DISCUSSION PAPER SERIES E

SHIGA UNIVERSITY

Discussion Paper No. E-31

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February 2024

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Regularized Robust Strategic Asset Allocation under Stochastic Variance-Covariance of Asset Returns

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January 9, 2024

Abstract

This study considers a finite-time robust consumption-investment problem under a quadratic security market model with stochastic variances and covariances of asset returns, as well as stochastic interest rates, market price of risk, and inflation rates. Since the optimal portfolio is proportional to the inverse of the stochastic variance-covariance matrix, it becomes unstable when the near-singularity of the variance-covariance matrix occurs. We propose a regularized consumption-investment problem in which the near-singularity risk is added to the variance-covariance matrix as a regularization term. We show that the optimal regularized portfolio is decomposed into the product of the "standard deviation," "correlation," and "investment control" factors. As the optimal regularized robust portfolio contains an unknown function that is a solution to a nonlinear PDE, we derive an approximate optimal regularized portfolio. Our numerical analysis shows that the market timing effects in the approximate optimal regularized asset allocation are significant and nonlinear, and all factors contribute to these market timing effects.

JEL classification: C61, G11

Keywords: Homothetic robust utility, Inflation-deflation risk, Stochastic variance-covariance, Regularization

1 Introduction

There are stylized facts in security markets that interest rates, market price of risk, variances and covariances of asset returns, and inflation rates are stochastic and mean-reverting. The class of quadratic models[1] independently developed by Ahn, Dittmar, and Gallant (2002) and Leippold and

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[1]Quadratic models are adopted in security pricing studies (Chen, Filipović, and Poor (2004), Kim and Singleton (2012), Filipović, Gourier, and Mancini (2016)) and opti-

Wu (2002) is a generalization of the affine models (Duffie and Kan (1996)). Batbold *et al.* (2022) consider a consumption-investment problem for long-term investors with constant relative risk aversion (CRRA) utility under a quadratic security market model in which interest rates, market price of risk, variance-covariance matrix of asset returns, and inflation rates are functions of a stochastic state process. They derive the optimal portfolio decomposed into the sum of myopic, intertemporal hedging, and inflation-deflation hedging demands; they show that all three demands are nonlinear functions of the state vector. Their numerical analysis shows the nonlinearity and significance of the market timing effects. Nonlinearity is attributed to the stochastic variance-covariance matrix of asset returns, while significance is attributed to inflation-deflation hedging demand in addition to myopic demand.

The global financial crisis reaffirmed the presence of Knightian uncertainty. Investors with robust utility (Hansen and Sargent (2001)) regard the "base probability" as the most likely probability; however, they also consider other probabilities because the true probability is unknown. Robust utility is not homothetic[2], unlike CRRA utility. Homothetic robust utility, proposed by Maenhout (2004) and theoretically justified by Skiadas (2003), is characterized by relative risk aversion and "relative ambiguity aversion," which represents investor's degree of distrust in the base probability.[3] Homothetic robust utility can be interpreted as homothetic robust CRRA utility because homothetic robust utility converges to CRRA utility as ambiguity aversion approaches zero. In this study, we assume homothetic robust Epstein-Zin utility introduced by Batbold, Kikuchi, and Kusuda (2023) and consider a finite-time consumption-investment problem under the quadratic security market model of Batbold *et al.* (2022).

Liu (2010) and Batbold *et al.* (2023) analyze infinite-time consumption-investment problems and derive the nonlinear nonhomogeneous partial differential equation (PDE) for the indirect utility function. Since the nonhomogeneous term appearing in the PDE is stable in infinite-time problems, they apply the loglinear approximation method of Campbell and Viceira (2002) to derive an approximate solution. Meanwhile, Kikuchi and Kusuda (2023) consider the finite-time consumption-investment problem with homothetic robust utility under the quadratic security market model of Batbold *et al.* (2022) and derive a nonlinear nonhomogeneous PDE. Under a finite-time setting, the nonhomogeneous term becomes time-dependent and

mal consumption-investment studies (Batbold, Kikuchi, and Kusuda (2022), Kikuchi and Kusuda (2023)).

[2] A utility function U is homothetic if, for any consumption plan c and $\tilde{c}$, and any scalar $\alpha > 0$, $U(\alpha\tilde{c}) \geq U(\alpha c) \Leftrightarrow U(\tilde{c}) \geq U(c)$.

[3] Homothetic robust utility is applied to robust control studies such as Skiadas (2003), Maenhout (2006), Liu (2010), Branger, Larsen, and Munk (2013), Munk and Rubtsov (2014), Yi, Viens, Law, and Li (2015), and Kikuchi and Kusuda (2023).

unstable. Kikuchi and Kusuda (2023) propose a time-dependent linear approximation method to derive an approximate solution. Their numerical analysis confirms that market timing effects are mainly due to inflation-deflation hedging demand in addition to myopic demand. Strategic asset allocation (Brennan, Schwartz, and Lagnado (1997), Campbell and Viceira (2002)) emphasizes the significance of the market timing effects of intertemporal hedging demand; however, the debate is ongoing.[4] The numerical analyses of Batbold *et al.* (2022) and Kikuchi and Kusuda (2023) shed new light on the effectiveness of strategic asset allocation from a different perspective: the market timing effects of inflation-deflation hedging demand.

In quadratic models, the variance-covariance matrix of asset returns is stochastic. Since the optimal portfolio is proportional to the inverse of the variance-covariance matrix, it becomes unstable when the near-singularity of the variance-covariance matrix occurs, *i.e.*, the minimum eigenvalue of the variance-covariance matrix is close to zero. Note that such near-singularity of the variance-covariance matrix does not arise in constant correlation models such as affine models but rather in stochastic correlation models. The importance of incorporating stochastic variance-covariance matrix of asset returns into security market models has been recognized in the context of option pricing and portfolio choice and actually incorporated into the models (Buraschi, Porchia, and Trojani (2010), Branger and Muck (2012), Bäuerle and Li (2013)) based on the Wishart process and the principal component stochastic volatility (PCSV) model (Escobar, Gotz, Seco, and Zagst (2010), Escobar and Olivares (2013)). Assuming the models (Buraschi *et al.* (2010), Bäuerle and Li (2013)) based on the Wishart process and the PCSV model (Escobar, Ferrando, Christoph, and Rubtso (2022)), they analyze the portfolio choice problem. However, these studies have not focused on the problem of he near-singularity of the variance-covariance matrix.

To address this near-singularity of the variance-covariance matrix, Kikuchi and Kusuda (2023) introduce a regularization term related to the inverse of the volatility matrix into the loss function to stabilize the optimal portfolio when estimating a quadratic security market model. Although Kikuchi and Kusuda (2023) estimate a quadratic security market model and confirm the nonlinearity and significance of the market timing effects, their estimation exhibits three problems. First, since the regularization term depends on the volatility of the optimal portfolio, the estimation results depend on the choice of underlying securities that make up the portfolio. Second, as the optimal portfolio depends on the investor's homothetic robust utility function, the estimation results depend on the utility function through the stabilization criterion of the optimal portfolio. Third, the regularization term biases

[4]Some empirical analyses, including Campbell and Viceira (1999, 2000), indicate that the magnitude of the market timing effects of intertemporal hedging demand is large, while others (Brandt (1999), Ang and Bekaert (2002)) indicate that it is small.

the estimation results.

Note that the near-singularity of the variance-covariance matrix of asset returns can be caused not only by multicollinearity, but also by the near-zero variance of an asset return. In real securities markets, asset returns can become highly correlated or the variance of an asset return can become close to zero; however, even in such cases, investors do not make significantly large rebalances of their portfolios. The main reason is that large portfolio rebalancing leads to high transaction costs. Since our study ignores transaction costs, our optimal portfolio deviates from the portfolio observed in reality when the near-singularity of the variance-covariance matrix of asset retunrs occur. However, introducing transaction costs into consumption-investment problems makes the problems difficult to solve. Moreover, the impact of transaction costs on the optimal portfolio is considered insignificant when the near-singularity of the variance-covariance matrix does not occur. Therefore, instead of introducing transaction costs into our consumption-investment problem, we propose a regularized consumption-investment problem. Since we do not introduce regularization terms for optimal portfolio stabilization in the loss function, we overcome the three problems mentioned above and obtain adequate estimation results for the security market model. We consider a finite-time consumption-investment problem assuming the quadratic security market model of Batbold *et al.* (2022). The objective of this study is to derive a regularized robust optimal portfolio and re-examine the market timing effects of the regularized optimal robust portfolio based on appropriate estimates of the quadratic security market model. The main results of this study are summarized as follows.

First, we propose a regularized consumption-investment problem in which the near-singularity risk of the variance-covariance matrix of asset returns is introduced into the budget constraint equation. We derive the optimal regularized robust portfolio and show that the maximum eigenvalue of the inverse of the regularized variance-covariance matrix in the optimal portfolio is controlled below a certain value. We show that the optimal regularized robust portfolio has two decompositions. One is the sum of myopic, intertemporal hedging, and inflation-deflation hedging demands—this is a regularized version of the decomposition shown by Batbold *et al.* (2023). The other is the product of the standard deviation, correlation, and investment control factors.

Second, since the optimal regularized robust portfolio contains an unknown function that is a solution to the nonlinear PDE, we apply the linear approximation method of Kikuchi and Kusuda (2023) to the nonlinear PDE and derive an approximate optimal regularized robust portfolio.

Third, we remove the regularization term for portfolio stabilization from the loss function used in Kikuchi and Kusuda (2023) and estimate the quadratic security market model. Subsequently, based on the estimated model, we confirm that an unregularized optimal portfolio becomes unstable in the presence of the near-singularity of the variance-covariance matrix of asset retunrs. Therefore, we set a regularization parameter to stabilize the optimal portfolio.

Fourth, we examine the market timing effects of the approximate optimal regularized robust portfolio. We assume a long-term investor who plans to invest in the S&P 500 and 10-year TIPS, in addition to the money market account. Our numerical analysis shows that the market timing effects of both S&P 500 and TIPS are significant and nonlinear. We then decompose the contribution of each demand to the magnitude of the market timing effects in the optimal regularized robust asset allocation based on the above sum decomposition. The results confirm that in addition to myopic demand, inflation-deflation hedging demand is a significant contributor to market timing effects; these are consistent with those of Batbold *et al.* (2022) and Kikuchi and Kusuda (2023). Finally, we decompose the contribution of each factor to the magnitude of the market timing effects based on product decomposition. The results indicate that all factors contribute to the market timing effects. These results suggest that incorporating stochastic variance-covariance of asset returns in addition to inflation-deflation risk into security market models is essential for the analysis of dynamic asset allocation.

The remainder of this paper is organized as follows: In Section 2, we explain the quadratic security market model. In Section 3, we derive the optimal control and nonlinear nonhomogeneous PDE. In Section 4, we propose a regularized consumption-investment problem. In Section 5, we derive a linear approximate optimal regularized robust portfolio. In Sections 6 and 7, we estimate a quadratic security market model and conduct a numerical analysis. In Section 8, we conclude this study. The Appendix includes the proofs of the lemmas and propositions.

2 Quadratic Security Market Model and Budget Constraint

We introduce the quadratic security market model of Batbold *et al.* (2022) and show the no-arbitrage dynamics of security price processes and real budget constraint.

