2.1, $\{P_{F(S)\cap F(T)} x_n\}$ converges in $F(S) \cap F(T)$. We denote the limit point as $\widehat{x} \equiv \lim_{n\to\infty} P_{F(S)\cap F(T)} x_n$.

Observe that $\{z_n\}$, $\{w_n\}$, $\{Sz_n\}$, and $\{Tw_n\}$ are bounded. Indeed, as $\{x_n\}$ is bounded, from (3.1), it follows that $\{z_n\}$ and $\{w_n\}$ are also bounded. Let $q \in F(S)$. As S is quasi-nonexpansive, we have

$$\begin{aligned}\|S z_n\| &\le \|S z_n - q\| + \|q\| \\ &\le \|z_n - q\| + \|q\|.\end{aligned}$$

As $\{z_n\}$ is bounded, $\{Sz_n\}$ is also bounded. Similarly, $\{Tw_n\}$ is also bounded.

We prove that

$$y_n - S z_n \to 0 \text{ and } y_n - T w_n \to 0. \tag{3.4}$$

Choose $q \in F(S) \cap F(T)$ arbitrarily. Using Lemma 2.2 and (3.1) yields

$$\begin{aligned}& \|x_{n+1} - q\|^2 \\ = \;& \|a_n (y_n - q) + b_n (S z_n - q) + c_n (T w_n - q)\|^2 \\ = \;& a_n \|y_n - q\|^2 + b_n \|S z_n - q\|^2 + c_n \|T w_n - q\|^2 \\ & - a_n b_n \|y_n - S z_n\|^2 - b_n c_n \|S z_n - T w_n\|^2 - c_n a_n \|T w_n - y_n\|^2 \\ \le \;& a_n \|y_n - q\|^2 + b_n \|z_n - q\|^2 + c_n \|w_n - q\|^2 \\ & - a_n b_n \|y_n - S z_n\|^2 - b_n c_n \|S z_n - T w_n\|^2 - c_n a_n \|T w_n - y_n\|^2 \\ \le \;& \|x_n - q\|^2 \\ & - a_n b_n \|y_n - S z_n\|^2 - b_n c_n \|S z_n - T w_n\|^2 - c_n a_n \|T w_n - y_n\|^2.\end{aligned}$$

As $b_n c_n \|S z_n - T w_n\|^2 \ge 0$, we obtain

$$a_n b_n \|y_n - S z_n\|^2 + a_n c_n \|y_n - T w_n\|^2 \le \|x_n - q\|^2 - \|x_{n+1} - q\|^2.$$

As $\{\|x_n - q\|\}$ is convergent, the right-hand side converges to 0. Using the hypotheses $\underline{\lim}_{n\to\infty} a_n b_n > 0$ and $\underline{\lim}_{n\to\infty} a_n c_n > 0$, we obtain (3.4), as asserted.

Our next aim is to demonstrate that

$$z_n - S z_n \to 0 \text{ and } w_n - T w_n \to 0. \tag{3.5}$$

Using (3.2) and (3.4) yields

$$\|z_n - Sz_n\| \leq \|z_n - x_n\| + \|x_n - y_n\| + \|y_n - Sz_n\| \to 0.$$

The statement $w_n - Tw_n \to 0$ can be verified similarly.

Our goal is to prove that $x_n \rightharpoonup \widehat{x} \left(\equiv \lim_{k\to\infty} P_{F(S)\cap F(T)} x_k\right)$. Let $\{x_{n_i}\}$ be a subsequence of $\{x_n\}$. As $\{x_{n_i}\}$ is bounded, there exists a subsequence $\left\{x_{n_j}\right\}$ of $\{x_{n_i}\}$ such that $x_{n_j} \rightharpoonup p$ for some $p \in H$. From (3.2), we have $z_{n_j} \rightharpoonup p$ and $w_{n_j} \rightharpoonup p$. As $I - S$ and $I - T$ are demiclosed (2.2), from (3.5), we obtain $p \in F(S) \cap F(T)$. Thus, from a property (2.3) of the metric projection, the following holds:

$$\left\langle x_{n_j} - P_{F(S)\cap F(T)} x_{n_j},\ P_{F(S)\cap F(T)} x_{n_j} - p \right\rangle \geq 0$$

for all $j \in \mathbb{N}$. As $x_{n_j} \rightharpoonup p$ and $P_{F(S)\cap F(T)} x_n \to \widehat{x}$, we have $\langle p - \widehat{x}, \widehat{x} - p\rangle \geq 0$, which implies that $p = \widehat{x}$. Therefore, for any subsequence $\{x_{n_i}\}$ of $\{x_n\}$, there exists a subsequence $\left\{x_{n_j}\right\}$ of $\{x_{n_i}\}$ such that $x_{n_j} \rightharpoonup p = \widehat{x}$. This indicates that $x_n \rightharpoonup \widehat{x}$. The proof is completed. □

A weak approximation method for a finite family of mappings can be established using a generalized version of Lemma 2.2.

4. Derivative Results

This section presents convergence results derived from Theorems 3.1. Throughout this section, we maintain the following setting:

($\star$) Let C be a nonempty, closed, and convex subset of a real Hilbert space H, let $S, T : C \to C$ be quasi-nonexpansive mappings such that $I - S$ and $I - T$ are demiclosed, where I is the identity mapping. Suppose that $F(S) \cap F(T) \neq \emptyset$. Let $P_{F(S)\cap F(T)}$ denote the metric projection from H onto $F(S) \cap F(T)$. Let $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ be sequences of real numbers in the interval $[0, 1]$ such that $a_n + b_n + c_n = 1$ for all $n \in \mathbb{N}$, $\underline{\lim}_{n\to\infty} a_n b_n > 0$, and $\underline{\lim}_{n\to\infty} a_n c_n > 0$.

