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WEAKLY ITERATIVE METHOD FOR SOLVING COMMON FIXED POINT AND SPLIT COMMON FIXED POINT PROBLEMS IN HILBERT SPACES

Atsumasa Kondo , Wataru Takahashi

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The Institute for Economic and Business Research
Faculty of Economics
SHIGA UNIVERSITY

1-1-1 BANBA, HIKONE,
SHIGA 522-8522, JAPAN

WEAKLY ITERATIVE METHOD FOR SOLVING COMMON FIXED POINT AND SPLIT COMMON FIXED POINT PROBLEMS IN HILBERT SPACES

ATSUMASA KONDO AND WATARU TAKAHASHI

Abstract. We investigate iterative methods for addressing common fixed point and split common fixed point problems in Hilbert spaces employing demimetric mappings. This class of mappings encompasses strict pseudo-contractions and generalized hybrid mappings as specifical instances and closely connects to inverse strongly monotone mappings. We initially establish a weak convergence theorem capable of resolving common fixed point and split common fixed point problems. Building upon this fundamental result, we derive a sequence of weak convergence theorems pertinent to common fixed point problems combined with split common fixed point problems, split feasibility problems, split common null point problems, and equilibrium problems.

1. Introduction

Throughout this study, we utilized the notations $\langle \cdot, \ \cdot \rangle$ and $\|\cdot\|$ to denote an inner product and the induced norm in Hilbert spaces, respectively. Let C be a nonempty subset of a Hilbert space H and let T be a mapping from C into H. The set of fixed points of T is denoted by

$$F(T) = \{x \in C : Tx = x\}.$$

A mapping T is termed *nonexpansive* if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. The identity and zero mappings are denoted by I and O, respectively.

Let H_1 and H_2 be real Hilbert spaces and let $A : H_1 \to H_2$ be a bounded linear operator such that $A \neq O$ with A^* denoting the adjoint operator of A. Let D and Q be nonempty, closed, and convex subsets of H_1 and H_2, respectively. *The split feasibility problem* [6] is defined as follows:

$$\text{Find an element } \overline{x} \in D \cap A^{-1}Q. \tag{1.1}$$

The split common null point problem [5] has also garnered attention from numerous researchers. Let $B : H_1 \to 2^{H_1}$ and $G : H_2 \to 2^{H_2}$ be multi-valued mappings on H_1 and H_2, respectively. The sets of null points of these mappings are denoted by $B^{-1}0$ and $G^{-1}0$, respectively. A simplified version of the split common null point problem is formulated as follows:

$$\text{Find an element } \overline{x} \in \left(B^{-1}0\right) \cap A^{-1}\left(G^{-1}0\right). \tag{1.2}$$

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Given nonlinear mappings $S : H_1 \to H_1$ and $T : H_2 \to H_2$, the *split common fixed point problem* [7, 21] is stated as:

$$\text{Find an element } \overline{x} \in F(S) \cap A^{-1}F(T). \tag{1.3}$$

Let P_D be the metric projection from H_1 onto D and let P_Q be the metric projection from H_2 onto Q. If $D \cap A^{-1}Q$ is nonempty, then it holds that

$$D \cap A^{-1}Q = F\left(P_D\left(I - \lambda A^*\left(I - P_Q\right)A\right)\right)$$

for all $\lambda > 0$. Moreover, if λ is sufficiently close to 0, the mapping $P_D(I - \lambda A^*\left(I - P_Q\right)A)$ is nonexpansive. These observations have spurred substantial interest among researchers in addressing split feasibility, split common null point, and split common fixed point problems; notable references include [1, 7, 21, 29].

Let T be a nonexpansive mapping from C into itself, where C is a nonempty subset of a real Hilbert space H. Various approximation methods have been explored extensively to find fixed points of nonexpansive mappings. Reich [22] used the following iterative method:

$$x_{n+1} = a_n x_n + (1 - a_n)\,Tx_n \quad \text{for all } n \in \mathbb{N}, \tag{1.4}$$

where $x_1 \in C$ is given, $\{a_n\} \subset [0, 1]$, and $\mathbb{N}$ represents the set of positive integers. It was established that the sequence $\{x_n\}$ converges weakly to a fixed point of T within the context of a Banach space. The iterative method, defined as (1.4), is commonly referred to as Mann type [18].

The generalization of the class of mappings has also received attention. In 2010, Kocourek et al. [11] introduced a broad class of nonlinear mappings: A mapping $T : C \to H$ is termed *generalized hybrid* if there exist $\alpha, \beta \in \mathbb{R}$ such that

$$\alpha\left\|Tx - Ty\right\|^2 + (1 - \alpha)\left\|x - Ty\right\|^2 \leq \beta\left\|Tx - y\right\|^2 + (1 - \beta)\left\|x - y\right\|^2 \tag{1.5}$$

for all $x, y \in C$. Setting $(\alpha, \beta) = (1, 0)$, we observe that T is nonexpansive. When $(\alpha, \beta) = (2, 1)$, T is categorized as *nonspreading* [12]:

$$2\left\|Tx - Ty\right\|^2 \leq \left\|Tx - y\right\|^2 + \left\|x - Ty\right\|^2$$

for all $x, y \in C$. If $(\alpha, \beta) = \left(\frac{3}{2}, \frac{1}{2}\right)$, then T is classified as a *hybrid mapping* [23]:

$$3\left\|Tx - Ty\right\|^2 \leq \left\|x - y\right\|^2 + \left\|Tx - y\right\|^2 + \left\|x - Ty\right\|^2$$

for all $x, y \in C$. According to Igarashi et al. [9], nonspreading and hybrid mappings are not necessarily continuous; see also [13, 29]. Kocourek et al. established weak convergence theorems of Baillon type [2] and Mann type [18] for finding fixed points of generalized hybrid mappings.

In 2017, Takahashi [24] introduced a class of nonlinear mappings encompassing generalized hybrid mappings. For a real number $\eta < 1$, a mapping $T : C \to H$ with $F(T) \neq \emptyset$ is called *η-demimetric* [24] if for any $x \in C$ and $q \in F(T)$,

$$\frac{1 - \eta}{2}\|x - Tx\|^2 \leq \langle x - q,\ x - Tx\rangle. \tag{1.6}$$

As will be pointed out in Example 2.1 in Section 2, this type of mappings includes generalized hybrid mappings as special cases, in addition to strict pseudo-contractions. The following theorem is a simplified version of Theorem 3.1 in Takahashi [25], which solves a split common fixed point problem by using the Mann type iterative method (1.4). In [25], H_2 is a smooth, strictly convex, and reflexive Banach space.

Theorem 1.1 ([25]). *Let H_1 and H_2 be real Hilbert spaces, let $A : H_1 \to H_2$ be a bounded linear operator with $A \neq O$, and let A^* be the adjoint operator of A. Let $U : H_1 \to H_1$ be a nonexpansive mapping and let $T : H_2 \to H_2$ be an η-demimetric and demiclosed mapping, where $\eta < 1$ is a real number. Suppose that*

$$\Omega \equiv F(U) \cap A^{-1}F(T) \neq \emptyset.$$

Let $\lambda \in \left(0,\ \frac{1-\eta}{\|A\|^2}\right)$. Let $a, b \in (0,1)$ with $a \leq b$ and let $\{\alpha_n\}$ be a sequence of real numbers that satisfies $0 < a \leq \alpha_n \leq b < 1$ for all $n \in \mathbb{N}$. Define a sequence $\{x_n\}$ in C as follows:

$$x_1 \in H_1 : \text{ given,}$$
$$x_{n+1} = \alpha_n x_n + (1-\alpha_n) U (I - \lambda A^* (I-T) A) x_n \in H_1$$

for all $n \in \mathbb{N}$. Then, the sequence $\{x_n\}$ converges weakly to a point $\overline{x} \in \Omega$, where $\overline{x} = \lim_{k\to\infty} P_\Omega x_k$ and P_Ω is the metric projection from H_1 onto Ω.