2.1 Quadratic Security Market Model

We consider frictionless US markets over the period $[0, T^*]$. Investors' common subjective probability and information structure are modeled by a complete filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathrm{P})$, where $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,\infty)}$ is the natural filtration generated by an N-dimensional standard Brownian motion B_t. We denote the expectation operator under P by E and the conditional expectation operator given $\mathcal{F}_t$ by E_t.

There are markets for a consumption commodity and securities at every date $t \in [0, \infty)$ and the consumer price index p_t is observed. The traded securities are the instantaneously nominal risk-free security called the money market account, non-bond indices (stock indices, REIT indices, *etc.*), and a continuum of zero-coupon bonds and zero-coupon inflation-indexed bonds, whose maturity dates are $(t, t + \tau^*]$. Each zero-coupon bond has a 1 USD payoff at maturity, and each zero-coupon inflation-indexed bond has a p_T USD payoff at maturity T.

At every date t, P_t, P_t^T, Q_t^T, and S_t^j denote the USD prices of the money market account, zero-coupon bond with maturity date T, zero-coupon inflation-indexed bond with maturity date T, and the j-th index, respectively. Let A' and I denote the transpose of A and the $N \times N$ identity matrix, respectively.

We assume the following quadratic latent factor security market model.

Assumption 1. *Let $(\rho_0, \iota_0, \delta_{0j}, \sigma_{0j})$ and $(\lambda, \rho, \iota, \lambda_I, \delta_j, \sigma_j)$ denote scalers and N-dimensional vectors, respectively.*

1. *State vector process X_t satisfies the following stochastic differential equation (SDE):*
$$dX_t = -\mathcal{K} X_t \, dt + I \, dB_t, \tag{2.1}$$
where $\mathcal{K}$ is an $N \times N$ positive lower triangular matrix.

2. *The market price λ_t of risk and the instantaneous nominal risk-free rate r_t are provided as*
$$\lambda_t = \lambda + \Lambda X_t, \tag{2.2}$$
$$r_t = \rho_0 + \rho' X_t + \frac{1}{2} X_t' \mathcal{R} X_t, \tag{2.3}$$
where Λ is an $N \times N$ matrix such that $\mathcal{K} + \Lambda$ is regular, $\mathcal{R}$ is a positive-definite symmetric matrix, and
$$\rho_0 \geq \frac{1}{2} \rho' \mathcal{R}^{-1} \rho. \tag{2.4}$$

3. *The consumer price index p_t satisfies*
$$\frac{dp_t}{p_t} = i_t \, dt + (\sigma_t^p)' dB_t, \qquad p_0 = 1, \tag{2.5}$$
where i_t and σ_t^p are given by
$$i_t = \iota_0 + \iota' X_t + \frac{1}{2} X_t' \mathcal{I} X_t, \tag{2.6}$$
$$\sigma_t^p = \sigma_p + \Sigma_p X_t. \tag{2.7}$$
For eq.(2.6), $\mathcal{I}$ is a positive-definite symmetric matrix and a matrix $\bar{\mathcal{R}}$ defined by
$$\bar{\mathcal{R}} = \mathcal{R} - \mathcal{I} + \Sigma_p' \Lambda + \Lambda' \Sigma_p \tag{2.8}$$
is positive-definite.

4. *The dividend of the l-th index is given by*

$$D_t^l = \left(\delta_{0l} + \delta_l' X_t + \frac{1}{2} X_t' \Delta_l X_t\right) \exp\left(\sigma_{0l} t + \sigma_l' X_t + \frac{1}{2} X_t' \Sigma_l X_t\right), \tag{2.9}$$

where $(\delta_{0l}, \delta_l, \Delta_l)$ *is such that* Δ_j *is a positive definite symmetric matrix and*

$$\delta_{0l} \geq \frac{1}{2} \delta_j' \Delta_l^{-1} \delta_l. \tag{2.10}$$

Note that $\delta_{0l} + \delta_l' X_t + \frac{1}{2} X_t' \Delta_l X_t$ *is the instantaneous dividend rate.*

5. *Markets are complete and arbitrage-free.*

2.2 No-arbitrage Dynamics of Security Price Processes

We define the real market price $\bar{\lambda}_t$ of risk and the real instantaneous interest rate $\bar{r}_t$ by

$$\bar{\lambda}_t = \lambda_t - \sigma_t^p, \tag{2.11}$$

$$\bar{r}_t = r_t - i_t + \lambda_t' \sigma_t^p. \tag{2.12}$$

Note that the real market price of risk is an affine function of X_t and $\bar{r}_t$ is a quadratic function of X_t:

$$\bar{\lambda}_t = \bar{\lambda} + \bar{\Lambda} X_t, \tag{2.13}$$

$$\bar{r}_t = \bar{\rho}_0 + \bar{\rho}' X_t + \frac{1}{2} X_t' \bar{\mathcal{R}} X_t, \tag{2.14}$$

where $\bar{\mathcal{R}}$ is given by eq.(2.8) and

$$\bar{\lambda} = \lambda - \sigma_p, \tag{2.15}$$

$$\bar{\Lambda} = \Lambda - \Sigma_p, \tag{2.16}$$

$$\bar{\rho}_0 = \rho_0 - \iota_0 + \lambda' \sigma^p, \tag{2.17}$$

$$\bar{\rho} = \rho - \iota + \Lambda' \sigma_p + \Sigma_p' \lambda. \tag{2.18}$$

Batbold *et al.* (2022) show the no-arbitrage security price processes.

Lemma 1. *Let* $\tau = T - t$ *denote the time to maturity of bond* P_t^T *or inflation-indexed bond* Q_t^T*. Under Assumption 1, the prices of securities and their return rates satisfy the following:*

1. *The default-free bond with time* τ *to maturity:*

$$P_t^T = \exp\left(\sigma_0(\tau) + \sigma(\tau)' X_t + \frac{1}{2} X_t' \Sigma(\tau) X_t\right), \tag{2.19}$$

$$\frac{dP_t^T}{P_t^T} = \left(r_t + \left(\sigma(\tau) + \Sigma(\tau)X_t\right)' \lambda_t\right) dt + \left(\sigma(\tau) + \Sigma(\tau)X_t\right)' dB_t, \quad (2.20)$$

where

$$\frac{d\Sigma(\tau)}{d\tau} = \Sigma(\tau)^2 - (\mathcal{K} + \Lambda)'\Sigma(\tau) - \Sigma(\tau)(\mathcal{K} + \Lambda) - \mathcal{R}, \quad (2.21)$$

$$\frac{d\sigma(\tau)}{d\tau} = -(\mathcal{K} + \Lambda - \Sigma(\tau))'\sigma(\tau) - (\Sigma(\tau)\lambda + \rho), \quad (2.22)$$

$$\frac{d\sigma_0(\tau)}{d\tau} = -\lambda'\sigma(\tau) + \frac{1}{2}\left(|\sigma(\tau)|^2 + \mathrm{tr}\left[\Sigma(\tau)\right]\right) - \rho_0, \quad (2.23)$$

with $(\Sigma, \sigma, \sigma_0)(0) = (0, 0, 0)$.

2. *The default-free inflation-indexed bond with time* τ *to maturity:*

$$Q_t^T = p_t \exp\left(\bar{\sigma}_{0q}(\tau) + \bar{\sigma}_q(\tau)'X_t + \frac{1}{2}X_t'\bar{\Sigma}_q(\tau)X_t\right), \quad (2.24)$$

$$\frac{dQ_t^T}{Q_t^T} = \left(r_t + \left(\sigma_q(\tau) + \Sigma_q(\tau)X_t\right)' \lambda_t\right) dt + \left(\sigma_q(\tau) + \Sigma_q(\tau)X_t\right)' dB_t, \quad (2.25)$$

where $\sigma_q(\tau) := \bar{\sigma}_q(\tau) + \sigma_p, \Sigma_q(\tau) := \bar{\Sigma}_q(\tau) + \Sigma_p$, *and*

$$\frac{d\bar{\Sigma}_q(\tau)}{d\tau} = \bar{\Sigma}_q(\tau)^2 - (\mathcal{K} + \bar{\Lambda})'\bar{\Sigma}_q(\tau) - \bar{\Sigma}_q(\tau)(\mathcal{K} + \bar{\Lambda}) - \bar{\mathcal{R}}, \quad (2.26)$$

$$\frac{d\bar{\sigma}_q(\tau)}{d\tau} = -(\mathcal{K} + \bar{\Lambda} - \bar{\Sigma}_q(\tau))'\bar{\sigma}_q(\tau) - (\bar{\Sigma}_q(\tau)\bar{\lambda} + \bar{\rho}), \quad (2.27)$$

$$\frac{d\bar{\sigma}_{0q}(\tau)}{d\tau} = -\bar{\lambda}'\bar{\sigma}_q(\tau) + \frac{1}{2}\left(|\bar{\sigma}_q(\tau)|^2 + \mathrm{tr}\left[\bar{\Sigma}_q(\tau)\right]\right) - \bar{\rho}_0, \quad (2.28)$$

with $(\bar{\Sigma}_q, \bar{\sigma}_q, \bar{\sigma}_{0q})(0) = (0, 0, 0)$.

3. *The* l*-th index:*

$$S_t^l = \exp\left(\sigma_{0l}t + \sigma_l'X_t + \frac{1}{2}X_t'\Sigma_l X_t\right), \quad (2.29)$$

$$\frac{dS_t^l + D_t^l dt}{S_t^l} = \left(r_t + (\sigma_l + \Sigma_l X_t)'\lambda_t\right) dt + (\sigma_l + \Sigma_l X_t)' dB_t, \quad (2.30)$$

where

$$\Sigma_l^2 - (\mathcal{K} + \Lambda)'\Sigma_l - \Sigma_l(\mathcal{K} + \Lambda) + \Delta_l - \mathcal{R}_l = 0, \quad (2.31)$$

$$\sigma_l = (\mathcal{K} + \Lambda - \Sigma_l)'^{-1}(\delta_l - \rho - \Sigma_l\lambda), \quad (2.32)$$

$$\sigma_{0l} = \lambda'\sigma_l - \frac{1}{2}\left(|\sigma_l|^2 + \mathrm{tr}\left[\Sigma_l\right]\right) + \rho_0 - \delta_{0l}. \quad (2.33)$$

Proof. See Appendix A.1 in Batbold *et al.* (2022). □

2.3 Real Budget Constraint

We assume that the investor invests in $P_t(\tau_1), \cdots, P_t(\tau_J), Q_t(\tau_1^Q), \cdots, Q_t(\tau_K^Q)$, and $S_t^1, \cdots, S_t^L$ where $J+K+L=N$. Let $\Phi(\tau)$ and $\Phi_t^Q(\tau^Q)$ denote the portfolio weight on a default-free bond with τ-time to maturity and a default-free inflation-indexed bond with τ^Q-time to maturity, respectively. Let Φ_t^l denote the portfolio weight on the l-th index. Let Φ_t and $\Sigma(X_t)$ denote the portfolio and volatility matrix. Φ_t and $\Sigma(X_t)$ are expressed as follows:

$$\Phi_t = \begin{pmatrix} \Phi_t(\tau_1) \\ \vdots \\ \Phi_t(\tau_J) \\ \Phi_t^Q(\tau_1^Q) \\ \vdots \\ \Phi_t^Q(\tau_K^Q) \\ \Phi_t^1 \\ \vdots \\ \Phi_t^L \end{pmatrix}, \quad \Sigma(X_t) = \begin{pmatrix} \big(\sigma(\tau_1) + \Sigma(\tau_1)X_t\big)' \\ \vdots \\ \big(\sigma(\tau_J) + \Sigma(\tau_J)X_t\big)' \\ \big(\sigma_q(\tau_1^Q) + \Sigma_q(\tau_1^Q)X_t\big)' \\ \vdots \\ \big(\sigma_q(\tau_K^Q) + \Sigma_q(\tau_K^Q)X_t\big)' \\ (\sigma_1 + \Sigma_1 X_t)' \\ \vdots \\ (\sigma_L + \Sigma_L X_t)' \end{pmatrix}. \tag{2.34}$$

Remark 1. *The volatility matrix expressed in eq.(2.34) shows that in the variance-covariance matrix $\Sigma(X_t)\Sigma(X_t)'$ of the asset returns, the variances and covariances are quadratic functions of the mean-reverting state process. Thus, our quadratic security market model satisfies the stylized fact that the variance-covariance matrix of security returns is stochastic and mean-reverting.*

Let c_t and $\bar{W}_t$ denote the consumption rate and real wealth processes, respectively. Batbold *et al.* (2022) show the real budget constraint.