We begin with a simple case. Set $y_n = z_n = w_n = x_n$ in Theorem 3.1, where the required conditions (3.1) and (3.2) are satisfied. This operation yields the following corollary:

Corollary 4.1. *Assume the setting ($\star$). Define a sequence $\{x_n\}$ in C as follows:*

$$x_1 \in C : \text{ given,}$$
$$x_{n+1} = a_n x_n + b_n S x_n + c_n T x_n$$

for all $n \in \mathbb{N}$. Then, the sequence $\{x_n\}$ converges weakly to a point $\widehat{x} \in F(S) \cap F(T)$, where $\widehat{x} \equiv \lim_{n\to\infty} P_{F(S)\cap F(T)} x_n$.

The iterative rule in Corollary 4.1 is a version of a Mann type iterative scheme; see Theorem 3.2 in Kondo and Takahashi [23].

Next, setting $z_n = w_n = x_n$ and $y_n = \lambda_n x_n + \mu_n S x_n + \nu_n T x_n$ in Theorem 3.1, we obtain the following corollary. For an illustration, a complete proof is provided without relying on Theorem 3.1.

Corollary 4.2. *Assume the setting* $(\star)$. *Let* $\{\lambda_n\}$, $\{\mu_n\}$, *and* $\{\nu_n\}$ *be sequences of real numbers in the interval* $[0,1]$ *such that* $\lambda_n + \mu_n + \nu_n = 1$ *for all* $n \in \mathbb{N}$ *and* $\lambda_n \to 1$. *Define a sequence* $\{x_n\}$ *in* C *as follows:*

$$\begin{aligned} x_1 &\in C: \text{ given,} \\ y_n &= \lambda_n x_n + \mu_n S x_n + \nu_n T x_n, \\ x_{n+1} &= a_n y_n + b_n S x_n + c_n T x_n \end{aligned}$$

for all $n \in \mathbb{N}$. *Then, the sequence* $\{x_n\}$ *converges weakly to a point* $\widehat{x} \in F(S) \cap F(T)$, *where* $\widehat{x} \equiv \lim_{n\to\infty} P_{F(S)\cap F(T)} x_n$.

Proof. First, we prove that

$$\|y_n - q\| \leq \|x_n - q\| \tag{4.1}$$

for all $q \in F(S) \cap F(T)$ and $n \in \mathbb{N}$. Select $q \in F(S) \cap F(T)$ and $n \in \mathbb{N}$ arbitrarily. As S and T are quasi-nonexpansive (2.1), it follows that

$$\begin{aligned} \|y_n - q\| &= \|\lambda_n x_n + \mu_n S x_n + \nu_n T x_n - q\| \\ &= \|\lambda_n (x_n - q) + \mu_n (S x_n - q) + \nu_n (T x_n - q)\| \\ &\leq \lambda_n \|x_n - q\| + \mu_n \|S x_n - q\| + \nu_n \|T x_n - q\| \\ &\leq \lambda_n \|x_n - q\| + \mu_n \|x_n - q\| + \nu_n \|x_n - q\| \\ &= \|x_n - q\|. \end{aligned}$$

Using (4.1), we show that

$$\|x_{n+1} - q\| \leq \|x_n - q\| \tag{4.2}$$

for all $q \in F(S) \cap F(T)$ and $n \in \mathbb{N}$. As S and T are quasi-nonexpansive, it follows that

$$\begin{aligned} \|x_{n+1} - q\| &= \|a_n (y_n - q) + b_n (S x_n - q) + c_n (T x_n - q)\| \\ &\leq a_n \|y_n - q\| + b_n \|S x_n - q\| + c_n \|T x_n - q\| \\ &\leq a_n \|x_n - q\| + b_n \|x_n - q\| + c_n \|x_n - q\| \\ &= \|x_n - q\|. \end{aligned}$$

Thus, (4.2) holds true, as claimed. Consequently, it follows that (a) $\{\|x_n - q\|\}$ is convergent in $\mathbb{R}$; (b) $\{x_n\}$ is bounded; (c) From Lemma 2.1, $\{P_{F(S)\cap F(T)} x_n\}$ is convergent in $F(S) \cap F(T)$. From (c), $\widehat{x} \equiv \lim_{n\to\infty} P_{F(S)\cap F(T)} x_n$ exists in $F(S) \cap F(T)$.

From (b), we can show that $\{S x_n\}$ and $\{T x_n\}$ are bounded. Indeed, for $q \in F(S)$,

$$\|S x_n\| \leq \|S x_n - q\| + \|q\| \leq \|x_n - q\| + \|q\|. \tag{4.3}$$

As $\{x_n\}$ is bounded, we can conclude that $\{S x_n\}$ is also bounded. The assertion that $\{T x_n\}$ is bounded can be demonstrated similarly.

Observe that

$$x_n - y_n \to 0. \tag{4.4}$$

It is true that

$$\begin{aligned} \|x_n - y_n\| &= \|x_n - (\lambda_n x_n + \mu_n S x_n + \nu_n T x_n)\| \\ &\leq (1-\lambda_n)\|x_n\| + \mu_n \|Sx_n\| + \nu_n \|Tx_n\|. \end{aligned}$$

As $\lambda_n \to 1$, we have $\mu_n \to 0$ and $\nu_n \to 0$. As both $\{Sx_n\}$ and $\{Tx_n\}$ are bounded, we conclude that $x_n - y_n \to 0$, as asserted.