Let $S_1, S_2 : C \to C$ be nonlinear mappings. In 2019, Kondo and Takahashi [16] investigated *common fixed point problems* by employing the following iterative method:

$$x_{n+1} = a_n x_n + b_n S_1 x_n + c_n S_2 x_n \quad \text{for all } n \in \mathbb{N}, \tag{1.7}$$

where $x_1 \in C$ is given, $a_n, b_n, c_n \in (0,1)$ with $a_n + b_n + c_n = 1$, and S_1, S_2 belong to a broader class than the generalized hybrid mappings. This is an extended version of the Mann type iterative method (1.4). A sequence generated by the rule (1.7) converges weakly to a common fixed point of S_1 and S_2. For additional results, refer to, for instance, [14, 15, 20, 28].

This study introduces an approximation method to concurrently solve *common fixed point and split common fixed point problems*. This class of problem is formulated as follows:

$$\text{Find an element } \overline{x} \in F(S_1) \cap F(S_2) \cap F(U) \cap A^{-1}F(T). \tag{1.8}$$

In addressing this type of problem, we establish a weak convergence theorem using mappings S_1, S_2, U, and T. The mappings S_1, S_2, U in (1.8) are in more general class than those in [16] and Theorem 1.1. We utilize a demimetric mapping as T, a class encompassing strict pseudo-contractions and generalized hybrid mappings as special cases. Additionally, demimetric mappings exhibit a close relationship with inverse strongly monotone mappings. Initially, we prove a weak convergence theorem employing an extended version of the Mann type iterative method, as described in (1.7). Building upon the main result, we derive a series of theorems about split common fixed point problems, split feasibility problems, split common null point problems, and equilibrium problems in real Hilbert spaces. These problems will be studied combined with common fixed point problems.

2. Preliminaries

This section provides preliminary information. Let $\{x_n\}$ be a sequence in a real Hilbert space H and let x be a point of H. Strong and weak convergence of $\{x_n\}$ to x are represented by $x_n \to x$ and $x_n \rightharpoonup x$, respectively. It is known that $x_n \rightharpoonup x$ if and only if for any subsequence $\{x_{n_i}\}$ of $\{x_n\}$, there is a subsequence $\{x_{n_j}\}$ of $\{x_{n_i}\}$ such that $x_{n_j} \rightharpoonup x$. If $x_n \rightharpoonup x$ and $y_n \to y$, then $\langle x_n, y_n \rangle \to \langle x, y \rangle$.

Let $a, b, c, d \in \mathbb{R}$ with $a+b+c+d=1$. For $x, y, z, w \in H$, it holds that

$$\begin{aligned}\|ax+by+cz+dw\|^2 &= a\|x\|^2+b\|y\|^2+c\|z\|^2+d\|w\|^2 \\ &\quad -ab\|x-y\|^2-ac\|x-z\|^2-ad\|x-w\|^2 \\ &\quad -bc\|y-z\|^2-bd\|y-w\|^2-cd\|z-w\|^2; \end{aligned} \tag{2.1}$$

see [20, 31]. Note that the conditions $a, b, c, d \in [0,1]$ are unnecessary.

A mapping $T : C \to H$ is termed *inverse strongly monotone* if there exists a positive real number $\alpha > 0$ such that

$$\alpha\|Tx-Ty\|^2 \leq \langle x-y,\ Tx-Ty\rangle \tag{2.2}$$

for all $x, y \in C$; refer to Liu and Nashed [17]. An inverse strongly monotone mapping $T : C \to H$ with $\alpha = 1$ is *firmly nonexpansive*:

$$\|Tx-Ty\|^2 \leq \langle x-y,\ Tx-Ty\rangle$$

for all $x, y \in C$. A firmly nonexpansive mapping is nonexpansive.

Let P_Ω be the *metric projection* from H onto Ω, where Ω be a nonempty, closed, and convex subset of H. This implies that $\|x-P_\Omega x\| = \inf_{v\in\Omega}\|x-v\|$ for all $x \in H$. For the metric projection P_Ω from H onto Ω, it holds that $F(P_\Omega) = \Omega$ and

$$\langle x-P_\Omega x,\ P_\Omega x-v\rangle \geq 0 \tag{2.3}$$

for all $x \in H$ and $v \in \Omega$. The metric projection is firmly nonexpansive and therefore, it is nonexpansive.

The following lemma is necessary to prove Theorem 3.1:

Lemma 2.1 ([26])**.** *Let Ω be a nonempty, closed, and convex subset of H, let P_Ω be the metric projection from H onto Ω, and let $\{x_n\}$ be a sequence in H. Assume that*

$$\|x_{n+1}-q\| \leq \|x_n-q\| \quad \text{for all } q \in \Omega \text{ and } n \in \mathbb{N}. \tag{2.4}$$

Then, $\{P_\Omega x_n\}$ is convergent in Ω.

A mapping $T : C \to H$ with $F(T) \neq \emptyset$ is termed *quasi-nonexpansive* if

$$\|Tx-q\| \leq \|x-q\| \quad \text{for all } x \in C \text{ and } q \in F(T), \tag{2.5}$$

where C is a nonempty subset of H. Assume that C is closed and convex in H. Itoh and Takahashi [10] proved that the set of fixed points of a quasi-nonexpansive mapping is closed and convex. A mapping $T : C \to H$ is called *demiclosed* if for a sequence $\{x_n\}$ in C, the following holds:

$$x_n - Tx_n \to 0 \text{ and } x_n \rightharpoonup v \Longrightarrow v \in F(T). \tag{2.6}$$

According to Kocourek et al. [11], a generalized hybrid mapping (1.5) is quasi-nonexpansive and demiclosed when it possesses a fixed point. More precisely, the following lemma holds:

Lemma 2.2 ([11, 30])**.** *Let C be a nonempty subset of H and let $T : C \to H$ be a generalized hybrid mapping. Then, the following assertions hold:*

(a) *The mapping T is quasi-nonexpansive if $F(T)$ is not empty;*
(b) *The mapping T is demiclosed if C is closed and convex.*

As a nonexpansive mapping belongs to the class of generalized hybrid mappings, it is quasi-nonexpansive and demiclosed when it possesses a fixed point and its domain is closed and convex.