Lemma 2. *The real budget constraint given $(c_t, \bar{\sigma}_t)$ is expressed as*

$$\frac{d\bar{W}_t}{\bar{W}_t} = \left(\bar{r}_t + \bar{\sigma}_t'\bar{\lambda}_t - \frac{c_t}{\bar{W}_t}\right) dt + \bar{\sigma}_t'\, dB_t, \tag{2.35}$$

where

$$\bar{\sigma}_t = \Sigma(X_t)'\Phi_t - \sigma_t^p. \tag{2.36}$$

Proof. See Appendix A.2 in Batbold *et al.* (2022). □

The real budget constraint equation (2.35) indicates that $(c_t, \bar{\sigma}_t)$ is the control in the optimal consumption-investment problem. Let $\mathbf{X}_t = (\bar{W}_t, X_t')'$ and let $\bar{W}_0 > 0$. We refer to the control satisfying the budget constraint equation (2.35) with the initial state $\mathbf{X}_0 = (\bar{W}_0, X_0')'$ as the admissible control and denote the set of admissible controls by $\mathcal{B}(\mathbf{X}_0)$.

3 Optimal Robust Control and PDE for Indirect Utility

We introduce our robust consumption-investment problem based on homothetic robust Epstein-Zin utility and derive the optimal control and PDE for indirect utility.

3.1 Robust Consumption-Investment Problem

The normalized aggregator (Duffie and Epstein (1992)) in Epstein-Zin utility is given by

$$f(c,v) = \begin{cases} \dfrac{\beta}{1-\psi^{-1}} c^{1-\psi^{-1}} \big((1-\gamma)v\big)^{1-\frac{1-\psi^{-1}}{1-\gamma}} - \dfrac{\beta(1-\gamma)}{1-\psi^{-1}} v, & \text{if} \quad \psi \neq 1, \\ \beta(1-\gamma) v \log c - \beta v \log\big((1-\gamma)v\big), & \text{if} \quad \psi = 1, \end{cases} \tag{3.1}$$

where $\beta > 0$ is the subjective discount rate, $\gamma > 1$ is the relative risk aversion, and $\psi > 0$ is the elasticity of intertemporal substitution.

Batbold *et al.* (2023) introduce the following homothetic robust Epstein-Zin (HREZ) utility.

$$u(c) = \inf_{\mathrm{P}^\xi \in \mathbb{P}} \mathrm{E}^\xi \left[\int_0^{T^*} \left(f(c_t, V_t) + \frac{(1-\gamma)V_t}{2\theta} |\xi_t|^2 \right) dt \right], \tag{3.2}$$

where E^ξ is the expectation under P^ξ, θ is the relative ambiguity aversion, and V_t is the utility process defined recursively as follows:

$$V_t = \mathrm{E}_t^\xi \left[\int_t^{T^*} \left(f(c_t, V_s) + \frac{(1-\gamma)V_s}{2\theta} |\xi_s|^2 \right) ds \right]. \tag{3.3}$$

Assumption 2. *The investor's utility is the HREZ utility (3.2).*

The investor's robust consumption-investment problem is given by

$$\sup_{(c,\bar{\sigma}) \in \mathcal{B}(\mathbf{X}_0)} \inf_{\mathrm{P}^\xi \in \mathbb{P}} V_0. \tag{3.4}$$

3.2 Optimal Control and PDE on Indirect Utility

As the standard Brownian motion under P^ξ is given by $B_t^\xi = B_t - \int_0^t \xi_s \, ds$, the SDE (2.1) for the state vector under P^ξ is rewritten as follows:

$$d\mathbf{X}_t = \left(\begin{pmatrix} \bar{W}_t(\bar{r}_t + \bar{\sigma}_t' \bar{\lambda}_t) - c_t \\ -\mathcal{K} X_t \end{pmatrix} + \begin{pmatrix} \bar{W}_t \bar{\sigma}_t' \\ I \end{pmatrix} \xi_t \right) dt + \begin{pmatrix} \bar{W}_t \bar{\sigma}_t' \\ I \end{pmatrix} dB_t^\xi. \tag{3.5}$$

Let J denote the indirect utility function. The Hamilton-Jacobi-Bellman (HJB) equation for problem (3.4) is expressed as

$$\begin{aligned} \sup_{(c,\bar{\sigma})\in\mathcal{B}(\mathbf{X}_0)} \inf_{\mathrm{P}^\xi\in\mathbb{P}} \Bigg\{ J_t &+ \begin{pmatrix} \bar{W}_t\left(\bar{r}_t + \bar{\sigma}_t'\bar{\lambda}_t\right) - c_t \\ -\mathcal{K}X_t \end{pmatrix}' \begin{pmatrix} J_W \\ J_X \end{pmatrix} \\ &+ \frac{1}{2}\operatorname{tr}\left[\begin{pmatrix} \bar{W}_t\bar{\sigma}_t' \\ I \end{pmatrix} \begin{pmatrix} \bar{W}_t\bar{\sigma}_t' \\ I \end{pmatrix}' \begin{pmatrix} J_{WW} & J_{WX} \\ J_{XW} & J_{XX} \end{pmatrix} \right] \\ &+ f(c_t, J) + \frac{(1-\gamma)J}{2\theta}|\xi_t|^2 + \xi_t' \begin{pmatrix} \bar{W}_t\bar{\sigma}_t' \\ I \end{pmatrix}' \begin{pmatrix} J_W \\ J_X \end{pmatrix} \Bigg\} = 0, \quad (3.6) \end{aligned}$$

$$\text{s.t.} \quad J(T^*, \mathbf{X}_{T^*}) = 0.$$

Let $\tau = T^* - t$. We obtain the following lemma.

Lemma 3. *Under Assumptions 1 and 2, the indirect utility function, optimal wealth, optimal consumption, and optimal investment for problem (3.4) satisfy eqs.(3.7), (3.8), (3.9), and (3.10), respectively, where $G(\tau, X_t)$ is a solution to the PDE (3.11).*

$$J(t, \mathbf{X}_t) = \frac{\bar{W}_t^{1-\gamma}}{1-\gamma}\left(G(\tau, X_t)\right)^{\frac{1-\gamma}{\psi-1}}, \tag{3.7}$$

$$\bar{W}_t^* = W_0 \exp\left(\int_0^t \left(\bar{r}_s + (\bar{\sigma}_s^*)'\bar{\lambda}_s - \frac{\beta^\psi}{G(\tau, s)} - \frac{1}{2}|\bar{\sigma}_s^*|^2 \right) ds + \int_0^t (\bar{\sigma}_s^*)'\, dB_s \right), \tag{3.8}$$

$$\begin{aligned} c_t^* &= \beta^\psi \frac{\bar{W}_t^*}{G(\tau, X_t)}, & (3.9) \\ \bar{\sigma}_t^* &= \frac{1}{\gamma+\theta}\bar{\lambda}_t + \left(1 - \frac{1}{\gamma+\theta}\right)\left(-\frac{1}{\psi-1}\frac{G_X(\tau, X_t)}{G(\tau, X_t)} \right), & (3.10) \end{aligned}$$

$$\begin{aligned} \frac{G_\tau}{G} &= \frac{1}{2}\operatorname{tr}\left[\frac{G_{XX}}{G}\right] - \frac{\psi - (\gamma+\theta)^{-1}}{2(\psi-1)}\left|\frac{G_X}{G}\right|^2 \\ &-\left(\mathcal{K}X_t + \left(1-(\gamma+\theta)^{-1}\right)\bar{\lambda}_t\right)'\frac{G_X}{G} + \frac{\beta^\psi}{G} + \left(\frac{(\psi-1)(\gamma+\theta)^{-1}}{2}|\bar{\lambda}_t|^2 + (\psi-1)\,\bar{r}_t - \beta\psi \right), \\ & G(0, X_{T^*}) = 0. \quad (3.11) \end{aligned}$$

Proof. See Appendix A.1. □

4 Regularized Problem and Optimal Regularized Portfolio

First, we explain how the near-singularity of the variance-covariance matrix of asset returns destabilizes optimal portfolios. Next, we propose a regularized robust consumption-investment problem. Then, we derive the optimal regularized robust portfolio and show that it is stable even in the presence of near-singularity of variance-covariance matrix of asset returns. Finally, we present two decompositions of the optimal regularized robust portfolio.

4.1 Near Singularity of Variance-Covariance Matrix of Asset Returns

The optimal robust portfolio Φ_t^* follows from eq.(3.10):

$$\Phi_t^* = \big(\Sigma(X_t)\Sigma(X_t)'\big)^{-1}\Sigma(X_t)\bigg\{\frac{1}{\gamma+\theta}\Big(\lambda + \Lambda X_t\Big) + \left(1 - \frac{1}{\gamma+\theta}\right)\left(-\frac{1}{\psi-1}\left(\frac{G_X(\tau, X_t)}{G(\tau, X_t)}\right) + \Big(\sigma_p + \Sigma_p X_t\Big)\right)\bigg\}. \quad (4.1)$$

Since the variance-covariance matrix $\Sigma(X_t)\Sigma(X_t)'$ of asset returns is positive-semidefinite, we have the eigenvalue decomposition $\Sigma(X_t)\Sigma(X_t)' = U(X_t)D(X_t)U(X_t)'$ where

$$D(x) = \mathrm{diag}\big(d_1(x), \cdots, d_N(x)\big), \quad (4.2)$$

where $d_1(x), \cdots, d_N(x) \geq 0$.