Let us prove that

$$y_n - Sx_n \to 0 \text{ and } y_n - Tx_n \to 0. \tag{4.5}$$

Using $q \in F(S) \cap F(T)$, Lemma 2.2, and (4.1), we have

$$\begin{aligned} & \|x_{n+1} - q\|^2 \\ = & \|a_n (y_n - q) + b_n (Sx_n - q) + c_n (Tx_n - q)\|^2 \\ = & a_n \|y_n - q\|^2 + b_n \|Sx_n - q\|^2 + c_n \|Tx_n - q\|^2 \\ & - a_n b_n \|y_n - Sx_n\|^2 - b_n c_n \|Sx_n - Tx_n\|^2 - c_n a_n \|Tx_n - y_n\|^2 \\ \leq & a_n \|x_n - q\|^2 + b_n \|x_n - q\|^2 + c_n \|x_n - q\|^2 \\ & - a_n b_n \|y_n - Sx_n\|^2 - b_n c_n \|Sx_n - Tx_n\|^2 - c_n a_n \|Tx_n - y_n\|^2 \\ = & \|x_n - q\|^2 \\ & - a_n b_n \|y_n - Sx_n\|^2 - b_n c_n \|Sx_n - Tx_n\|^2 - c_n a_n \|Tx_n - y_n\|^2. \end{aligned}$$

As $b_n c_n \|Sx_n - Tx_n\|^2 \geq 0$, it holds that

$$a_n b_n \|y_n - Sx_n\|^2 + a_n c_n \|y_n - Tx_n\|^2 \leq \|x_n - q\|^2 - \|x_{n+1} - q\|^2.$$

From (a) and the hypotheses $\underline{\lim}_{n\to\infty} a_n b_n > 0$ and $\underline{\lim}_{n\to\infty} a_n c_n > 0$, we obtain (4.5), as claimed. From (4.4) and (4.5), it follows that

$$x_n - Sx_n \to 0 \text{ and } x_n - Tx_n \to 0. \tag{4.6}$$

Our aim is to show that $x_n \rightharpoonup \widehat{x} \left(\equiv \lim_{k\to\infty} P_{F(S)\cap F(T)} x_k\right)$. Let $\{x_{n_i}\}$ be a subsequence of $\{x_n\}$. From (b), $\{x_{n_i}\}$ is bounded. Thus, there exists a subsequence $\{x_{n_j}\}$ of $\{x_{n_i}\}$ such that $x_{n_j} \rightharpoonup p$ for some $p \in H$. As $I - S$ and $I - T$ are demiclosed (2.2), from (4.6), we have that $p \in F(S) \cap F(T)$. From (2.3), it holds that

$$\left\langle x_{n_j} - P_{F(S)\cap F(T)} x_{n_j},\ P_{F(S)\cap F(T)} x_{n_j} - p \right\rangle \geq 0$$

for all $j \in \mathbb{N}$. As $x_{n_j} \rightharpoonup p$ and $P_{F(S)\cap F(T)} x_n \to \widehat{x}$, it follows that $\langle p - \widehat{x}, \widehat{x} - p\rangle \geq 0$, which implies that $p = \widehat{x}$. This completes the proof. □

Compare the iterative scheme in Corollary 4.2 with the Ishikawa type (1.1).

Additionally, we obtain the following result:

Corollary 4.3. *Assume the setting* $(\star)$. *Let* $\{\lambda_n\}$ *and* $\{\mu_n\}$ *be sequences of real numbers in the interval* $[0,1]$ *such that*

$$\lambda_n \to 1 \text{ and } \mu_n \to 1. \tag{4.7}$$

Define a sequence $\{x_n\}$ *in* C *as follows:*

$$\begin{aligned} x_1 &\in C: \text{ given,} \\ z_n &= \lambda_n x_n + (1-\lambda_n) T x_n, \\ w_n &= \mu_n x_n + (1-\mu_n) S x_n, \\ x_{n+1} &= a_n x_n + b_n S z_n + c_n T w_n \end{aligned} \tag{4.8}$$

for all $n \in \mathbb{N}$. *Then, the sequence* $\{x_n\}$ *converges weakly to a point* $\widehat{x} \in F(S) \cap F(T)$, *where* $\widehat{x} \equiv \lim_{n\to\infty} P_{F(S)\cap F(T)} x_n$.

Proof. From Theorem 3.1, it is sufficient to demonstrate that

$$\|z_n - q\| \le \|x_n - q\| \text{ and } \|w_n - q\| \le \|x_n - q\| \tag{4.9}$$

for all $q \in F(S) \cap F(T)$ and $n \in \mathbb{N}$ and

$$x_n - z_n \to 0 \text{ and } x_n - w_n \to 0. \tag{4.10}$$

First, we verify (4.9). Choose $q \in F(S) \cap F(T)$ and $n \in \mathbb{N}$ arbitrarily. As T is quasi-nonexpansive (2.1) and $q \in F(T)$, we have

$$\begin{aligned} \|z_n - q\| &= \|\lambda_n (x_n - q) + (1-\lambda_n)(T x_n - q)\| \\ &\le \lambda_n \|x_n - q\| + (1-\lambda_n) \|T x_n - q\| \\ &\le \lambda_n \|x_n - q\| + (1-\lambda_n) \|x_n - q\| \\ &= \|x_n - q\|. \end{aligned}$$

Similarly, employing the hypothesis that S is quasi-nonexpansive, we can also show that $\|w_n - q\| \le \|x_n - q\|$, as claimed.