Let k be a real number with $0 \leq k < 1$. A mapping $T : C \to H$ is termed a *k-strict pseudo-contraction* [4] if

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + k\| (I - T) x - (I - T) y\|^2 \tag{2.7}$$

for all $x, y \in C$. A strict pseudo-contraction with $k = 0$ is nonexpansive. A strict pseudo-contraction mapping is demiclosed:

Lemma 2.3 ([19, 27])**.** *Let C be a nonempty, closed, and convex subset of H. Let $T : C \to H$ be a k-strict pseudo-contraction, where $0 \leq k < 1$. Then, T is demiclosed.*

Let $B : H \to 2^H$ be a multi-valued mapping defined on H, also denoted by $B \subset H \times H$. Its effective domain is represented by $D(B) = \{x \in H : Bx \neq \emptyset\}$. A multi-valued mapping $B \subset H \times H$ is termed *monotone* if $\langle x - y,\ u - v\rangle \geq 0$ for all $x, y \in D(B)$, $u \in Bx$, and $v \in By$. For a monotone multi-valued mapping B on H and $r > 0$, we define

$$J_r \equiv (I + rB)^{-1},$$

known as the *resolvent* of B for $r > 0$. The resolvent is single-valued and firmly nonexpansive. Furthermore, $F(J_r) = B^{-1}0$ for all $r > 0$, where $B^{-1}0$ is the set of null points of B, that is, $B^{-1}0 = \{x \in H : 0 \in Bx\}$.

A monotone mapping is *maximal* if any other monotone mappings on H do not adequately contain its graph. For a maximal monotone multi-valued mapping $B \subset H \times H$, its null point set $B^{-1}0$ is a closed and convex subset of its effective domain $D(B)$. If a multi-valued mapping B is maximal monotone, its resolvent J_r is defined over the entire domain of H.

Some examples of η-demimetric mappings, which is defined as (1.6), were provided by Takahashi [24, 25]:

Example 2.1. (*demimetric mappings*)

(1) *Let $T : C \to H$ be a generalized hybrid mapping* (1.5) *with $F(T) \neq \emptyset$. Then, T is* 0*-demimetric; see* [25]*. Consequently, a nonexpansive mapping with $F(T) \neq \emptyset$ is* 0*-demimetric, as it is a special case of a generalized hybrid mapping.*

(2) *Let P_C be the metric projection of H onto C. Then, P_C is* (-1)*-demimetric; see* [24]*.*

(3) *Let T be a k-strict pseudo-contraction* (2.7) *with $F(T) \neq \emptyset$. Then, T is k-demimetric; see* [24]*.*

(4) *Let B be a maximal monotone mapping with $B^{-1}0 \neq \emptyset$ and let $r > 0$. Then, the resolvent J_r of B for r is* (-1)*-demimetric; see* [24]*. Note that $B^{-1}0 = F(J_r)$.*

(5) *Let $V : C \to H$ be an α-inverse strongly monotone mapping* (2.2) *such that $V^{-1}0 \neq \emptyset$, where $\alpha > 0$. Then, $I - V$ is* $(1 - 2\alpha)$*-demimetric; see* [24]*. Note that $V^{-1}0 = F(I - V)$.*

The subsequent lemma plays a crucial role in establishing Theorem 3.1:

Lemma 2.4 ([24])**.** *Let C be a nonempty, closed, and convex subset of H. Suppose η is a real number with $\eta < 1$ and let T be an η-demimetric mapping of C into H with $F(T) \neq \emptyset$. Then, $F(T)$ is closed and convex.*

3. Main theorem

In this section, we establish a weak convergence theorem employing a Mann type iterative method such as (1.7). The theorem addresses the simultaneous resolution of common fixed point and split common fixed point problems (1.8) in two real Hilbert spaces:

Theorem 3.1. *Let H_1 and H_2 be real Hilbert spaces, let $A : H_1 \to H_2$ be a bounded linear operator with $A \neq O$, and let A^* be the adjoint operator of A. Let C be a nonempty, closed, and convex subset of H_1, let S_1 and S_2 be quasi-nonexpansive and demiclosed mappings from C into itself, and let U be a quasi-nonexpansive and demiclosed mapping from H_1 into C. Let T be an η-demimetric and demiclosed mapping from H_2 into itself, where $\eta < 1$ is a real number. Suppose that*

$$\Omega \equiv F(S_1) \cap F(S_2) \cap F(U) \cap A^{-1}F(T) \neq \emptyset.$$

Let $\underline{\lambda}, \overline{\lambda} \in \left(0, \frac{1-\eta}{\|A\|^2}\right)$ with $\underline{\lambda} \leq \overline{\lambda}$ and let $\{\lambda_n\}$ be a sequence of real numbers such that $0 < \underline{\lambda} \leq \lambda_n \leq \overline{\lambda} < \frac{1-\eta}{\|A\|^2}$ for all $n \in \mathbb{N}$. Let $a, b \in (0,1)$ with $a \leq b$ and let $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, and $\{d_n\}$ be sequences of real numbers that satisfy $0 < a \leq a_n, b_n, c_n, d_n \leq b < 1$ and $a_n + b_n + c_n + d_n = 1$ for all $n \in \mathbb{N}$. Define a sequence $\{x_n\}$ in C as follows:

$$\begin{aligned} &x_1 \in C : \text{ given}, \\ &x_{n+1} = a_n x_n + b_n S_1 x_n + c_n S_2 x_n + d_n U\left(I - \lambda_n A^*(I-T)A\right)x_n \in C \end{aligned}$$

for all $n \in \mathbb{N}$. Then, the sequence $\{x_n\}$ converges weakly to a point $\overline{x} \in \Omega$, where $\overline{x} = \lim_{k\to\infty} P_\Omega x_k$ and P_Ω is the metric projection from H_1 onto Ω.

Proof. First, note that $\{x_n\}$ is a sequence in C as $U : H_1 \to C$ and C is convex. The set $\Omega = F(S_1) \cap F(S_2) \cap F(U) \cap A^{-1}F(T) \, (\subset C)$ is closed and convex. Indeed, as S_1, S_2, and U are quasi-nonexpansive (2.5), $F(S_1)$, $F(S_2)$, and $F(U)$ are closed and convex subsets of C. As T is η-demimetric (1.6), it follows from Lemma 2.4 that $F(T) \, (\subset H_2)$ is closed and convex. As A is linear and continuous, the inverse image $A^{-1}F(T) \, (\subset H_1)$ of $F(T)$ is also closed and convex. Therefore, Ω is closed and convex, as claimed. As $\Omega \neq \emptyset$ is assumed, the metric projection P_Ω from H_1 onto Ω exists.

Define

$$\begin{aligned} y_n &= \left(I - \lambda_n A^*(I-T)A\right)x_n \\ &= x_n - \lambda_n A^*(Ax_n - TAx_n) \in H_1 \end{aligned} \tag{3.1}$$

for each $n \in \mathbb{N}$. Then, we can simply write

$$x_{n+1} = a_n x_n + b_n S_1 x_n + c_n S_2 x_n + d_n U y_n \in C.$$

Observe that

$$\|Uy_n - q\| \leq \|x_n - q\| \quad \text{for all } q \in \Omega \text{ and } n \in \mathbb{N}. \tag{3.2}$$

Let $q \in \Omega \subset F(U) \cap A^{-1}F(T)$ and $n \in \mathbb{N}$. As $q \in A^{-1}F(T)$, it holds that $Aq \in F(T)$. Furthermore, recalling that U is quasi-nonexpansive (2.5), T is η-demimetric (1.6), and $0 < \lambda_n < \frac{1-\eta}{\|A\|^2}$, we can utilize these facts to obtain

$$\begin{aligned}
&\|Uy_n - q\|^2\\
&\le \|y_n - q\|^2\\
&= \|x_n - \lambda_n A^*(Ax_n - TAx_n) - q\|^2\\
&= \|x_n - q\|^2 - 2\lambda_n\langle x_n - q,\ A^*(Ax_n - TAx_n)\rangle + \|\lambda_n A^*(Ax_n - TAx_n)\|^2\\
&= \|x_n - q\|^2 - 2\lambda_n\langle Ax_n - Aq,\ Ax_n - TAx_n\rangle + \|\lambda_n A^*(Ax_n - TAx_n)\|^2\\
&\le \|x_n - q\|^2 - \lambda_n(1-\eta)\|Ax_n - TAx_n\|^2 + (\lambda_n)^2\|A\|^2\|Ax_n - TAx_n\|^2\\
&= \|x_n - q\|^2 - \lambda_n\left(1 - \eta - \lambda_n\|A\|^2\right)\|Ax_n - TAx_n\|^2 \qquad (3.3)\\
&\le \|x_n - q\|^2.
\end{aligned}$$

This demonstrates that (3.2) holds, as asserted.