Let $d_{\min}(x) = \min_{n\in\{1,\cdots,N\}} d_n(x)$. We say that the near-singularity of $\Sigma(X_t)\Sigma(X_t)'$ occurs if and only if $d_{\min}(X_t)$ is close to zero. When the near-singularity of $\Sigma(X_t)\Sigma(X_t)'$ occurs, the maximum eigenvalue $d_{\min}^{-1}(X_t)$ of the inverse matrix $\big(\Sigma(X_t)\Sigma(X_t)'\big)^{-1}$ diverges and the optimal robust portfolio becomes unstable. Note that the near-singularity of $\Sigma(X_t)\Sigma(X_t)'$ can be caused not only by multicollinearity, but also by the near-zero variance of an asset return. In real securities markets, asset returns can become highly correlated or the variance of an asset return can become close to zero; however, even in such cases, investors do not rebalance their portfolios significantly. This is because the near-singularity of the variance-covariance matrix of asset returns is only a temporary problem, and considering the high transaction costs of portfolio rebalancing, it is reasonable not to rebalance the portfolio significantly.

Since our study ignores transaction costs, our optimal portfolio deviates from the portfolio observed in reality when the near-singularity of $\Sigma(X_t)\Sigma(X_t)'$ occurs. However, introducing transaction costs into consumption-investment problems makes the problems difficult to solve. Moreover, the impact of transaction costs on the optimal portfolio is considered insignificant when the near-singularity of $\Sigma(X_t)\Sigma(X_t)'$ does not occur. Therefore, instead of introducing transaction costs into our consumption-investment problem, we propose a regularized consumption-investment problem.

4.2 Regularized Consumption-Investment Problem

It follows from eq.(2.36) that the real budget constraint equation (2.35) given $(c_t, \bar{\sigma}_t)$ is rewritten as

$$\frac{d\bar{W}_t}{\bar{W}_t} = \left(\bar{r}_t + \left(\Sigma(X_t)'\Phi_t - \sigma_t^p\right)'\bar{\lambda}_t - \frac{c_t}{\bar{W}_t}\right) dt + \left(\Sigma(X_t)'\Phi_t - \sigma_t^p\right)' dB_t. \tag{4.3}$$

The budget constraint equation shows that the investor views the volatility $\Sigma(X_t)$ of asset returns as the risk per unit of portfolio Φ_t. For investors, the near-singularity of $\Sigma(X_t)\Sigma(X_t)'$ is an additional risk that leads to high transaction costs due to significant portfolio rebalancing.

We assume that the investor judges that the near-singularity of $\Sigma(X_t)\Sigma(X_t)'$ occurs when $d_{\min}(X_t) < \varepsilon_0$ for some $\varepsilon_0 > 0$. We define the investor's subjective volatility $\Sigma_\varepsilon(X_t)$ of asset returns by

$$\Sigma_\varepsilon(X_t) = \left(I + \varepsilon(X_t)\big(\Sigma(X_t)\Sigma(X_t)'\big)^{-1}\right)\Sigma(X_t), \tag{4.4}$$

where

$$\varepsilon(x) = \max\big\{\varepsilon_0 - d_{\min}(x),\, 0\big\}. \tag{4.5}$$

Remark 2. *The investor's subjective variance-covariance matrix of asset returns is expressed as*

$$\begin{aligned}\Sigma_\varepsilon(X_t)\Sigma_\varepsilon(X_t)' &= \Sigma(X_t)\Sigma(X_t)' + 2\varepsilon(X_t)I + \varepsilon^2(X_t)\big(\Sigma(X_t)\Sigma(X_t)'\big)^{-1} \\ &= U(X_t)\,\mathrm{diag}\big(d_1^\varepsilon(X_t), \cdots, d_N^\varepsilon(X_t)\big)U(X_t)',\end{aligned} \tag{4.6}$$

where

$$d_n^\varepsilon(x) = \left(\sqrt{d_n(x)} + \varepsilon(x)\sqrt{d_n^{-1}(x)}\right)^2. \tag{4.7}$$

Since $d_n^\varepsilon(X_t) > d_n(X_t)$ when $\varepsilon_0 > d_{\min}(X_t)$, $\Sigma_\varepsilon(X_t)\Sigma_\varepsilon(X_t)'$ is interpreted as a regularized variance-covariance matrix of asset returns.

The regularized real budget constraint equation is defined as

$$\frac{d\bar{W}_t}{\bar{W}_t} = \left(\bar{r}_t + \varsigma_t'\bar{\lambda}_t - \frac{c_t}{\bar{W}_t}\right) dt + \varsigma_t'\, dB_t, \tag{4.8}$$

where

$$\bar{\varsigma}_t = \Sigma_\varepsilon(X_t)'\Phi_t - \sigma_t^p. \tag{4.9}$$

We call $\bar{\varsigma}_t$ the regularized investment control. Let $\mathcal{B}_\varepsilon(\mathbf{X}_0)$ denote the set of admissible controls satisfying the regularized real budget constraint equation (4.8).

The investor's regularized robust consumption-investment problem is given by

$$\sup_{(c,\bar{\varsigma})\in\mathcal{B}_\varepsilon(\mathbf{X}_0)} \inf_{\mathrm{P}^\xi\in\mathbb{P}} V_0. \tag{4.10}$$

Remark 3. *In the context of machine learning, regularization is a parameter estimation method used to reduce the variance of generalization errors. In contrast, our regularization can be interpreted as a regularization method similar to the Tikhonov regularization method (Tikhonov and Arsenin (1977)) for ill-posed problems, since the unregularized consumption-investment problem beomes a kind of ill-conditioned problem when $d_{\min}(X_t)$ is close to zero.*

4.3 Optimal Regularized Robust Portfolio

In the regularized problem (4.10), investment control $\bar{\sigma}_t$ in the original problem (3.4) is merely replaced by regularized investment control $\bar{\varsigma}$. Thus, we immediately obtain Proposition 1 from Lemma 3 for the original problem.

Proposition 1. *Under Assumptions 1 and 2, the indirect utility function, optimal wealth, optimal consumption, and optimal investment for problem (4.10) satisfy eqs.(3.7), (4.11), (4.12), and (4.13), respectively. $G(\tau, X_t)$ is a solution to the PDE (3.11).*

$$\bar{W}_t^* = W_0 \exp\left(\int_0^t \left(\bar{r}_s + (\bar{\varsigma}_s^*)'\bar{\lambda}_s - \frac{\beta^\psi}{G(\tau,s)} - \frac{1}{2}|\bar{\varsigma}_s^*|^2\right) ds + \int_0^t (\bar{\varsigma}_s^*)'\, dB_s\right), \tag{4.11}$$

$$c_t^* = \beta^\psi \frac{\bar{W}_t^*}{G(\tau, X_t)}, \tag{4.12}$$

$$\bar{\varsigma}_t^* = \frac{1}{\gamma+\theta}\bar{\lambda}_t + \left(1 - \frac{1}{\gamma+\theta}\right)\left(-\frac{1}{\psi-1}\frac{G_X(\tau,X_t)}{G(\tau,X_t)}\right). \tag{4.13}$$

The optimal regularized portfolio $\varPhi_\varepsilon^*(X_t) := \Sigma_\varepsilon(X_t)'^{-1}(\bar{\varsigma}_t^* + \sigma_t^p)$ satisfies

$$\begin{aligned}\varPhi_\varepsilon^*(X_t) = \Sigma_\varepsilon(X_t)'^{-1}\Bigg\{&\frac{1}{\gamma+\theta}\Big(\lambda + \Lambda X_t\Big) \\ &+ \left(1 - \frac{1}{\gamma+\theta}\right)\left(-\frac{1}{\psi-1}\frac{G_X(\tau,X_t)}{G(\tau,X_t)} + \Big(\sigma_p + \Sigma_p X_t\Big)\right)\Bigg\},\end{aligned} \tag{4.14}$$

Furthermore, $\Sigma_\varepsilon(X_t)'^{-1}$ is expressed as

$$\Sigma_\varepsilon(X_t)'^{-1} = \Big(\Sigma(X_t)\Sigma(X_t)' + \varepsilon(X_t)I\Big)^{-1}\Sigma(X_t). \tag{4.15}$$

Proof. See Appendix A.2. □

Remark 4. $\Big(\Sigma(X_t)\Sigma(X_t)' + \varepsilon(X_t)I\Big)^{-1}$ *in eq.(4.15) is expressed as*

$$\Big(\Sigma(X_t)\Sigma(X_t)' + \varepsilon(X_t)I\Big)^{-1} = U(X_t)\,\mathrm{diag}\big(e_1^\varepsilon(X_t), \cdots, e_N^\varepsilon(X_t)\big)U(X_t)', \tag{4.16}$$

where $e_n^\varepsilon(x) = \big(d_n(x)+\varepsilon(x)\big)^{-1}$. *Thus, the optimal regularized robust portfolio can be stable even in the presence of the near-singularity of* $\Sigma(X_t)\Sigma(X_t)'$ *because*

$$\max_{n\in\{1,\cdots,N\}} e_n^\varepsilon(X_t) = \max_{n\in\{1,\cdots,N\}} \big(d_n(X_t) + \varepsilon(X_t)\big)^{-1} \leq \frac{1}{\varepsilon_0}. \tag{4.17}$$

Consequently, the investor does not need to rebalance the portfolio significantly and can maintain low transaction costs. This justifies our method of introducing the near-singularity risk of $\Sigma(X_t)\Sigma(X_t)'$ *instead of transaction costs into the budget constraint equation.*

4.4 Two Decompositions of the Optimal Portfolio

We show two decomposions of the optimal regularized robust portfolio.

4.4.1 Sum Decomposition

It immediately follows from eq.(4.14) that the optimal regularized robust portfolio is decomposed into the sum of myopic, intertemporal hedging, and inflation-deflation hedging demands.

$$\varPhi_\varepsilon^*(X_t) = \frac{1}{\gamma+\theta}\Sigma_\varepsilon(X_t)'^{-1}\lambda_t + \left(1 - \frac{1}{\gamma+\theta}\right)\Sigma_\varepsilon(X_t)'^{-1}\left(-\frac{1}{\psi-1}\frac{G_X(\tau,X_t)}{G(\tau,X_t)}\right) + \left(1 - \frac{1}{\gamma+\theta}\right)\Sigma_\varepsilon(X_t)'^{-1}\sigma_t^p. \tag{4.18}$$

The sum decomposition shown above is a regularized version of the decomposition shown in Batbold *et al.* (2023).

4.4.2 Product Decomposition

The regularized variance-covariance matrix of the asset returns is decomposed as $\Sigma_\varepsilon(X_t)\Sigma_\varepsilon(X_t)' = D_\varepsilon(X_t)R_\varepsilon(X_t)D_\varepsilon(X_t)$ where $D_\varepsilon(X_t)$ is the diagonal matrix of the standard deviations and $R_\varepsilon(X_t)$ is the correlation matrix. Let $\varsigma^*(X_t) = \bar{\varsigma}_t^* + \sigma_t^p$ and call it the "optimal regularized nominal investment control." Then, the optimal regularized portfolio is decomposed as

$$\varPhi_\varepsilon^*(X_t) = \Sigma_\varepsilon(X_t)'^{-1}\varsigma^*(X_t) = D_\varepsilon(X_t)^{-1}\sqrt{R_\varepsilon(X_t)}'^{-1}\varsigma^*(X_t), \tag{4.19}$$

where $\sqrt{R_\varepsilon(X_t)} = D_\varepsilon(X_t)^{-1}\Sigma_\varepsilon(X_t)$.