It follows that

$$\|x_{n+1} - q\| \le \|x_n - q\| \tag{4.11}$$

for all $q \in F(S) \cap F(T)$ and $n \in \mathbb{N}$. In fact, as S and T are quasi-nonexpansive, using (4.9) yields

$$\begin{aligned} \|x_{n+1} - q\| &\le a_n \|x_n - q\| + b_n \|S z_n - q\| + c b_n \|T w_n - q\| \\ &\le a_n \|x_n - q\| + b_n \|z_n - q\| + c b_n \|w_n - q\| \\ &\le a_n \|x_n - q\| + b_n \|x_n - q\| + c b_n \|x_n - q\| \\ &= \|x_n - q\|. \end{aligned}$$

This shows that (4.11) holds true, as stated. From (4.11), it is evident that $\{x_n\}$ is bounded. Applying the same reasoning as in the proof of Corollary 4.2 concerning (4.3), we conclude that $\{S x_n\}$ and $\{T x_n\}$ are also bounded.

Finally, we demonstrate (4.10). It holds that

$$\begin{aligned} \|x_n - z_n\| &= \|x_n - [\lambda_n x_n + (1-\lambda_n) T x_n]\| \\ &= (1-\lambda_n) \|x_n - T x_n\|. \end{aligned}$$

As $\{x_n\}$ and $\{Tx_n\}$ are bounded, using the condition $\lambda_n \to 1$ in (4.7), we deduce that $x_n - z_n \to 0$. Similarly, we can show that $x_n - w_n \to 0$. The proof is completed. □

Remark 4.1. *Notice that* z_n (*respectively* w_n) *in* (4.8) *only depends on the mapping* T (*respectively* S), *at least directly. When comparing the iterative scheme in* Corollary 4.3 *with* (1.5), *it is apparent that in* Corollary 4.3, *the additional conditions in* (4.7) *are required. However,* Corollary 4.3 *can be established without relying on mean-valued sequences such as those in* (1.5).

A multi-step iterative scheme is also derived:

Corollary 4.4. *Assume the setting* ($\star$). *Let* $\{\lambda_n\}$, $\{\mu_n\}$, $\{\nu_n\}$, $\{\lambda'_n\}$, $\{\mu'_n\}$, $\{\nu'_n\}$, $\{\lambda''_n\}$, $\{\mu''_n\}$, *and* $\{\nu''_n\}$ *be sequences of real numbers in the interval* $[0,1]$ *such that* $\lambda_n + \mu_n + \nu_n = 1$, $\lambda'_n + \mu'_n + \nu'_n = 1$, $\lambda''_n + \mu''_n + \nu''_n = 1$ *for all* $n \in \mathbb{N}$,

$$\lambda_n \to 1,\ \lambda'_n \to 1,\ \text{and}\ \lambda''_n \to 1. \tag{4.12}$$

Define a sequence $\{x_n\}$ *in* C *as follows:*

$$\begin{aligned} x_1 &\in C: \ \text{given}, \\ w_n &= \lambda''_n x_n + \mu''_n S x_n + \nu''_n T x_n, \\ z_n &= \lambda'_n w_n + \mu'_n S w_n + \nu'_n T w_n, \\ y_n &= \lambda_n z_n + \mu_n S z_n + \nu_n T z_n, \\ x_{n+1} &= a_n y_n + b_n S y_n + c_n T y_n \end{aligned}$$

for all $n \in \mathbb{N}$. *Then, the sequence* $\{x_n\}$ *converges weakly to a point* $\widehat{x} \in F(S) \cap F(T)$, *where* $\widehat{x} \equiv \lim_{n\to\infty} P_{F(S)\cap F(T)} x_n$.

Proof. According to Theorem 3.1, it is sufficient to demonstrate that

$$\|y_n - q\| \le \|x_n - q\| \quad \text{for all } q \in F(S) \cap F(T) \text{ and } n \in \mathbb{N} \text{ and}$$
$$x_n - y_n \to 0.$$

Let $q \in F(S) \cap F(T)$ and $n \in \mathbb{N}$. As S and T are quasi-nonexpansive, it follows that

$$\|w_n - q\| \le \|x_n - q\|. \tag{4.13}$$

This verification can be conducted as follows:

$$\begin{aligned} \|w_n - q\| &= \left\|\lambda''_n x_n + \mu''_n S x_n + \nu''_n T x_n - q\right\| \\ &\le \lambda''_n \|x_n - q\| + \mu''_n \|S x_n - q\| + \nu''_n \|T x_n - q\| \\ &\le \|x_n - q\|. \end{aligned} \tag{4.14}$$

Similarly, we have

$$\|z_n - q\| \le \|w_n - q\|,\ \|y_n - q\| \le \|z_n - q\|, \text{ and} \tag{4.15}$$

$$\|x_{n+1} - q\| \le \|y_n - q\|. \tag{4.16}$$

From (4.13) and (4.15), we derive the inequality $\|y_n - q\| \leq \|x_n - q\|$ for all $q \in F(S) \cap F(T)$ and $n \in \mathbb{N}$.

From (4.13), (4.15), and (4.16), we have

$$\|x_{n+1} - q\| \leq \|x_n - q\|,$$

which implies that the sequence $\{x_n\}$ is bounded. Additionally, considering (4.13) and (4.15), we conclude that the sequences $\{y_n\}$, $\{z_n\}$, and $\{w_n\}$ are bounded as well. Consequently, the sequences $\{Sx_n\}$, $\{Tx_n\}$, $\{Sy_n\}$, $\{Ty_n\}$, and so forth, are also bounded. These facts can be ascertained as (4.3) in the proof of Corollary 4.2.