Using (3.2), we can show that

$$\|x_{n+1} - q\| \le \|x_n - q\| \quad \text{for all } q \in \Omega \text{ and } n \in \mathbb{N}. \qquad (3.4)$$

Indeed, as S_1 and S_2 are quasi-nonexpansive and $q \in \Omega$, it holds that

$$\begin{aligned}
&\|x_{n+1} - q\|\\
&= \|a_n x_n + b_n S_1 x_n + c_n S_2 x_n + d_n U y_n - q\|\\
&= \|a_n (x_n - q) + b_n (S_1 x_n - q) + c_n (S_2 x_n - q) + d_n (Uy_n - q)\|\\
&\le a_n \|x_n - q\| + b_n \|S_1 x_n - q\| + c_n \|S_2 x_n - q\| + d_n \|Uy_n - q\|\\
&\le a_n \|x_n - q\| + b_n \|x_n - q\| + c_n \|x_n - q\| + d_n \|x_n - q\|\\
&= \|x_n - q\|.
\end{aligned}$$

This shows that (3.4) holds true, as described above. Thus, the following consequences emerge: (i) $\{\|x_n - q\|\}$ is convergent for all $q \in \Omega$; (ii) $\{x_n\}$ is bounded; (iii) From Lemma 2.1, $\{P_\Omega x_n\}$ is convergent in Ω, implying that $\overline{x} = \lim_{n\to\infty} P_\Omega x_n$ exists.

We show that

$$Ax_n - TAx_n \to 0. \qquad (3.5)$$

Using (3.3) yields

$$\begin{aligned}
&\quad \|x_{n+1} - q\|^2\\
&= \|a_n (x_n - q) + b_n (S_1 x_n - q) + c_n (S_2 x_n - q) + d_n (Uy_n - q)\|^2\\
&\le a_n\|x_n - q\|^2 + b_n\|S_1 x_n - q\|^2 + c_n\|S_2 x_n - q\|^2 + d_n\|Uy_n - q\|^2\\
&\le a_n\|x_n - q\|^2 + b_n\|x_n - q\|^2 + c_n\|x_n - q\|^2\\
&\quad + d_n\left\{\|x_n - q\|^2 - \lambda_n(1 - \eta - \lambda_n\|A\|^2)\|Ax_n - TAx_n\|^2\right\}\\
&= \|x_n - q\|^2 - d_n\lambda_n(1 - \eta - \lambda_n\|A\|^2)\|Ax_n - TAx_n\|^2.
\end{aligned}$$

Consequently,

$$d_n\lambda_n(1 - \eta - \lambda_n\|A\|^2)\|Ax_n - TAx_n\|^2 \le \|x_n - q\|^2 - \|x_{n+1} - q\|^2$$

for all $q \in \Omega$ and $n \in \mathbb{N}$. As $\left\{\|x_n - q\|^2\right\}$ is convergent, based on the assumptions regarding d_n and λ_n, we obtain (3.5).

Observe that

$$x_n - y_n \to 0. \tag{3.6}$$

Indeed, from (3.1) and (3.5), it holds that

$$\begin{aligned}\|x_n - y_n\| &= \|\lambda_n A^*(Ax_n - TAx_n)\| \\ &\leq \overline{\lambda}\,\|A^*\|\,\|Ax_n - TAx_n\| \to 0.\end{aligned}$$

This confirms that (3.6) holds true, as claimed.

Next, we verify that

$$x_n - S_1x_n \to 0, \quad x_n - S_2x_n \to 0, \tag{3.7}$$

$$x_n - Uy_n \to 0 \tag{3.8}$$

as $n \to \infty$. Indeed, using (2.1) and (3.2) results in

$$\begin{aligned}
& \|x_{n+1} - q\|^2 \\
= \; & \|a_n(x_n - q) + b_n(S_1x_n - q) + c_n(S_2x_n - q) + d_n(Uy_n - q)\|^2 \\
= \; & a_n\|x_n - q\|^2 + b_n\|S_1x_n - q\|^2 + c_n\|S_2x_n - q\|^2 + d_n\|Uy_n - q\|^2 \\
& -a_nb_n\|x_n - S_1x_n\|^2 - a_nc_n\|x_n - S_2x_n\|^2 - a_nd_n\|x_n - Uy_n\|^2 \\
& -b_nc_n\|S_1x_n - S_2x_n\|^2 - b_nd_n\|S_1x_n - Uy_n\|^2 - c_nd_n\|S_2x_n - Uy_n\|^2 \\
\leq \; & a_n\|x_n - q\|^2 + b_n\|x_n - q\|^2 + c_n\|x_n - q\|^2 + d_n\|x_n - q\|^2 \\
& -a_nb_n\|x_n - S_1x_n\|^2 - a_nc_n\|x_n - S_2x_n\|^2 - a_nd_n\|x_n - Uy_n\|^2 \\
\leq \; & \|x_n - q\|^2 \\
& -a_nb_n\|x_n - S_1x_n\|^2 - a_nc_n\|x_n - S_2x_n\|^2 - a_nd_n\|x_n - Uy_n\|^2.
\end{aligned}$$

This implies that

$$\begin{aligned} & a_nb_n\|x_n - S_1x_n\|^2 + a_nc_n\|x_n - S_2x_n\|^2 + a_nd_n\|x_n - Uy_n\|^2 \\ & \leq \|x_n - q\|^2 - \|x_{n+1} - q\|^2 \end{aligned} \tag{3.9}$$

for all $q \in \Omega$ and $n \in \mathbb{N}$. As the sequence $\left\{\|x_n - q\|^2\right\}$ is convergent, we obtain (3.7) and (3.8), as stated. From (3.6) and (3.8), we have

$$y_n - Uy_n \to 0. \tag{3.10}$$

Our objective is to prove that $x_n \rightharpoonup \overline{x}\,(=\lim_{k\to\infty} P_\Omega x_k)$. Let $\{x_{n_i}\}$ be a subsequence of $\{x_n\}$. According to (ii), $\{x_{n_i}\}$ is bounded. Hence, there exist a subsequence $\{x_{n_j}\}$ of $\{x_{n_i}\}$ and $v \in H_1$ such that $x_{n_j} \rightharpoonup v$. Our task is to demonstrate that $v \in \Omega$. As S_1 and S_2 are demiclosed (2.6), according to (3.7), we have $v \in F(S_1) \cap F(S_2)$. As A is bounded and linear, from $x_{n_j} \rightharpoonup v$, it follows that $Ax_{n_j} \rightharpoonup Av$. As T is demiclosed, from (3.5), we conclude that $Av \in F(T)$, indicating that $v \in A^{-1}F(T)$. Moreover, from $x_{n_j} \rightharpoonup v$ and (3.6), it follows that $y_n \rightharpoonup v$. As U is demiclosed, $v \in F(U)$ also holds from (3.10). Therefore, $v \in \Omega$, as asserted. Consequently, from (2.3),

$$\left\langle x_{n_j} - P_\Omega x_{n_j},\; P_\Omega x_{n_j} - v\right\rangle \geq 0$$

for all $j \in \mathbb{N}$. As $x_{n_j} \rightharpoonup v$ and $P_\Omega x_n \to \overline{x}$, we obtain $\langle v - \overline{x},\; \overline{x} - v\rangle \geq 0$. This implies that $v = \overline{x}$. Therefore, we can conclude that $x_n \rightharpoonup \overline{x}$. The proof is completed. □