Remark 5. *The third term $\varsigma^*(X_t)$ in eq.(4.19) is expressed as*

$$\varsigma^*(X_t) = \frac{1}{\gamma+\theta}\lambda_t + \left(1 - \frac{1}{\gamma+\theta}\right)\left(-\frac{1}{\psi-1}\frac{G_X(\tau,X_t)}{G(\tau,X_t)}\right) + \left(1-\frac{1}{\gamma+\theta}\right)\sigma_t^p. \tag{4.20}$$

The first and third terms are independent of the variance-covariance matrix of asset returns. The PDE (3.11) shows that the second term is also independent of the variance-covariance matrix of asset returns. Thus, $\varsigma^*(X_t)$ is independent of the variance-covariance matrix of asset returns. We refer to $D_\varepsilon(X_t)^{-1}$, $\sqrt{R_\varepsilon(X_t)}'^{-1}$, and $\varsigma^*(X_t)$ as the "standard deviation factor," the "correlation factor," and the "investment control factor," respectively.

5 Approximate Optimal Regularized Robust Portfolio

Following Kikuchi and Kusuda (2023), we derive a linear approximate optimal regularized robust portfolio.

5.1 Linear Approximation for the PDE

The PDE (3.11) is rewritten as

$$G_\tau = \frac{1}{2}\operatorname{tr}[G_{XX}] - \frac{1}{2(\psi-1)}\frac{G_X'}{G}\left(\psi - (\gamma+\theta)^{-1}\right)G_X$$
$$-\left(\mathcal{K}X_t + \left(1-(\gamma+\theta)^{-1}\right)\bar{\lambda}_t\right)' G_X + \left(\frac{(\psi-1)(\gamma+\theta)^{-1}}{2}|\bar{\lambda}_t|^2 + (\psi-1)\,\bar{r}_t - \beta\psi\right)G + \beta^\psi. \tag{5.1}$$

Let $\tilde{G}$ denote a time-dependent linear approximate solution of the PDE (5.1). Kikuchi and Kusuda (2023) approximate $\frac{G_X}{G}$ in the nonlinear term of the PDE (5.1) by a linear function of X_t.

$$\frac{G_X}{G} \approx \frac{\tilde{G}_X}{\tilde{G}} := a(\tau) + A(\tau)X_t, \tag{5.2}$$

where $(a(\tau), A(\tau))$ is specified at the end of this subsection.

Then, we obtain the following approximate nonhomogeneous linear PDE:

$$\tilde{G}_\tau = \mathcal{L}\tilde{G} + \beta^\psi, \quad \tilde{G}(0, X) = 0. \tag{5.3}$$

where $\mathcal{L}$ is the linear differential operator defined by

$$\mathcal{L}\tilde{G} = \frac{1}{2}\,\mathrm{tr}\left[\tilde{G}_{XX}\right]$$
$$-\left(\frac{\psi-(\gamma+\theta)^{-1}}{2(\psi-1)}\big(a(\tau)+A(\tau)X_t\big)+\mathcal{K}X_t+\big(1-(\gamma+\theta)^{-1}\big)\left(\bar{\lambda}+\bar{\Lambda}X_t\right)\right)'\tilde{G}_X$$
$$+\left\{\frac{(\psi-1)}{2(\gamma+\theta)}|\bar{\lambda}+\bar{\Lambda}X_t|^2+(\psi-1)\left(\bar{\rho}_0+\bar{\rho}'X_t+\frac{1}{2}X_t'\bar{\mathcal{R}}X_t\right)-\beta\psi\right\}\tilde{G}. \tag{5.4}$$

To solve the nonhomogeneous linear PDE (5.3), we first consider the following homogeneous linear PDE:

$$\frac{\partial}{\partial\tau}\tilde{g}(\tau,X)=\mathcal{L}\tilde{g}(\tau,X),\quad \tilde{g}(0,X)=1. \tag{5.5}$$

An analytical solution of the PDE (5.5) is expressed as

$$\tilde{g}(\tau,X)=\exp\left(b_0(\tau)+b(\tau)'X+\frac{1}{2}X'B(\tau)X\right),\quad (b_0,b,B)(0)=0, \tag{5.6}$$

where $B(\tau)$ is a symmetric matrix. Then, a semi-analytical solution to the PDE (5.3) is expressed as

$$\tilde{G}(\tau,X_t)=\int_0^\tau \tilde{g}(s,X_t)\,ds. \tag{5.7}$$

Define $\tilde{b}(\tau,X_t)$ and $\tilde{B}(\tau,X_t)$ by

$$\begin{aligned}\tilde{b}(\tau,X_t)&=\frac{1}{\tilde{G}(\tau,X_t)}\int_0^\tau \tilde{g}(s,X_t)b(s)\,ds,\\ \tilde{B}(\tau,X_t)&=\frac{1}{\tilde{G}(\tau,X_t)}\int_0^\tau \tilde{g}(s,X_t)B(s)\,ds.\end{aligned} \tag{5.8}$$

In eq.(5.2), we set $(a(\tau),A(\tau))=(\tilde{b}(\tau,0),\tilde{B}(\tau,0))$, that is,

$$\frac{\tilde{G}_X}{\tilde{G}}=\tilde{b}(\tau,0)+\tilde{B}(\tau,0)X_t. \tag{5.9}$$

5.2 Approximate Optimal Regularized Robust Portfolio

Define functions m_2, m_1, and m_0 by

$$\begin{aligned}m_2(B)=B^2&-\Big(\mathcal{K}+\big(1-(\gamma+\theta)^{-1}\big)\bar{\Lambda}\Big)'B-B\Big(\mathcal{K}+\big(1-(\gamma+\theta)^{-1}\big)\bar{\Lambda}\Big)\\&+(\psi-1)\Big((\gamma+\theta)^{-1}\bar{\Lambda}'\bar{\Lambda}+\bar{\mathcal{R}}\Big),\end{aligned} \tag{5.10}$$

$$
\begin{aligned}
m_1(B,b) = \Big(B - \big(\mathcal{K} + \big(1-(\gamma+\theta)^{-1}\big)\bar{\Lambda}\big)'\Big)b \\
- \big(1-(\gamma+\theta)^{-1}\big)B\bar{\lambda} + (\psi-1)\Big((\gamma+\theta)^{-1}\bar{\Lambda}'\bar{\lambda} + \bar{\rho}\Big),
\end{aligned} \tag{5.11}
$$

$$
m_0(B,b) = \frac{1}{2}\big(\mathrm{tr}[B]+|b|^2\big)-\bar{\lambda}'\big(1-(\gamma+\theta)^{-1}\big)b+(\psi-1)\left(\frac{1}{2}(\gamma+\theta)^{-1}|\bar{\lambda}|^2+\bar{\rho}_0\right)-\beta\psi. \tag{5.12}
$$

When the solution to the PDE (5.1) is approximated by the solution to the approximate PDE (5.3), the optimal wealth and control are called the approximate optimal regularized wealth and control, and are denoted by $\tilde{W}_t^*$ and $(\tilde{c}_t^*, \tilde{\varsigma}_t^*)$. We obtain Proposition 2.

Proposition 2. *Under Assumptions 1 and 2, the approximate optimal wealth, optimal consumption, and optimal investment for problem (3.4) satisfy eqs.(5.13), (5.14), and (5.15).*

$$
\tilde{W}_t^* = W_0 \exp\left(\int_0^t \left(\bar{r}_s + (\tilde{\varsigma}_s^*)'\bar{\lambda}_s - \frac{\beta^\psi}{\tilde{G}(\tau,s)} - \frac{1}{2}|\tilde{\varsigma}_s^*|^2\right) ds + \int_0^t (\tilde{\varsigma}_s^*)'\, dB_s\right), \tag{5.13}
$$

$$
\tilde{c}_t^* = \frac{\beta^\psi \tilde{W}_t^*}{\tilde{G}(\tau,t)}, \tag{5.14}
$$

where $\tilde{G}$ is given by eqs.(5.6) and (5.7), and

$$
\tilde{\varsigma}_t^* = \frac{1}{\gamma+\theta}\big(\bar{\lambda}+\bar{\Lambda}X_t\big) + \left(1-\frac{1}{\gamma+\theta}\right)\left(-\frac{1}{\psi-1}\Big(\tilde{b}(\tau,X_t)+\tilde{B}(\tau,X_t)X_t\Big)\right), \tag{5.15}
$$

where $(\tilde{b},\tilde{B})$ is given by eq.(5.8), while (B,b,b_0) is a solution to the ODEs:

$$
\begin{aligned}
\frac{dB}{d\tau} &= m_2(B) - \frac{\psi-(\gamma+\theta)^{-1}}{\psi-1}\tilde{B}(\tau,0)'B(\tau), \\
\frac{db}{d\tau} &= m_1(B,b) - \frac{\psi-(\gamma+\theta)^{-1}}{2(\psi-1)}\left(\tilde{B}(\tau,0)'b(\tau)+B(\tau)'\tilde{b}(\tau,0)\right), \\
\frac{db_0}{d\tau} &= m_0(B,b) - \frac{\psi-(\gamma+\theta)^{-1}}{2(\psi-1)}\tilde{b}(\tau,0)'b(\tau),
\end{aligned} \tag{5.16}
$$

with $(B,b,b_0)(0) = (0,0,0)$, where m_2, m_1, and m_0 are given by eqs.(5.10)–(5.12).

Proof. See Appendix A.3. □

6 Estimation of the Quadratic Model

In this section, we introduce the estimation methods of our quadratic model and present the estimation results.

6.1 Estimation Methods

In the next section, we quantitatively analyze the market timing effects in the optimal regularized robust asset allocations for an investor who plans to invest in the 10-year TIPS Q_t^{10} and S&P 500 S_t, in addition to the money market account. Thus, we estimate a two-factor quadratic security market model. To estimate the model, we use the 6-month and 5-year treasury spot rate, 10-year TIPS real spot rates $\bar{s}_t^Q(10)$, and dividend D_t/S_t of S&P 500. We use the following notations:

$$
\begin{aligned}
Y_t &= \begin{pmatrix} s_t(0.5) \\ s_t(5) \\ \bar{s}_t^Q(10) \\ D_t/S_t \end{pmatrix}, \qquad H_2(X_t) = \frac{1}{2}\begin{pmatrix} -0.5^{-1}X_t'\Sigma(0.5)X_t \\ -5^{-1}X_t'\Sigma(5)X_t \\ -10^{-1}X_t'\bar{\Sigma}_q(10)X_t \\ X_t'\mathit{\Delta}X_t \end{pmatrix}, \\
H_1 &= \begin{pmatrix} -0.5^{-1}\sigma(0.5)' \\ -5^{-1}\sigma(5)' \\ -10^{-1}\bar{\sigma}_q(10)' \\ \delta' \end{pmatrix}, \qquad H_0 = \begin{pmatrix} -0.5^{-1}\sigma_0(0.5) \\ -5^{-1}\sigma_0(5) \\ -10^{-1}\bar{\sigma}_{0q}(10) \\ \delta_0 \end{pmatrix},
\end{aligned} \tag{6.1}
$$

where $s_t(\tau)$ is the treasury spot rates with time τ to maturity at time t; $(\Sigma(\tau), \sigma(\tau), \sigma_0(\tau))$ and $(\bar{\Sigma}_q(\tau), \bar{\sigma}_q(\tau), \bar{\sigma}_{0q})$ are solutions to eqs.(2.21)-(2.23) and (2.26)-(2.28); and $(\Delta, \delta, \delta_0)$ is given by eq.(2.9). Assume that we have M observables: $Y_0, Y_h, \cdots, Y_{nh}, \cdots, Y_{(M-1)h}$ where h is the observation time interval. We use the following notations:

$$
\begin{aligned}
&\begin{pmatrix} x_n \\ y_n \end{pmatrix} = \begin{pmatrix} X_{nh} \\ Y_{nh} \end{pmatrix}, \quad F = e^{-h\mathcal{K}}, \quad w_n = \int_0^h e^{(s-h)\mathcal{K}}\, dB_s, \\
&\Omega_w = \int_0^h e^{(s-h)\mathcal{K}}\big(e^{(s-h)\mathcal{K}}\big)' ds = (\mathcal{K} + \mathcal{K}')^{-1}\big(I_2 - e^{-h(\mathcal{K}+\mathcal{K}')}\big), \\
&\Omega_\varepsilon = \mathrm{diag}(\omega_1, \omega_2, \omega_3, \omega_4).
\end{aligned} \tag{6.2}
$$

Batbold *et al.* (2022) derive the following nonlinear Gaussian state-space model representation of the quadratic security market model.[5]

$$
x_{n+1} = F\, x_n + w_n, \qquad w_n \overset{i.i.d.}{\sim} N(0, \Omega_w), \tag{6.3}
$$

$$
y_n = H_2(x_n) + H_1 x_n + H_0 + \varepsilon_n, \qquad \varepsilon_n \overset{i.i.d.}{\sim} N(0, \Omega_\varepsilon), \tag{6.4}
$$

where w_n and ε_n are independent of each other.