Observe that

$$w_n - x_n \to 0. \tag{4.17}$$

Indeed, as $\{x_n\}$, $\{Sx_n\}$, $\{Tx_n\}$ are bounded and $\lambda''_n \to 1$ it follows that

$$\begin{aligned} \|w_n - x_n\| &= \left\|\lambda''_n x_n + \mu''_n Sx_n + \nu''_n Tx_n - x_n\right\| \\ &\leq \left(1 - \lambda''_n\right) \|x_n\| + \mu''_n \|Sx_n\| + \nu''_n \|Tx_n\| \to 0. \end{aligned}$$

Similarly, by considering $\lambda'_n \to 1$ and $\lambda_n \to 1$, we can demonstrate that

$$z_n - w_n \to 0 \text{ and } y_n - z_n \to 0. \tag{4.18}$$

Finally, using (4.17) and (4.18), we obtain

$$\|x_n - y_n\| \leq \|x_n - w_n\| + \|w_n - z_n\| + \|z_n - y_n\| \to 0.$$

This concludes the proof. □

In Corollary 4.4, if $\lambda''_n = 1$, then $w_n = x_n$ and the following iterative scheme is deduced:

$$\begin{aligned} z_n &= \lambda'_n x_n + \mu'_n Sx_n + \nu'_n Tx_n, \\ y_n &= \lambda_n z_n + \mu_n Sz_n + \nu_n Tz_n, \\ x_{n+1} &= a_n y_n + b_n Sy_n + c_n Ty_n, \end{aligned}$$

where an initial point $x_1 \in C$ is given. This is a version of the three-step iterative scheme; see Noor [28], Dashputre and Diwan [7], Phuengrattana and Suantai [29], and Chugh et al. [6]. The next corollary is also derived from Theorem 3.1:

Corollary 4.5. *Assume the setting* $(\star)$. *Let* $\{\lambda_n\}$, $\{\mu_n\}$, $\{\nu_n\}$, $\left\{\lambda'_n\right\}$, $\{\mu'_n\}$, $\{\nu'_n\}$, $\left\{\lambda''_n\right\}$, $\{\mu''_n\}$, *and* $\{\nu''_n\}$ *be sequences of real numbers in the interval* $[0,1]$ *such that* $\lambda_n + \mu_n + \nu_n = 1$, $\lambda'_n + \mu'_n + \nu'_n = 1$, $\lambda''_n + \mu''_n + \nu''_n = 1$ *for all* $n \in \mathbb{N}$, *and*

$$\lambda_n \to 1. \tag{4.19}$$

Define a sequence $\{x_n\}$ *in* C *as follows:*

$$\begin{aligned} x_1 &\in C: \text{ given}, \\ w_n &= \lambda''_n x_n + \mu''_n S x_n + \nu''_n T x_n, \\ z_n &= \lambda'_n x_n + \mu'_n S w_n + \nu'_n T w_n, \\ y_n &= \lambda_n x_n + \mu_n S z_n + \nu_n T z_n, \\ x_{n+1} &= a_n x_n + b_n S y_n + c_n T y_n \end{aligned}$$

for all $n \in \mathbb{N}$*. Then, the sequence* $\{x_n\}$ *converges weakly to a point* $\widehat{x} \in F(S) \cap F(T)$*, where* $\widehat{x} \equiv \lim_{n\to\infty} P_{F(S)\cap F(T)} x_n$*.*

Proof. From Theorem 3.1, it is sufficient to show that

$$\|y_n - q\| \le \|x_n - q\| \quad \text{for all } q \in F(S) \cap F(T) \text{ and } n \in \mathbb{N} \text{ and}$$
$$x_n - y_n \to 0.$$

Choose $q \in F(S) \cap F(T)$ and $n \in \mathbb{N}$ arbitrarily. We demonstrate that $\|y_n - q\| \le \|x_n - q\|$. Following the same steps as in (4.14), we have

$$\|w_n - q\| \le \|x_n - q\|. \tag{4.20}$$

Using (4.20), we have

$$\|z_n - q\| \le \|x_n - q\|. \tag{4.21}$$

Indeed,

$$\begin{aligned} \|z_n - q\| &\le \lambda'_n \|x_n - q\| + \mu'_n \|S w_n - q\| + \nu'_n \|T w_n - q\| \\ &\le \lambda'_n \|x_n - q\| + \left(\mu'_n + \nu'_n\right) \|w_n - q\| \\ &\le \lambda'_n \|x_n - q\| + \left(\mu'_n + \nu'_n\right) \|x_n - q\| \\ &= \|x_n - q\|. \end{aligned}$$

Similarly, using (4.21), we obtain

$$\|y_n - q\| \le \|x_n - q\|, \tag{4.22}$$

as claimed.

Next, observe that

$$\|x_{n+1} - q\| \le \|x_n - q\| \tag{4.23}$$

for all $q \in F(S) \cap F(T)$ and $n \in \mathbb{N}$. Indeed, from (4.22),

$$\begin{aligned} \|x_{n+1} - q\| &\le a_n \|x_n - q\| + b_n \|S y_n - q\| + c_n \|T y_n - q\| \\ &\le a_n \|x_n - q\| + b_n \|y_n - q\| + c_n \|y_n - q\| \\ &\le a_n \|x_n - q\| + b_n \|x_n - q\| + c_n \|x_n - q\| \\ &= \|x_n - q\|. \end{aligned}$$

From (4.23), $\{x_n\}$ is bounded. Thus, from (4.21), $\{z_n\}$ is also bounded. As S and T are quasi-nonexpansive, $\{S z_n\}$ and $\{T z_n\}$ are also bounded, which can be verified as in (4.3) in the proof of Corollary 4.2.