4. Applications

In this section, leveraging Theorem 3.1, we derive weak convergence theorems concerning split common fixed point problems, split feasibility problems, split common null point problems, and equilibrium problems. These problems will be analyzed in conjunction with common fixed point problems. Recall the following facts: (i) metric projections and resolvents are firmly nonexpansive and (-1)-demimetric; (ii) a firmly nonexpansive mapping is nonexpansive; (iii) a nonexpansive mapping is a specific case of a generalized hybrid mapping (1.5); (iv) a generalized hybrid mapping with a fixed point is 0-demimetric and demiclosed. Refer to Section 2 for more details.

4.1. **Common fixed point and split common fixed point problems.** The following theorem solves common fixed point problems for generalized hybrid mappings with the iterative scheme (1.7):

Theorem 4.1 ([16]). *Let C be a nonempty, closed, and convex subset of a real Hilbert space H and let $S_1, S_2 : C \to C$ be generalized hybrid mappings such that*

$$\Omega \equiv F(S_1) \cap F(S_2) \neq \emptyset.$$

Let $a, b \in (0,1)$ with $a \leq b$ and let $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$ be sequences of real numbers such that $0 < a \leq \alpha_n, \beta_n, \gamma_n \leq b < 1$ and $\alpha_n + \beta_n + \gamma_n = 1$ for all $n \in \mathbb{N}$. Define a sequence $\{x_n\}$ in C as follows:

$$\begin{aligned} &x_1 \in C : \text{ given,} \\ &x_{n+1} = \alpha_n x_n + \beta_n S_1 x_n + \gamma_n S_2 x_n \in C \end{aligned}$$

for all $n \in \mathbb{N}$. Then, the sequence $\{x_n\}$ converges weakly to a point $\overline{x} \in \Omega$, where $\overline{x} = \lim_{k\to\infty} P_\Omega x_k$ and P_Ω is the metric projection from H onto Ω.

Proof. In Theorem 3.1, set $H_1 = H_2 = H$ and $A = U = T = I$. Then, we have

$$F(S_1) \cap F(S_2) \cap F(U) \cap A^{-1}F(T) = F(S_1) \cap F(S_2).$$

As S_1 and S_2 are generalized hybrid mappings (1.5) from C into itself with nonempty fixed point sets $F(S_1)$ and $F(S_2)$, they are quasi-nonexpansive and demiclosed according to Lemma 2.2. Furthermore, set $a_n + d_n = \alpha_n$, $b_n = \beta_n$, and $c_n = \gamma_n$. Then, it holds that

$$\begin{aligned} x_{n+1} &= a_n x_n + b_n S_1 x_n + c_n S_2 x_n + d_n U\left(I - \lambda_n A^*\left(I - T\right)A\right)x_n \\ &= a_n x_n + b_n S_1 x_n + c_n S_2 x_n + d_n x_n \\ &= \alpha_n x_n + \beta_n S_1 x_n + \gamma_n S_2 x_n. \end{aligned}$$

Therefore, we obtain the desired result. □

We address weak convergence theorems that solve split common fixed point problems. First, note that Theorem 1.1 is derived from Theorem 3.1 by setting $S_1 = S_2 = I$, $a_n + b_n + c_n = \alpha_n$, and $\lambda_n = \lambda$ as a nonexpansive mapping U is quasi-nonexpansive and demiclosed if it has a fixed point.

The following theorem is also derived from Theorem 3.1:

Theorem 4.2. *Let H_1 and H_2 be real Hilbert spaces, let $A : H_1 \to H_2$ be a bounded linear operator with $A \neq O$, and let A^* be the adjoint operator of A. Let C be a nonempty, closed, and convex subset of H_1 and let P_C be the metric projection from H_1 onto C. Let S_1 and S_2 be generalized hybrid mappings from C into itself. Let*

T be a k-strict pseudo-contraction from H_2 into itself, where $0 \le k < 1$. Suppose that

$$\Omega \equiv F(S_1) \cap F(S_2) \cap A^{-1}F(T) \neq \emptyset.$$

Let $\underline{\lambda}, \overline{\lambda} \in \left(0, \frac{1-k}{\|A\|^2}\right)$ with $\underline{\lambda} \le \overline{\lambda}$ and let $\{\lambda_n\}$ be a sequence of real numbers such that $0 < \underline{\lambda} \le \lambda_n \le \overline{\lambda} < \frac{1-k}{\|A\|^2}$ for all $n \in \mathbb{N}$. Let $a, b \in (0,1)$ with $a \le b$ and let $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, and $\{d_n\}$ be sequences of real numbers that satisfy $0 < a \le a_n, b_n, c_n, d_n \le b < 1$ and $a_n + b_n + c_n + d_n = 1$ for all $n \in \mathbb{N}$. Define a sequence $\{x_n\}$ in C as follows:

$$x_1 \in C: \text{ given,}$$
$$x_{n+1} = a_n x_n + b_n S_1 x_n + c_n S_2 x_n + d_n P_C (I - \lambda_n A^* (I-T) A) x_n \in C$$

for all $n \in \mathbb{N}$. Then, the sequence $\{x_n\}$ converges weakly to a point $\overline{x} \in \Omega$, where $\overline{x} = \lim_{k\to\infty} P_\Omega x_k$ and P_Ω is the metric projection of H_1 onto Ω.

Proof. The generalized hybrid mappings S_1 and S_2 with fixed points are quasi-nonexpansive and demiclosed. Let $U = P_C$. As $U : H_1 \to C$ is firmly nonexpansive and $F(U) = C \neq \emptyset$, by Lemma 2.2, U is also quasi-nonexpansive and demiclosed. From Lemma 2.3, the k-strict pseudo-contraction $T : H_2 \to H_2$ is demiclosed. Additionally, from (3) in Example 2.1, the mapping T is k-demimetric. Therefore, the desired result follows from Theorem 3.1. □

If S is a nonexpansive mapping, then S^2 is also nonexpansive. Furthermore, it holds that $F(S) \cap F(S^2) = F(S)$. Based on these observations, we obtain the following theorem from Theorem 3.1:

Theorem 4.3. *Let H_1 and H_2 be real Hilbert spaces, let $A : H_1 \to H_2$ be a bounded linear operator with $A \neq O$, and let A^* be the adjoint operator of A. Let C be a nonempty, closed, and convex subset of H_1. Let S be a nonexpansive mapping from C into itself, let U be a nonexpansive mapping from H_1 into C, and let T be a nonexpansive mapping from H_2 into itself. Suppose that*

$$\Omega \equiv F(S) \cap F(U) \cap A^{-1}F(T) \neq \emptyset.$$

Let $\underline{\lambda}, \overline{\lambda} \in \left(0, \frac{1}{\|A\|^2}\right)$ with $\underline{\lambda} \le \overline{\lambda}$ and let $\{\lambda_n\}$ be a sequence of real numbers such that $0 < \underline{\lambda} \le \lambda_n \le \overline{\lambda} < \frac{1}{\|A\|^2}$ for all $n \in \mathbb{N}$. Let $a, b \in (0,1)$ with $a \le b$ and let $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, and $\{d_n\}$ be sequences of real numbers that satisfy $0 < a \le a_n, b_n, c_n, d_n \le b < 1$ and $a_n + b_n + c_n + d_n = 1$ for all $n \in \mathbb{N}$. Define a sequence $\{x_n\}$ in C as follows:

$$x_1 \in C: \text{ given,} \tag{4.1}$$
$$x_{n+1} = a_n x_n + b_n S x_n + c_n S^2 x_n + d_n U (I - \lambda_n A^* (I-T) A) x_n \in C$$

for all $n \in \mathbb{N}$. Then, the sequence $\{x_n\}$ converges weakly to a point $\overline{x} \in \Omega$, where $\overline{x} = \lim_{k\to\infty} P_\Omega x_k$ and P_Ω is the metric projection from H_1 onto Ω.

Proof. As S and S^2 are nonexpansive mappings from C into itself with $F(S)$ and $F(S^2)$ being nonempty, from Lemma 2.2, they are quasi-nonexpansive and demiclosed. Similarly, as $U : H_1 \to C$ is nonexpansive with $F(U) \neq \emptyset$, it is also quasi-nonexpansive and demiclosed. Additionally, according to Lemma 2.2 and Example 2.1, a nonexpansive mapping $T : H_2 \to H_2$ is demiclosed and 0-demimetric. Hence, we obtain the desired result from Theorem 3.1. □

For iterative methods such as (4.1), see [14, 15, 16, 20].

4.2. **Common fixed point and split feasibility problems.** Next, we derive a weak convergence theorem that simultaneously solves common fixed point and split feasibility problems:

Theorem 4.4. *Let H_1 and H_2 be real Hilbert spaces, let $A : H_1 \to H_2$ be a bounded linear operator with $A \neq O$, and let A^* be the adjoint operator of A. Let $C\,(\subset H_1)$, $D\,(\subset C \subset H_1)$, and $Q\,(\subset H_2)$ be nonempty, closed, and convex sets. Let P_D be the metric projection from H_1 onto D and let P_Q be the metric projection from H_2 onto Q. Let S_1 and S_2 be generalized hybrid mappings from C into itself. Suppose that*

$$\Omega \equiv F(S_1) \cap F(S_2) \cap D \cap A^{-1}Q \neq \emptyset.$$

Let $\underline{\lambda}, \overline{\lambda} \in \left(0, \frac{2}{\|A\|^2}\right)$ with $\underline{\lambda} \leq \overline{\lambda}$ and let $\{\lambda_n\}$ be a sequence of real numbers such that $0 < \underline{\lambda} \leq \lambda_n \leq \overline{\lambda} < \frac{2}{\|A\|^2}$ for all $n \in \mathbb{N}$. Let $a, b \in (0,1)$ with $a \leq b$ and let $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, and $\{d_n\}$ be sequences of real numbers that satisfy $0 < a \leq a_n, b_n, c_n, d_n \leq b < 1$ and $a_n + b_n + c_n + d_n = 1$ for all $n \in \mathbb{N}$. Define a sequence $\{x_n\}$ in C as follows:

$$\begin{aligned} &x_1 \in C: \text{ given,} \\ &x_{n+1} \equiv a_n x_n + b_n S_1 x_n + c_n S_2 x_n + d_n P_D \left(I - \lambda_n A^* \left(I - P_Q\right) A\right) x_n \in C \end{aligned}$$

for all $n \in \mathbb{N}$. Then, the sequence $\{x_n\}$ converges weakly to a point $\overline{x} \in \Omega$, where $\overline{x} = \lim_{k\to\infty} P_\Omega x_k$ and P_Ω is the metric projection from H_1 onto Ω.

Proof. The generalized hybrid mappings $S_1, S_2 : C \to C$ are quasi-nonexpansive and and demiclosed as $F(S_1)$ and $F(S_2)$ are nonempty. As P_D is firmly nonexpansive with $F(P_D) = D \neq \emptyset$, according to Lemma 2.2, it is a quasi-nonexpansive and demiclosed. Similarly, as P_Q is firmly nonexpansive, it is demiclosed. Furthermore, as $F(P_Q)\,(= Q)$ is nonempty, from (2) in Example 2.1, it is (-1)-demimetric. Therefore, setting $U = P_D$ and $T = P_Q$ in Theorem 3.1, we obtain the desired result. □

4.3. **Common fixed point and split common null point problems.** The subsequent two theorems are pertinent for addressing common fixed point and split common null point problems. According to (5) in Example 2.1, if V is an α-inverse strongly monotone mapping with $V^{-1}0 \neq \emptyset$, then $I - V$ is $(1 - 2\alpha)$-demimetric, where $\alpha > 0$. Therefore, the following theorem is obtained:

Theorem 4.5. *Let H_1 and H_2 be real Hilbert spaces, let $A : H_1 \to H_2$ be a bounded linear operator with $A \neq O$, and let A^* be the adjoint operator of A. Let C be a nonempty, closed, and convex subset of H_1. Let $B \subset H_1 \times H_1$ be a maximal monotone mapping such that its effective domain is included in C and let $J_r^B = (I + rB)^{-1}$ be the resolvent of B for $r > 0$. Let S_1 and S_2 be generalized hybrid mappings from C into itself. Let V be an α-inverse strongly monotone mapping from H_2 into itself, where $\alpha > 0$. Suppose that*

$$\Omega \equiv F(S_1) \cap F(S_2) \cap B^{-1}0 \cap A^{-1}\left(V^{-1}0\right) \neq \emptyset.$$

Let $\underline{\lambda}, \overline{\lambda} \in \left(0, \frac{2\alpha}{\|A\|^2}\right)$ with $\underline{\lambda} \leq \overline{\lambda}$ and let $\{\lambda_n\}$ be a sequence of real numbers such that $0 < \underline{\lambda} \leq \lambda_n \leq \overline{\lambda} < \frac{2\alpha}{\|A\|^2}$ for all $n \in \mathbb{N}$. Let $a, b \in (0,1)$ with $a \leq b$ and let $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, and $\{d_n\}$ be sequences of real numbers that satisfy $0 < a \leq$

$a_n, b_n, c_n, d_n \le b < 1$ and $a_n + b_n + c_n + d_n = 1$ for all $n \in \mathbb{N}$. Define a sequence $\{x_n\}$ in C as follows:

$$x_1 \in C : \text{ given,}$$
$$x_{n+1} = a_n x_n + b_n S_1 x_n + c_n S_2 x_n + d_n J_r^B \left(I - \lambda_n A^* V A\right) x_n \in C$$

for all $n \in \mathbb{N}$. Then, the sequence $\{x_n\}$ converges weakly to a point $\overline{x} \in \Omega$, where $\overline{x} = \lim_{k\to\infty} P_\Omega x_k$ and P_Ω is the metric projection of H_1 onto Ω.