Since the state-space model shown above is not a general state-space model but a nonlinear Gaussian state-space model, and since the state equation is linear, we judged that local approximation methods, such as extended Kalman filter, would be more efficient than global approximation methods, such as particle filters. Among the local approximation methods, we use

[5] For the derivation of the state-space model representation, see Appendix B.1.

the unscented Kalman filter (Julier, Uhlmann, and Durrant-Whyte (2000)), which is considered to perform better than the extended Kalman filter (Julier and Uhlmann (2004)).

In the above state-space model, the state process can be interpreted as time-varying coefficients. Thus, the parameter set becomes large, which may cause overtraining. To address this overtraining, we introduce a regularization term in the loss function. Since eq.(2.1) is transformed into $d\big(e^{t\mathcal{K}}X_t\big) = e^{t\mathcal{K}}\,dB_t$, X_t is solved as $X_t = e^{-t\mathcal{K}}X_0 + \int_0^t e^{(s-t)\mathcal{K}}\,dB_s$. Hence, the stationary distribution of X_t is given by $N(0, (\mathcal{K}+\mathcal{K}')^{-1})$. Let $\mathcal{C}\mathcal{C}' = (\mathcal{K}+\mathcal{K}')^{-1}$ be the Cholesky decomposition and define $Z = \mathcal{C}^{-1}X$. Then, since $Z \sim N(0, I)$, we call Z the standardized state vector. Let L denote the likelihood. To avoid overtraining, we introduce the following regularization term into the loss function, expressed as

$$-\log L + \nu \sum_{n=0}^{M-1} |z_n|^2, \tag{6.5}$$

where $z_n = \mathcal{C}^{-1}x_n$.

In addition to the above regularization term, Kikuchi and Kusuda (2023) introduce a regularization term in the loss function to stabilize the optimal portfolio, which distorts the estimation efficiency. We obtain appropriate estimates with the above loss function by removing the regularization term for portfolio stabilization from their loss function.

6.2 Estimation Results

Using 291 month-end data observed in U.S. securities markets from January 1999 through March 2023, we estimated the two-factor quadratic security market model by minimizing the above loss function. The time-series data used for the estimation are 6-month and 5-year treasury spot rates[6] and 10-year TIPS real spot rates[7], and the dividends of the S&P 500[8].

To reduce the estimation burden, we assume that the second-order term of the instantaneous expected inflation rate is zero; that is, $\mathcal{I} = 0$. We set the hyperparameter ν in eq.(6.5) after trial and error, albeit arbitrarily, to: $\nu = 1$. We obtained the parameter estimates as shown in Appendix B.2.

[6] These spot rates data are available on the Federal Reserve Board (FRB) website. They are computed based on the estimation method by Gürkaynak, Sack, and Wright Gürkaynak, Sack, and Wright (2007).

[7] These TIPS real spot rate data are available on the FRB website. They are computed based on the estimation method by Gürkaynak, Sack, and Wright Gürkaynak, Sack, and Wright (2010).

[8] These data are available on the website of Robert Shiller.

7 Numerical Analysis

We quantitatively analyze the market timing effects in optimal regularized robust asset allocations. We consider a long-term investor who plans to invest in the S&P 500 and 10-year TIPS in addition to the money market account over 35 years. We assume $(\beta, \gamma, \theta) = (0.04, 2.5, 1.5)$ following Kikuchi and Kusuda (2023). We also assume $\psi = 1.5$ following Bansal and Yaron (2004).

To analyze the variation in the optimal robust portfolio allocations due to the change in the state vector, we assume, based on the results of the analysis shown above, that the state vector $X = \mathcal{C}Z$ changes, as shown in the following equation:

$$z = j\left(\cos\frac{\pi k}{4}, \sin\frac{\pi k}{4}\right)', \tag{7.1}$$

where $j = -2.50, -2.25, -2.00, \cdots, 2.25, 2.50$ and $k = 0, 1, 2, 3$.

7.1 Determination of the Regularization Parameter

In our optimal regularized portfolio, the near-singularity of $\Sigma(X_t)\Sigma(X_t)'$ is judged to occur when $d_{\min}(X_t)$ is less than the regularization parameter ε_0 in eq.(4.5). Suppose that the regularization parameter is set to define the region where the unregularized optimal portfolio is unstable. Then, the regularization parameter may not be large enough to stabilize the optimal portfolio. Therefore, the regularization parameter that defines the unstable region of the unregularized optimal portfolio is set as the lower bound of the candidate regularization parameters, and the appropriate regularization parameter that stabilizes the optimal portfolio is determined among the candidate regularization parameters that are greater than or equal to the lower bound.

First, we examined the relationship between the minimum eigenvalues of the variance-covariance matrix and the optimal unregularized allocations to the S&P 500 and TIPS when the standardized state vector moves in the above region (Table 1). The minimum eigenvalues are less than 0.0025 in cells with a red background, 0.0025–0.0050 in orange cells, 0.0050–0.0075 in yellow cells, 0.0075–0.0100 in green cells, and greater than or equal to 0.0100 in the remaining cells.

	minimum eigenvalue				S&P 500				TIPS			
(j, k)	0	1	2	3	0	1	2	3	0	1	2	3
-2.50	0.00328	0.00618	0.05175	0.03056	-11.1	13.1	-9.2	34.9	-141.3	127.4	30.5	-54.1
-2.25	0.00392	0.00453	0.03657	0.03315	-10.0	13.8	-14.6	32.9	-95.2	121.3	25.1	-42.8
-2.00	0.00502	0.00273	0.01796	0.03624	-7.9	14.8	-27.0	30.9	-52.1	116.8	13.9	-31.6
-1.75	0.00685	0.00100	0.00284	0.03985	-5.3	17.4	-83.5	28.7	-15.2	118.2	-33.1	-20.6
-1.50	0.00994	0.00001	0.00190	0.04394	-2.3	84.2	119.4	26.5	13.6	377.3	130.3	-10.1
-1.25	0.01510	0.00149	0.02111	0.04834	0.9	7.3	39.9	24.3	33.6	59.1	64.6	-0.2
-1.00	0.02345	0.00904	0.05241	0.05275	4.0	10.5	25.7	22.0	45.3	62.6	51.8	9.0
-0.75	0.03555	0.02667	0.06587	0.05667	6.8	11.7	19.9	19.8	49.8	57.9	45.7	17.3
-0.50	0.04921	0.04934	0.06596	0.05949	9.3	12.4	16.7	17.5	48.8	51.2	41.6	24.6
-0.25	0.05849	0.06066	0.06335	0.06072	11.5	12.9	14.7	15.4	43.7	43.7	38.5	30.8
0.00	0.06019	0.06019	0.06019	0.06019	13.4	13.4	13.4	13.4	35.9	35.9	35.9	35.9
0.25	0.05736	0.05618	0.05697	0.05815	14.9	13.8	12.4	11.5	26.4	27.8	33.5	39.7
0.50	0.05353	0.05204	0.05381	0.05512	16.2	14.1	11.8	9.7	16.1	19.6	31.1	42.3
0.75	0.05022	0.04852	0.05077	0.05165	17.3	14.5	11.3	8.1	5.4	11.3	28.9	43.7
1.00	0.04777	0.04567	0.04784	0.04815	18.1	14.8	10.9	6.6	-5.2	3.0	26.6	43.9
1.25	0.04615	0.04336	0.04502	0.04487	18.8	15.2	10.6	5.4	-15.6	-5.3	24.3	43.0
1.50	0.04521	0.04149	0.04232	0.04196	19.4	15.5	10.4	4.3	-25.4	-13.5	21.9	41.1
1.75	0.04482	0.03995	0.03972	0.03944	19.9	15.8	10.2	3.4	-34.6	-21.7	19.5	38.2
2.00	0.04487	0.03867	0.03723	0.03733	20.3	16.1	10.1	2.6	-43.2	-29.9	16.9	34.5
2.25	0.04529	0.03759	0.03483	0.03557	20.6	16.4	10.1	2.1	-51.2	-37.9	14.2	30.0
2.50	0.04601	0.03669	0.03252	0.03415	20.9	16.7	10.1	1.7	-58.4	-45.8	11.4	24.9

Table 1: The relationship between minimum eigenvalues of the variance-covariance matrix and the optimal unregularized allocations (%) to S&P 500 and the TIPS.

The optimal unregularized portfolio destabilizes when the minimum eigenvalues are less than 0.002. For ease of analysis, we evaluate the relationship between the minimum eigenvalue and the optimal unregularized portfolio as follows: less than 0.0025, unstable; 0.0025–0.0075, locally unstable; 0.0075–0.0100, generally stable; greater than or equal to 0.0100, stable. We then compared the optimal regularized allocations to the S&P 500 (Table 2) and TIPS (Table 3) for $\varepsilon_0 = 0.0025, 0.0050, 0.0075, 0.0100$, and 0.0125, respectively. Note that $\varepsilon_0 = 0.0000$ in the figure shows the optimal unregularized allocations for comparison.