Finally, we show that $x_n - y_n \to 0$. As $\{x_n\}$, $\{Sz_n\}$, and $\{Tz_n\}$ are bounded and $\lambda_n \to 1$, we obtain

$$\begin{aligned}\|x_n - y_n\| &= \|x_n - (\lambda_n x_n + \mu_n S z_n + \nu_n T z_n)\| \\ &\leq (1-\lambda_n)\|x_n\| + \mu_n \|Sz_n\| + \nu_n \|Tz_n\| \to 0.\end{aligned}$$

The desired result follows from Theorem 3.1. □

Remark 4.2. *By comparing* (4.19) *with* (4.12)*, we find that the conditions* $\lambda_n' \to 1$ *and* $\lambda_n'' \to 1$ *are dispensable in* Corollary 4.5 *by adopting the iterative scheme in* Corollary 4.5 *instead of that in* Corollary 4.4*.*

We present the following corollary, also derived from Theorem 3.1:

Corollary 4.6. *Assume the setting* $(\star)$*. Let* $\{\lambda_n\}$*,* $\{\mu_n\}$*, and* $\{\nu_n\}$ *be sequences of real numbers in the interval* $[0,1]$ *such that* $\lambda_n + \mu_n + \nu_n = 1$ *for all* $n \in \mathbb{N}$ *and* $\lambda_n \to 1$*. Define a sequence* $\{x_n\}$ *in* C *as follows:*

$$\begin{aligned}x_1 &\in C : \ \textit{given}, \\ y_n &= \lambda_n x_n + \mu_n \frac{1}{n}\sum_{k=0}^{n-1} S^k x_n + \nu_n \frac{1}{n}\sum_{l=0}^{n-1} T^l x_n, \\ x_{n+1} &= a_n y_n + b_n S y_n + c_n T y_n\end{aligned}$$

for all $n \in \mathbb{N}$*. Then, the sequence* $\{x_n\}$ *converges weakly to a point* $\widehat{x} \in F(S) \cap F(T)$*, where* $\widehat{x} \equiv \lim_{n\to\infty} P_{F(S)\cap F(T)} x_n$*.*

Proof. From Theorem 3.1, it is sufficient to demonstrate that

$$\begin{aligned}&\|y_n - q\| \leq \|x_n - q\| \quad \text{for all } q \in F(S) \cap F(T) \text{ and } n \in \mathbb{N} \text{ and} \\ &x_n - y_n \to 0.\end{aligned}$$

First, let us verify that

$$\|y_n - q\| \leq \|x_n - q\|. \tag{4.24}$$

Let $q \in F(S) \cap F(T)$ and $n \in \mathbb{N}$. As S is quasi-nonexpansive, it holds that

$$\begin{aligned}\left\|\frac{1}{n}\sum_{k=0}^{n-1} S^k x_n - q\right\| &= \frac{1}{n}\left\|\sum_{k=0}^{n-1} S^k x_n - nq\right\| \\ &= \frac{1}{n}\left\|\sum_{k=0}^{n-1}\left(S^k x_n - q\right)\right\| \leq \frac{1}{n}\sum_{k=0}^{n-1}\left\|S^k x_n - q\right\| \\ &\leq \frac{1}{n}\sum_{k=0}^{n-1}\|x_n - q\| = \|x_n - q\|.\end{aligned} \tag{4.25}$$

Similarly, we can establish that $\left\|\frac{1}{n}\sum_{l=0}^{n-1}T^{l}x_n - q\right\| \leq \|x_n - q\|$. Utilizing these observations, we have

$$
\begin{aligned}
&\|y_n - q\| \\
&= \left\|\lambda_n (x_n - q) + \mu_n \left(\frac{1}{n}\sum_{k=0}^{n-1}S^{k}x_n - q\right) + \nu_n \left(\frac{1}{n}\sum_{l=0}^{n-1}T^{l}x_n - q\right)\right\| \\
&\leq \lambda_n \|x_n - q\| + \mu_n \left\|\frac{1}{n}\sum_{k=0}^{n-1}S^{k}x_n - q\right\| + \nu_n \left\|\frac{1}{n}\sum_{l=0}^{n-1}T^{l}x_n - q\right\| \\
&\leq \|x_n - q\|,
\end{aligned}
$$

as claimed.

We aim to prove that $x_n - y_n \to 0$. Given that S and T are quasi-nonexpansive, employing (4.24) yields

$$
\|x_{n+1} - q\| \leq \|x_n - q\| \tag{4.26}
$$

for all $q \in F(S) \cap F(T)$ and $n \in \mathbb{N}$. Consequently, $\{x_n\}$ is bounded. Then, $\left\{\frac{1}{n}\sum_{k=0}^{n-1}S^{k}x_n\right\}$ and $\left\{\frac{1}{n}\sum_{l=0}^{n-1}T^{l}x_n\right\}$ are also bounded. Indeed, it follows from (4.25) that

$$
\left\|\frac{1}{n}\sum_{k=0}^{n-1}S^{k}x_n\right\| \leq \left\|\frac{1}{n}\sum_{k=0}^{n-1}S^{k}x_n - q\right\| + \|q\| \leq \|x_n - q\| + \|q\|.
$$

As $\{x_n\}$ is bounded, $\left\{\frac{1}{n}\sum_{k=0}^{n-1}S^{k}x_n\right\}$ is also bounded. Similarly, $\left\{\frac{1}{n}\sum_{l=0}^{n-1}T^{l}x_n\right\}$ is also bounded.