Proof. From Lemma 2.2, the generalized hybrid mappings $S_1, S_2 : C \to C$ are quasi-nonexpansive and demiclosed. As the effective domain of B is included in C, the range of J_r^B is contained in C, that is, $J_r^B : H_1 \to C$. Moreover, as J_r^B is firmly nonexpansive with $F\left(J_r^B\right) = B^{-1}0 \neq \emptyset$, from Lemma 2.2, it is quasi-nonexpansive and demiclosed. From (5) in Example 2.1, $I - V$ is $(1-2\alpha)$-demimetric as $V^{-1}0 \neq \emptyset$. Setting $U = J_r^B$ and $T = I - V$ in Theorem 3.1, we have the desired result. □

Theorem 4.6. *Let H_1 and H_2 be real Hilbert spaces, let $A : H_1 \to H_2$ be a bounded linear operator with $A \neq O$, and let A^* be the adjoint operator of A. Let C be a nonempty, closed, and convex subset of H_1. Let $B \subset H_1 \times H_1$ be a maximal monotone mapping such that its effective domain is included in C and let $J_r^B = (I + rB)^{-1}$ be the resolvent of B for $r > 0$. Let $G \subset H_2 \times H_2$ be a maximal monotone mapping and let $J_s^G = (I + sG)^{-1}$ be the resolvent of G for $s > 0$. Let S_1 and S_2 be generalized hybrid mappings from C into itself. Suppose that*

$$\Omega \equiv F(S_1) \cap F(S_2) \cap B^{-1}0 \cap A^{-1}\left(G^{-1}0\right) \neq \emptyset.$$

Let $\underline{\lambda}, \overline{\lambda} \in \left(0, \frac{2}{\|A\|^2}\right)$ with $\underline{\lambda} \le \overline{\lambda}$ and let $\{\lambda_n\}$ be a sequence of real numbers such that $0 < \underline{\lambda} \le \lambda_n \le \overline{\lambda} < \frac{2\alpha}{\|A\|^2}$ for all $n \in \mathbb{N}$. Let $a, b \in (0,1)$ with $a \le b$ and let $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, and $\{d_n\}$ be sequences of real numbers that satisfy $0 < a \le a_n, b_n, c_n, d_n \le b < 1$ and $a_n + b_n + c_n + d_n = 1$ for all $n \in \mathbb{N}$. Define a sequence $\{x_n\}$ in C as follows:

$$x_1 \in C : \text{ given,}$$
$$x_{n+1} = a_n x_n + b_n S_1 x_n + c_n S_2 x_n + d_n J_r^B \left(I - \lambda_n A^* \left(I - J_s^G\right) A\right) x_n \in C$$

for all $n \in \mathbb{N}$. Then, the sequence $\{x_n\}$ converges weakly to a point $\overline{x} \in \Omega$, where $\overline{x} = \lim_{k\to\infty} P_\Omega x_k$ and P_Ω is the metric projection of H_1 onto Ω.

Proof. First, note that the generalized hybrid mappings $S_1, S_2 : C \to C$ with fixed points are quasi-nonexpansive and demiclosed. Set $U = J_r^B$ and $T = J_s^G$ in Theorem 3.1. Then, $J_r^B : H_1 \to C$ is firmly nonexpansive and thus, it is quasi-nonexpansive and demiclosed. Furthermore, J_s^G is demiclosed and (-1)-demimetric. Thus, we have the desired result. □

4.4. **common fixed point and equilibrium problems.** Finally, we address common fixed point and equilibrium problems. Let C be a nonempty, closed, and convex subset of a real Hilbert space H and let f be a bifunction from $C \times C$ into $\mathbb{R}$. The *equilibrium problem* is as follows:

Find an element $\overline{x} \in C$ such that $f(\overline{x}, y) \ge 0$ for all $y \in C$.

The set of solutions is denoted by

$$EP(f) = \{z \in C : f(z,y) \ge 0 \text{ for all } y \in C\}.$$

Following the established literature, we make the following assumptions:

(A1) $f(x,x)=0$ for all $x\in C$;
(A2) f is monotone, that is, $f(x,y)+f(y,x)\leq 0$ for all $x,y\in C$;
(A3) $f(x,\cdot):C\to\mathbb{R}$ is convex and lower semi-continuous for all $x\in C$;
(A4) $\limsup_{t\downarrow 0} f(tz+(1-t)x,\ y)\leq f(x,y)$ for all $x,y,z\in C$.

We are aware of the following results:

Lemma 4.1 ([3]). *Let C be a nonempty, closed, and convex subset of H and let $f:C\times C\to\mathbb{R}$ be a bifunction satisfying the assumptions (A1)–(A4). Let $r>0$ and $x\in H$. Then, there exists $z\in C$ such that*

$$f(z,y)+\frac{1}{r}\langle y-z,\ z-x\rangle\geq 0$$

for all $y\in C$.

Lemma 4.2 ([8]). *Let C be a nonempty, closed, and convex subset of H and let $f:C\times C\to\mathbb{R}$ be a bifunction satisfying the assumptions (A1)–(A4). For $r>0$, define the resolvent $T_r:H\to C$ of f for $r>0$ as follows:*

$$T_r x=\left\{z\in C: f(z,y)+\frac{1}{r}\langle y-z,\ z-x\rangle\geq 0 \text{ for all } y\in C\right\} \tag{4.2}$$

for all $x\in H$. Then, the following assertions hold:
(a) *T_r is single-valued;*
(b) *T_r is firmly nonexpansive;*
(c) *$F(T_r)=EP(f)$ for all $r>0$;*
(d) *$EP(f)$ is closed and convex.*

Utilizing these lemmas and Theorem 3.1, we derive the subsequent weak convergence theorem, which effectively tackles both common fixed point and equilibrium problems:

Theorem 4.7. *Let C be a nonempty, closed, and convex subset of a real Hilbert space H. Let $f:C\times C\to\mathbb{R}$ be a bifunction satisfying* (A1)–(A4)*. Let $T_r:H\to C$ be the resolvent (defined as (4.2)) of f for $r>0$. Let S_1 and S_2 be generalized hybrid mappings from C into itself. Suppose that*

$$\Omega\equiv F(S_1)\cap F(S_2)\cap EP(f)\neq\emptyset.$$

Let $a,b\in(0,1)$ with $a\leq b$ and let $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, and $\{d_n\}$ be sequences of real numbers that satisfy $0<a\leq a_n,b_n,c_n,d_n\leq b<1$ and $a_n+b_n+c_n+d_n=1$ for all $n\in\mathbb{N}$. Define a sequence $\{x_n\}$ in C as follows:

$$x_1\in C:\ \text{given},$$
$$x_{n+1}=a_nx_n+b_nS_1x_n+c_nS_2x_n+d_nT_rx_n\in C$$

for all $n\in\mathbb{N}$. Then, the sequence $\{x_n\}$ converges weakly to a point $\overline{x}\in\Omega$, where $\overline{x}=\lim_{k\to\infty}P_\Omega x_k$ and P_Ω is the metric projection from H_1 onto Ω. .