ε_0	0.0000				0.0025				0.0050			
(j, k)	0	1	2	3	0	1	2	3	0	1	2	3
-2.50	-11.1	13.1	-9.2	34.9	-11.1	13.1	-9.2	34.9	-7.2	13.1	-9.2	34.9
-2.25	-10.0	13.8	-14.6	32.9	-10.0	13.8	-14.6	32.9	-8.0	12.3	-14.6	32.9
-2.00	-7.9	14.8	-27.0	30.9	-7.9	14.8	-27.0	30.9	-7.9	6.4	-27.0	30.9
-1.75	-5.3	17.4	-83.5	28.7	-5.3	3.5	-83.5	28.7	-5.3	-1.1	-56.3	28.7
-1.50	-2.3	84.2	119.4	26.5	-2.3	-7.5	86.5	26.5	-2.3	-7.6	34.5	26.5
-1.25	0.9	7.3	39.9	24.3	0.9	0.5	39.9	24.3	0.9	-4.5	39.9	24.3
-1.00	4.0	10.5	25.7	22.0	4.0	10.5	25.7	22.0	4.0	10.5	25.7	22.0
-0.75	6.8	11.7	19.9	19.8	6.8	11.7	19.9	19.8	6.8	11.7	19.9	19.8
-0.50	9.3	12.4	16.7	17.5	9.3	12.4	16.7	17.5	9.3	12.4	16.7	17.5
-0.25	11.5	12.9	14.7	15.4	11.5	12.9	14.7	15.4	11.5	12.9	14.7	15.4
0.00	13.4	13.4	13.4	13.4	13.4	13.4	13.4	13.4	13.4	13.4	13.4	13.4
0.25	14.9	13.8	12.4	11.5	14.9	13.8	12.4	11.5	14.9	13.8	12.4	11.5
0.50	16.2	14.1	11.8	9.7	16.2	14.1	11.8	9.7	16.2	14.1	11.8	9.7
0.75	17.3	14.5	11.3	8.1	17.3	14.5	11.3	8.1	17.3	14.5	11.3	8.1
1.00	18.1	14.8	10.9	6.6	18.1	14.8	10.9	6.6	18.1	14.8	10.9	6.6
1.25	18.8	15.2	10.6	5.4	18.8	15.2	10.6	5.4	18.8	15.2	10.6	5.4
1.50	19.4	15.5	10.4	4.3	19.4	15.5	10.4	4.3	19.4	15.5	10.4	4.3
1.75	19.9	15.8	10.2	3.4	19.9	15.8	10.2	3.4	19.9	15.8	10.2	3.4
2.00	20.3	16.1	10.1	2.6	20.3	16.1	10.1	2.6	20.3	16.1	10.1	2.6
2.25	20.6	16.4	10.1	2.1	20.6	16.4	10.1	2.1	20.6	16.4	10.1	2.1
2.50	20.9	16.7	10.1	1.7	20.9	16.7	10.1	1.7	20.9	16.7	10.1	1.7
ε_0	0.0075				0.0100				0.0125			
(j, k)	0	1	2	3	0	1	2	3	0	1	2	3
-2.50	-4.8	10.7	-9.2	34.9	-3.5	7.8	-9.2	34.9	-2.4	6.8	-9.2	34.9
-2.25	-5.7	7.4	-14.6	32.9	-4.5	5.0	-14.6	32.9	-3.3	4.2	-14.6	32.9
-2.00	-6.0	3.0	-27.0	30.9	-5.0	1.3	-27.0	30.9	-3.7	0.7	-27.0	30.9
-1.75	-5.1	-2.6	-44.2	28.7	-4.7	-3.4	-37.9	28.7	-3.3	-3.6	-32.5	28.7
-1.50	-2.3	-7.7	17.5	26.5	-2.3	-7.7	9.1	26.5	-0.9	-7.7	4.3	26.5
-1.25	0.9	-6.2	39.9	24.3	0.9	-7.0	39.9	24.3	0.9	-7.0	39.9	24.3
-1.00	4.0	10.5	25.7	22.0	4.0	8.5	25.7	22.0	4.0	7.8	25.7	22.0
-0.75	6.8	11.7	19.9	19.8	6.8	11.7	19.9	19.8	6.8	11.7	19.9	19.8
-0.50	9.3	12.4	16.7	17.5	9.3	12.4	16.7	17.5	9.3	12.4	16.7	17.5
-0.25	11.5	12.9	14.7	15.4	11.5	12.9	14.7	15.4	11.5	12.9	14.7	15.4
0.00	13.4	13.4	13.4	13.4	13.4	13.4	13.4	13.4	13.4	13.4	13.4	13.4
0.25	14.9	13.8	12.4	11.5	14.9	13.8	12.4	11.5	14.9	13.8	12.4	11.5
0.50	16.2	14.1	11.8	9.7	16.2	14.1	11.8	9.7	16.2	14.1	11.8	9.7
0.75	17.3	14.5	11.3	8.1	17.3	14.5	11.3	8.1	17.3	14.5	11.3	8.1
1.00	18.1	14.8	10.9	6.6	18.1	14.8	10.9	6.6	18.1	14.8	10.9	6.6
1.25	18.8	15.2	10.6	5.4	18.8	15.2	10.6	5.4	18.8	15.2	10.6	5.4
1.50	19.4	15.5	10.4	4.3	19.4	15.5	10.4	4.3	19.4	15.5	10.4	4.3
1.75	19.9	15.8	10.2	3.4	19.9	15.8	10.2	3.4	19.9	15.8	10.2	3.4
2.00	20.3	16.1	10.1	2.6	20.3	16.1	10.1	2.6	20.3	16.1	10.1	2.6
2.25	20.6	16.4	10.1	2.1	20.6	16.4	10.1	2.1	20.6	16.4	10.1	2.1
2.50	20.9	16.7	10.1	1.7	20.9	16.7	10.1	1.7	20.9	16.7	10.1	1.7

Table 2: The optimal regularized allocations (%) to S&P 500 for $\varepsilon_0 = 0.0025, 0.0050, 0.0075, 0.0100$, and 0.0125.

ε_0	0.0000				0.0025				0.0050			
(j, k)	0	1	2	3	0	1	2	3	0	1	2	3
-2.50	-141.3	127.4	30.5	-54.1	-141.3	127.4	30.5	-54.1	-92.7	127.4	30.5	-54.1
-2.25	-95.2	121.3	25.1	-42.8	-95.2	121.3	25.1	-42.8	-74.7	109.9	25.1	-42.8
-2.00	-52.1	116.8	13.9	-31.6	-52.1	116.8	13.9	-31.6	-52.1	64.0	13.9	-31.6
-1.75	-15.2	118.2	-33.1	-20.6	-15.2	47.8	-33.1	-20.6	-15.2	24.5	-9.0	-20.6
-1.50	13.6	377.3	130.3	-10.1	13.6	2.9	104.6	-10.1	13.6	2.4	63.5	-10.1
-1.25	33.6	59.1	64.6	-0.2	33.6	36.4	64.6	-0.2	33.6	19.6	64.6	-0.2
-1.00	45.3	62.6	51.8	9.0	45.3	62.6	51.8	9.0	45.3	62.6	51.8	9.0
-0.75	49.8	57.9	45.7	17.3	49.8	57.9	45.7	17.3	49.8	57.9	45.7	17.3
-0.50	48.8	51.2	41.6	24.6	48.8	51.2	41.6	24.6	48.8	51.2	41.6	24.6
-0.25	43.7	43.7	38.5	30.8	43.7	43.7	38.5	30.8	43.7	43.7	38.5	30.8
0.00	35.9	35.9	35.9	35.9	35.9	35.9	35.9	35.9	35.9	35.9	35.9	35.9
0.25	26.4	27.8	33.5	39.7	26.4	27.8	33.5	39.7	26.4	27.8	33.5	39.7
0.50	16.1	19.6	31.1	42.3	16.1	19.6	31.1	42.3	16.1	19.6	31.1	42.3
0.75	5.4	11.3	28.9	43.7	5.4	11.3	28.9	43.7	5.4	11.3	28.9	43.7
1.00	-5.2	3.0	26.6	43.9	-5.2	3.0	26.6	43.9	-5.2	3.0	26.6	43.9
1.25	-15.6	-5.3	24.3	43.0	-15.6	-5.3	24.3	43.0	-15.6	-5.3	24.3	43.0
1.50	-25.4	-13.5	21.9	41.1	-25.4	-13.5	21.9	41.1	-25.4	-13.5	21.9	41.1
1.75	-34.6	-21.7	19.5	38.2	-34.6	-21.7	19.5	38.2	-34.6	-21.7	19.5	38.2
2.00	-43.2	-29.9	16.9	34.5	-43.2	-29.9	16.9	34.5	-43.2	-29.9	16.9	34.5
2.25	-51.2	-37.9	14.2	30.0	-51.2	-37.9	14.2	30.0	-51.2	-37.9	14.2	30.0
2.50	-58.4	-45.8	11.4	24.9	-58.4	-45.8	11.4	24.9	-58.4	-45.8	11.4	24.9
ε_0	0.0075				0.0100				0.0125			
(j, k)	0	1	2	3	0	1	2	3	0	1	2	3
-2.50	-61.8	104.9	30.5	-54.1	-46.4	78.7	30.5	-54.1	-37.1	63.0	30.5	-54.1
-2.25	-49.8	73.4	25.1	-42.8	-37.3	55.1	25.1	-42.8	-29.8	44.1	25.1	-42.8
-2.00	-34.8	42.9	13.9	-31.6	-26.0	32.3	13.9	-31.6	-20.8	26.0	13.9	-31.6
-1.75	-13.8	16.7	1.3	-20.6	-10.3	12.8	6.2	-20.6	-8.1	10.5	9.0	-20.6
-1.50	13.6	2.2	49.5	-10.1	13.6	2.1	42.3	-10.1	11.0	2.1	37.8	-10.1
-1.25	33.6	14.0	64.6	-0.2	33.6	11.2	64.6	-0.2	33.6	9.5	64.6	-0.2
-1.00	45.3	62.6	51.8	9.0	45.3	56.9	51.8	9.0	45.3	46.2	51.8	9.0
-0.75	49.8	57.9	45.7	17.3	49.8	57.9	45.7	17.3	49.8	57.9	45.7	17.3
-0.50	48.8	51.2	41.6	24.6	48.8	51.2	41.6	24.6	48.8	51.2	41.6	24.6
-0.25	43.7	43.7	38.5	30.8	43.7	43.7	38.5	30.8	43.7	43.7	38.5	30.8
0.00	35.9	35.9	35.9	35.9	35.9	35.9	35.9	35.9	35.9	35.9	35.9	35.9
0.25	26.4	27.8	33.5	39.7	26.4	27.8	33.5	39.7	26.4	27.8	33.5	39.7
0.50	16.1	19.6	31.1	42.3	16.1	19.6	31.1	42.3	16.1	19.6	31.1	42.3
0.75	5.4	11.3	28.9	43.7	5.4	11.3	28.9	43.7	5.4	11.3	28.9	43.7
1.00	-5.2	3.0	26.6	43.9	-5.2	3.0	26.6	43.9	-5.2	3.0	26.6	43.9
1.25	-15.6	-5.3	24.3	43.0	-15.6	-5.3	24.3	43.0	-15.6	-5.3	24.3	43.0
1.50	-25.4	-13.5	21.9	41.1	-25.4	-13.5	21.9	41.1	-25.4	-13.5	21.9	41.1
1.75	-34.6	-21.7	19.5	38.2	-34.6	-21.7	19.5	38.2	-34.6	-21.7	19.5	38.2
2.00	-43.2	-29.9	16.9	34.5	-43.2	-29.9	16.9	34.5	-43.2	-29.9	16.9	34.5
2.25	-51.2	-37.9	14.2	30.0	-51.2	-37.9	14.2	30.0	-51.2	-37.9	14.2	30.0
2.50	-58.4	-45.8	11.4	24.9	-58.4	-45.8	11.4	24.9	-58.4	-45.8	11.4	24.9

Table 3: The optimal regularized allocations (%) to the TIPS for $\varepsilon_0 = 0.0025, 0.0050, 0.0075, 0.0100$, and 0.0125.