From the above, we obtain $x_n - y_n \to 0$. Indeed, as $\lambda_n \to 1$, it follows that

$$
\begin{aligned}
\|x_n - y_n\| &= \left\|x_n - \left(\lambda_n x_n + \mu_n \frac{1}{n}\sum_{k=0}^{n-1}S^{k}x_n + \nu_n \frac{1}{n}\sum_{l=0}^{n-1}T^{l}x_n\right)\right\| \\
&\leq (1 - \lambda_n)\|x_n\| + \mu_n \left\|\frac{1}{n}\sum_{k=0}^{n-1}S^{k}x_n\right\| + \nu_n \left\|\frac{1}{n}\sum_{l=0}^{n-1}T^{l}x_n\right\| \to 0.
\end{aligned}
$$

This completes the proof. □

Apart from Corollaries 4.1–4.6, various iterative schemes can be derived from Theorem 3.1.

5. Application

In this section, we apply a result obtained in this study to the variational inequality problem (VIP). The following classes of mappings are frequently referred in the literature. A mapping $A : C \to H$ is called *K-Lipschitz continuous* if there exists $K > 0$ such that $\|Ax - Ay\| \leq K\|x - y\|$ for all $x, y \in C$, where C is a nonempty subset of a real Hilbert space H. A mapping $A : C \to H$ is called *monotone* if $0 \le \langle x-y,\ Ax-Ay\rangle$ for all $x, y \in C$. A mapping $A : C \to H$ is called *α-inverse strongly monotone* if there exists $\alpha > 0$ such that

$$\alpha \|Ax - Ay\|^2 \le \langle x-y,\ Ax-Ay\rangle \tag{5.1}$$

for all $x, y \in C$; see [5, 25]. An α-inverse strongly monotone mapping $A : C \to H$ is monotone and $(1/\alpha)$-Lipschitz continuous.

For a mapping $A : C \to H$, the set of solutions to the VIP is denoted by

$$VI(C, A) = \{x \in C : \langle y-x,\ Ax\rangle \ge 0 \text{ for all } y \in C\}. \tag{5.2}$$

The VIPs are directly linked to optimization problems under appropriate settings. For instance, this connection is evidenced in the work of Toyoda and Takahashi [33]:

Proposition 5.1. *For a mapping $A : H \to H$, the set* (5.2) *of solutions to the VIP coincides with the set of null points of A, that is, $VI(H, A) = A^{-1}0$, where $A^{-1}0 = \{x \in H : Ax = 0\}$.*

Proof. The inclusion $VI(H, A) \supset A^{-1}0$ follows from the definition of $VI(H, A)$. Let $x \in VI(H, A)$ be arbitrary to show inverse inclusion. Then, it follows from (5.2) that

$$\langle y-x,\ Ax\rangle \ge 0 \text{ for all } y \in H.$$

Setting $y = x - Ax \in H$, we have $\langle -Ax,\ Ax\rangle \ge 0$. Consequently, we obtain $Ax = 0$, which means that $x \in A^{-1}0$. This completes the proof. □

Let $H = \mathbb{R}$ and interpret A as a derivative f' of a real-valued function f defined on $\mathbb{R}$. Then, $VI(\mathbb{R}, f')$ is the set of points $x \in \mathbb{R}$ that satisfies $f'(x) = 0$.

The following facts are crucial for applying fixed point theory to VIPs:

Proposition 5.2. *Let A be a mapping from C into H, where C is a nonempty subset of H. Then, the following holds:*

(a) *If A is α-inverse strongly monotone, then $I - \eta A$ is a nonexpansive mapping from C into H for all $\eta \in [0, 2\alpha]$, where I is the identity mapping defined on C.*

(b) *Suppose that C is closed and convex. Then, it holds that $VI(C, A) = F(P_C(I - \eta A))$ for all $\eta > 0$, where P_C is the metric projection from H onto C.*

Proof. (a) Let $\eta \in [0, 2\alpha]$. As A is α-inverse strongly monotone (5.1), it holds that

$$\begin{aligned}
&\|(I-\eta A)x-(I-\eta A)y\|^2 \\
&= \|x-y-\eta(Ax-Ay)\|^2 \\
&= \|x-y\|^2-2\eta\langle x-y,\ Ax-Ay\rangle+\eta^2\|Ax-Ay\|^2 \\
&\leq \|x-y\|^2-2\eta\alpha\|Ax-Ay\|^2+\eta^2\|Ax-Ay\|^2 \\
&= \|x-y\|^2-\eta(2\alpha-\eta)\|Ax-Ay\|^2 \\
&\leq \|x-y\|^2
\end{aligned}$$

for all $x, y \in C$. This implies that $I - \eta A$ is nonexpansive.

(b) Let $\eta > 0$. From the properties of metric projections, we have the desired result as follows:

$$\begin{aligned}
&x \in F(P_C(I-\eta A)) \\
&\quad\Longleftrightarrow x = P_C(x-\eta Ax) \\
&\quad\Longleftrightarrow \langle (x-\eta Ax)-x,\ x-y\rangle \geq 0 \text{ for all } y \in C \\
&\quad\Longleftrightarrow \langle -\eta Ax,\ x-y\rangle \geq 0 \text{ for all } y \in C \\
&\quad\Longleftrightarrow \langle Ax,\ x-y\rangle \leq 0 \text{ for all } y \in C \\
&\quad\Longleftrightarrow x \in VI(C,A).
\end{aligned}$$

□

From (a) in Proposition 5.2, $P_C(I - \eta A)$ is a nonexpansive mapping from C into itself if $\eta \in [0, 2\alpha]$. Consequently, from (b) and Theorem 2.1, the set $VI(C, A)$ is a nonempty, closed, and convex subset of C if C is a nonempty, closed, convex, and bounded subset of H. For contributions regarding the VIPs, see Yamada [36], Takahashi and Toyoda [33], Xu and Kim [35], and Truong et al. [34].