Proof. In Theorem 3.1, set $H_1=H_2=H$ and $A=T=I$. Then, it follows that $I-\lambda_nA^*(I-T)A=I$ for all $n\in\mathbb{N}$. Set $U=T_r$. Then, it holds from Lemma 4.2 that $U:H\to C$ is firmly nonexpansive with $F(U)\,(=EP(f))\neq\emptyset$. Consequently, from Lemma 2.2, $U\,(=T_r)$ is quasi-nonexpansive and demiclosed. Furthermore, it holds that $F(S_1)\cap F(S_2)\cap F(U)\cap A^{-1}F(T)=F(S_1)\cap F(S_2)\cap EP(f)=\Omega$. Therefore, we obtain the desired result. □

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References

[1] S.M. Alsulami and W. Takahashi, *The split common null point problem for maximal monotone mappings in Hilbert spaces and applications*, J. Nonlinear Convex Anal. **15**(4) (2014), 793-808.

[2] J.B. Baillon, *Un théorème de type ergodique pour les contractions non linéaires dans un espace de Hilbert*, C. R. Acad. Sci. Ser. A–B **280** (1975), 1511-1514.

[3] E. Blum and W. Oettli, *From optimization and variational inequalities to equilibrium problems*, Math. student **63**(1) (1994), 123-145.

[4] F.E. Browder and W.V. Petryshyn, *Construction of fixed points of nonlinear mappings in Hilbert spaces*, J. Math. Anal. Appl. **20**(2) (1967), 197–228.

[5] C. Byrne, Y. Censor, A. Gibali, and S. Reich, *The split common null point problem*, J. Nonlinear Convex Anal. **13**(4) (2012), 759–775.

[6] Y. Censor and T. Elfving, *A multiprojection algorithm using Bregman projections in a product space*, Numerical Algorithms **8** (1994), 221-239.

[7] Y. Censor and A. Segal, *The split common fixed point problem for directed operators*, J. Convex Anal. **16**(2) (2009), 587-600.

[8] P.L. Combettes and S.A. Hirstoaga, *Equilibrium programming in Hilbert spaces*, J. Nonlinear Convex Anal. **6**(1) (2005), 117-136.

[9] T. Igarashi, W. Takahashi, and K. Tanaka, *Weak convergence theorems for nonspreading mappings and equilibrium problems*, in Nonlinear Analysis and Optimization (S. Akashi, W. Takahashi and T. Tanaka Eds.), Yokohama Publishers, Yokohama, 2008, 75–85.

[10] S. Itoh and W. Takahashi, *The common fixed point theory of singlevalued mappings and multivalued mappings*, Pacific J. Math. **79**(2) (1978), 493–508.

[11] P. Kocourek, W. Takahashi, and J.-C. Yao, *Fixed point theorems and weak convergence theorems for generalized hybrid mappings in Hilbert Spaces*, Taiwanese J. Math. **14**(6) (2010), 2497-2511.

[12] F. Kohsaka and W. Takahashi, *Fixed point theorems for a class of nonlinear mappings related to maximal monotone operators in Banach spaces*, Arch. Math. **91**(2) (2008), 166–177.

[13] A. Kondo, *Generalized common fixed point theorem for generalized hybrid mappings in Hilbert spaces*, Demonstr. Math., **55** (2022), 752-759.

[14] A. Kondo, *Strong approximation using hybrid methods to find common fixed points of noncommutative nonlinear mappings in Hilbert spaces*, J. Nonlinear Convex Anal. **23**(1) (2022), 33–58.

[15] A. Kondo and W. Takahashi, *Attractive point and weak convergence theorems for normally N-generalized hybrid mappings in Hilbert spaces*, Linear Nonlinear Anal. **3**(2) (2017), 297-310.

[16] A. Kondo and W. Takahashi, *Approximation of a common attractive point of noncommutative normally 2-generalized hybrid mappings in Hilbert spaces*, Linear Nonlinear Anal. **5**(2) (2019), 279–297.

[17] F. Liu and M.Z. Nashed, *Regularization of nonlinear ill-posed variational inequalities and convergence rates*, Set-Valued Var. Anal. **6** (1998), 313-344.

[18] W.R. Mann, *Mean value methods in iteration*, Proc. Amer. Math. Soc. **4** (1953), 506-510.

[19] G. Marino and H.-K. Xu, *Weak and strong convergence theorems for strich pseudo-contractions in Hilbert spaces*, J. Math. Anal. Appl. **329**(1) (2007), 336–346.

[20] T. Maruyama, W. Takahashi, and M. Yao, *Fixed point and mean ergodic theorems for new nonlinear mappings in Hilbert spaces*, J. Nonlinear Convex Anal. **12**(1) (2011), 185-197.

[21] A. Moudafi, *The split common fixed-point problem for demicontractive mappings*, Inverse Problems **26**(5) (2010), 1-6.

[22] S. Reich, *Weak convergence theorems for nonexpansive mappings in Banach spaces*, J. Math. Anal. Appl. **67**(2) (1979), 274–276.

[23] W. Takahashi, *Fixed point theorems for new nonlinear mappings in a Hilbert space*, J. Nonlinear Convex Anal. **11**(1) (2010), 79-88.

[24] W. Takahashi, *The split common fixed point problem and the shrinking projection method in Banach spaces*, J. Convex Anal. **24**(3) (2017), 1015–1028.
[25] W. Takahashi, *Mann and Halpern iterations for the split common fixed point problem in Banach spaces*, Linear Nonlinear Anal. **3**(1) (2017), 1–18.
[26] W. Takahashi and M. Toyoda, *Weak convergence theorems for nonexpansive mappings and monotone mappings*, J. Optim. Theory Appl. **118**(2) (2003), 417–428.
[27] W. Takahashi, N.-C. Wong, and J.-C. Yao, *Weak and strong mean convergence theorems for extended hybrid mappings in Hilbert spaces*, J. Nonlinear Convex Anal. **12**(3) (2011), 553–575.
[28] W. Takahashi, N.-C. Wong, and J.-C. Yao, *Attractive point and weak convergence theorems for new generalized hybrid mappings in Hilbert spaces*, J. Nonlinear Convex Anal. **13**(4) (2012), 745-757.
[29] W. Takahashi, H.-K. Xu, and J.-C. Yao, *Iterative methods for generalized split feasibility problems in Hilbert spaces*, Set-Valued Var. Anal. **23** (2015), 205-221.
[30] W. Takahashi, J.-C. Yao, and P. Kocourek, *Weak and strong convergence theorems for generalized hybrid nonself-mappings in Hilbert spaces*, J. Nonlinear Convex Anal. **11**(3) (2010), 567–586.
[31] H. Zegeye and N. Shahzad, *Convergence of Mann's type iteration method for generalized asymptotically nonexpansive mappings*, Computers & Mathematics with Applications **62**(11) (2011), 4007-4014.

(Atsumasa Kondo) Department of Economics, Shiga University, Banba 1-1-1, Hikone, Shiga 522-0069, Japan

E-mail address: a-kondo@biwako.shiga-u.ac.jp

(Wataru Takahashi) Research Center for Interneural Computing, China Medical University, Hospital, China Medical University, Taichung 40447, Taiwan;, Keio Research and Education Center for Natural Sciences,, Keio University, Kouhoku-ku, Yokohama 223-8521, Japan;, and Department of Mathematical and Computing Sciences,, Tokyo Institute of Technology,, Ookayama, Meguro-ku, Tokyo 152-8552, Japan

E-mail address: wataru@is.titech.ac.jp; wataru@a00.itscom.net