Proposition 2.1 states that a nonexpansive mapping with a fixed point is quasi-nonexpansive and fulfills the condition (2.2). By integrating Corollary 4.2 with Proposition 5.2, we derive result that explicitly illustrates how to approximate a common solution to a fixed point problem and a VIP:

Theorem 5.1. *Let C be a nonempty, closed, and convex subset of a real Hilbert space H and let $S : C \to C$ be a nonexpansive mapping. Let $A : C \to H$ be an α-inverse strongly monotone mapping and let $\eta \in [0, 2\alpha]$. Suppose that*

$$\Omega \equiv F(S) \cap VI(C, A) \neq \emptyset.$$

Let $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ be sequences of real numbers in the interval $[0, 1]$ such that $a_n+b_n+c_n = 1$ for all $n \in \mathbb{N}$, $\underline{\lim}_{n\to\infty} a_n b_n > 0$, and $\underline{\lim}_{n\to\infty} a_n c_n > 0$. Let $\{\lambda_n\}$, $\{\mu_n\}$, and $\{\nu_n\}$ be sequences of real numbers in the interval $[0, 1]$ such that $\lambda_n+\mu_n+\nu_n = 1$ for all $n \in \mathbb{N}$ and $\lambda_n \to 1$. Define a sequence

$\{x_n\}$ in C as follows:

$$x_1 \in C: \text{ given,}$$
$$y_n = \lambda_n x_n + \mu_n S x_n + \nu_n P_C (I - \eta A) x_n,$$
$$x_{n+1} = a_n y_n + b_n S x_n + c_n P_C (I - \eta A) x_n$$

for all $n \in \mathbb{N}$. Then, the sequence $\{x_n\}$ converges weakly to a point $\widehat{x} \in \Omega$, where $\widehat{x} \equiv \lim_{n\to\infty} P_\Omega x_n$.

6. Appendix

In this study, our focus is on quasi-nonexpansive mappings that satisfy (2.2). This category encompasses nonexpansive mappings and broader classes of mappings with fixed points. In this section, we introduce the diverse classes of mappings addressed in this study, illustrating the extensive applicability of the findings presented herein.

Let H be a real Hilbert space and C be a nonempty subset of H. A mapping $S: C \to H$ is called

(i) *nonexpansive* if $\|Sx - Sy\| \le \|x - y\|$ for all $x, y \in C$,

(ii) *nonspreading* [15] if $2\|Sx - Sy\|^2 \le \|x - Sy\|^2 + \|Sx - y\|^2$ for all $x, y \in C$,

(iii) *hybrid* [32] if $3\|Sx - Sy\|^2 \le \|x - y\|^2 + \|x - Sy\|^2 + \|Sx - y\|^2$ for all $x, y \in C$.

The definition of a nonexpansive mapping overlaps with Section 2. We provide Example 6.1 to illustrate that the class of nonspreading mappings includes mappings that are not continuous. The definition of hybrid mapping (iii) is obtained by summing up the conditions (i) and (ii).

The class of *generalized hybrid mappings* encompasses these three types (i)–(iii) of mappings simultaneously. A mapping $S: C \to H$ is said to be

(iv) *generalized hybrid* [14] if there exist $\alpha, \beta \in \mathbb{R}$ such that

$$\alpha\|Sx - Sy\|^2 + (1-\alpha)\|x - Sy\|^2 \le \beta\|Sx - y\|^2 + (1-\beta)\|x - y\|^2 \tag{6.1}$$

for all $x, y \in C$. Setting $\alpha = 1$ and $\beta = 0$ in (6.1), we have the condition (i) of nonexpansive mappings. Therefore, the class of generalized hybrid mappings contains nonexpansive mappings as special cases. Substituting $(\alpha, \beta) = (2, 1)$ and $(3/2, 1/2)$ in (6.1), we derive the conditions of nonspreading mappings (ii) and hybrid mappings (iii), respectively. Thus, nonspreading and hybrid mappings are also included in the class of generalized hybrid mappings as special cases. Another type of mappings called *λ-hybrid* [1] is also within the class of generalized hybrid mappings.

According to Kocourek et al. [14], the following holds:

Proposition 6.1 ([14])**.** *Let C be a nonempty subset of H and let $S: C \to H$ be a generalized hybrid mapping. Then, the following assertions hold:*

(i) *The mapping S is quasi-nonexpansive when $F(S) \ne \emptyset$;*

(ii) *Assume that C is closed and convex. Then, $I - S$ is demiclosed.*

Kocourek et al. proved weak convergence theorems for finding fixed points of S. According to Proposition 6.1, mappings such as nonexpansive, nonspreading, hybrid, and λ-hybrid with fixed points are quasi-nonexpansive and satisfy the condition (2.2). Consequently, the mappings of these classes fall within in the category addressed in this study.

Finally, we provide an example of a nonspreading mapping to show that the class of mappings includes a mapping that is not continuous:

Example 6.1 ([21])**.** *Let $H = C = \mathbb{R}$ and define a mapping $S : \mathbb{R} \to \mathbb{R}$ as follows:*

$$Sx = \begin{cases} 1 & \text{if } x > A, \\ 0 & \text{if } x \leq A, \end{cases}$$

where $A \in \mathbb{R}$ is a constant.

According to Kondo [21], the mapping S is nonspreading if and only if $A \geq \sqrt{2}$. Assume that $A \geq \sqrt{2}$. In this scenario, the mapping S is nonspreading and it is generalized hybrid. As S has a fixed point $0 \in \mathbb{R}$, from Proposition 6.1, S is quasi-nonexpansive and $I - S$ is demiclosed. Thus, it belongs to the class discussed in this study, even though it is not continuous. For other examples, refer to Igarashi et al. [10], Hojo et al. [9], Kondo [16], and articles cited therein.

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