As the regularization parameter increases, the stability of the optimal regularized portfolio increases. However, the marginal stabilizing effect decreases; meanwhile, when the regularization parameter exceeds 0.0075, the regularization appears as a distortion in the generally stabilized regions (yellow-green and green cells), that is, regions that do not require modification. We judge that at 0.0100, the distortion is small, and the disadvantage of the distortion is smaller than the benefit of the stabilization effect, whereas at 0.0125, the distortion becomes larger, and the disadvantage of the distortion exceeds the benefit of the stabilization effect. Therefore, we set the regularization parameter to $\varepsilon_0 = 0.0100$.

7.2 Market Timing Effects

The estimated optimal allocations to the S&P 500, TIPS, and money market account are plotted against the state vector in Figs.1-3. Markers in Figs.1-3 indicate the optimal asset allocations which the regularization is applied to.

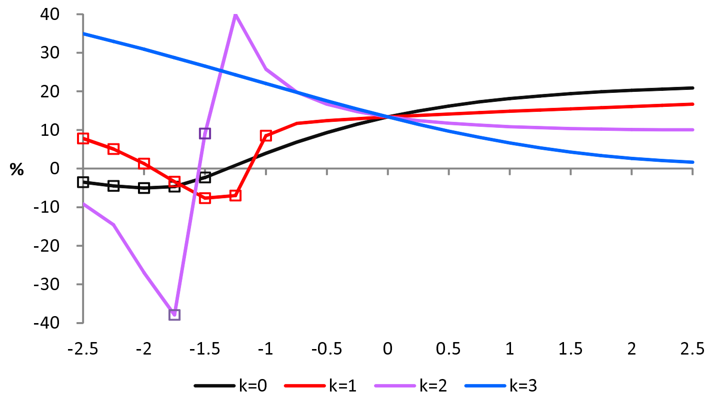


Figure 1: Optimal allocation (%) to S&P 500 plotted against the state vector.

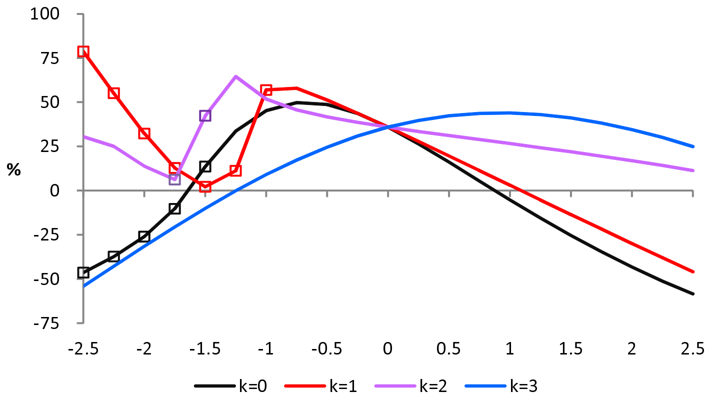


Figure 2: Optimal allocation (%) to the TIPS plotted against the state vector.

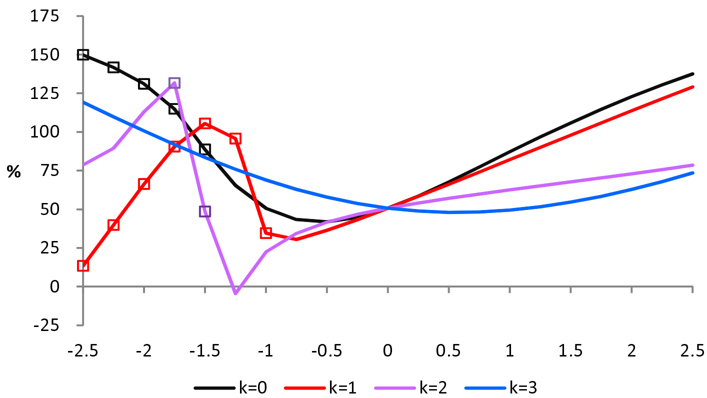


Figure 3: Optimal allocation (%) to the money market account plotted against the state vector.

The optimal allocations to both the S&P 500 and TIPS respond significantly and nonlinearly to changes in the state vector, suggesting that market timing effects are significant and nonlinear. In particular, even after stabilization by regularization, market timing effects are significant and nonlinear where the near-singularity of $\Sigma(X_t)\Sigma(X_t)'$ occurs.

7.3 Factor Decomposition

We first analyze the contribution of each type of demand to the magnitude of market timing effects in optimal regularized robust asset allocation.

Then, we analyze the contribution of asset return standard deviations and correlations, as well as the optimal investment control, to the magnitude of market timing effects.

7.3.1 By Demand Type

First, we analyze the contribution of each demand to the magnitude of the market timing effects in optimal asset allocation by breaking down the market timing effects by demand factor. It follows from eqs.(4.9) and (5.15) that the sum decomposition of the approximate optimal regularized robust asset allocation $\tilde{\Phi}^*_\varepsilon(X_t)$ is given by

$$\tilde{\Phi}^*_\varepsilon(X_t) = \frac{1}{\gamma+\theta}\Sigma_\varepsilon(X_t)'^{-1}\Big(\lambda + \Lambda X_t\Big) + \left(1-\frac{1}{\gamma+\theta}\right)\Sigma_\varepsilon(X_t)'^{-1}\left(-\frac{1}{\psi-1}\Big(\tilde{b}(\tau,X_t)+\tilde{B}(\tau,X_t)X_t\Big)\right) + \left(1-\frac{1}{\gamma+\theta}\right)\Sigma_\varepsilon(X_t)'^{-1}\Big(\sigma_p+\Sigma_p X_t\Big). \quad (7.2)$$

Figs.4 and 5 show the factor decomposition of optimal allocation to S&P 500 plotted against the state vector.

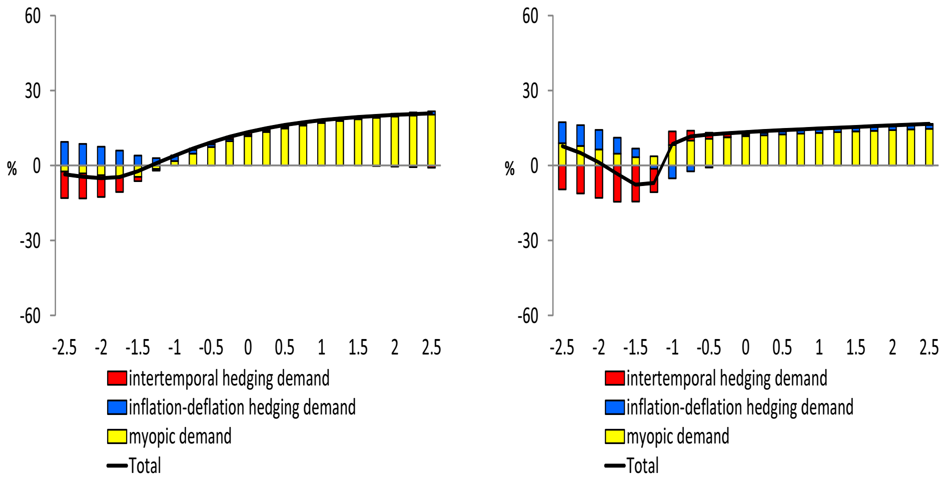


Figure 4: Factor decomposition of optimal allocation (%) to S&P 500 plotted against the state vector by demand type ($k = 0, 1$).

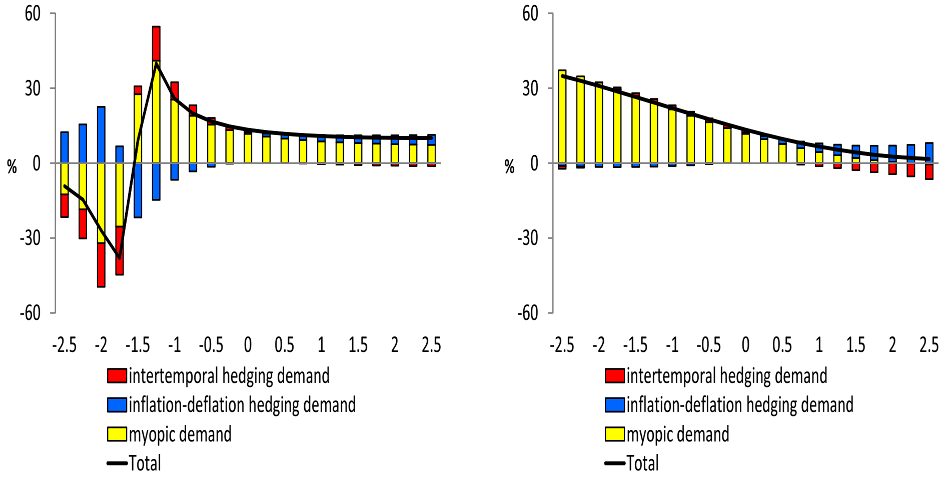


Figure 5: Factor decomposition of optimal allocation (%) to S&P 500 plotted against the state vector by demand type ($k = 2, 3$).

As the state vector changes, myopic demand changes the most, but intertemporal hedging demand and inflation-deflation hedging demand also change considerably. Figs.6 and 7 illustrate the factor decomposition of optimal allocation to the TIPS plotted against the state vector.

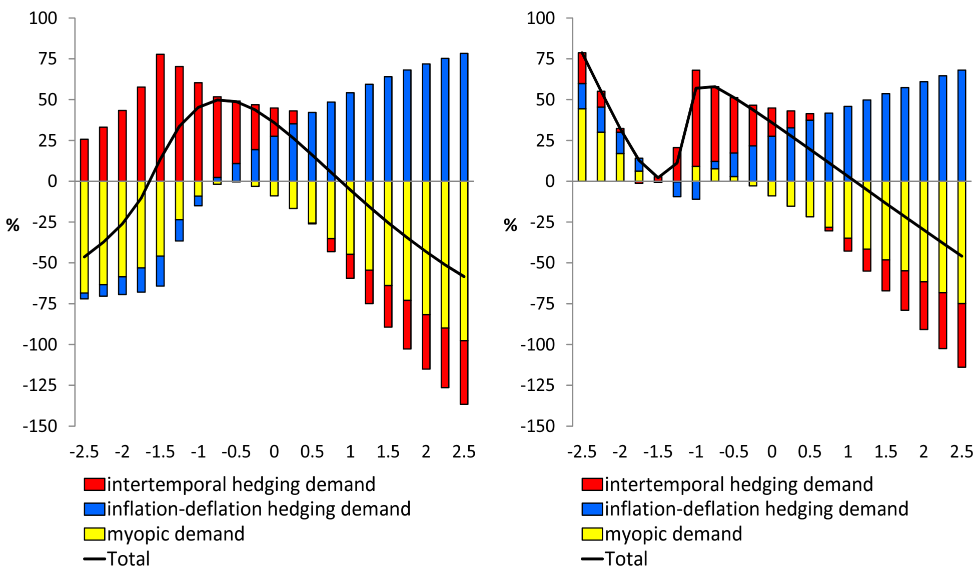


Figure 6: Factor decomposition of optimal allocation (%) to the TIPS plotted against the state vector by demand type ($k = 0, 1$).

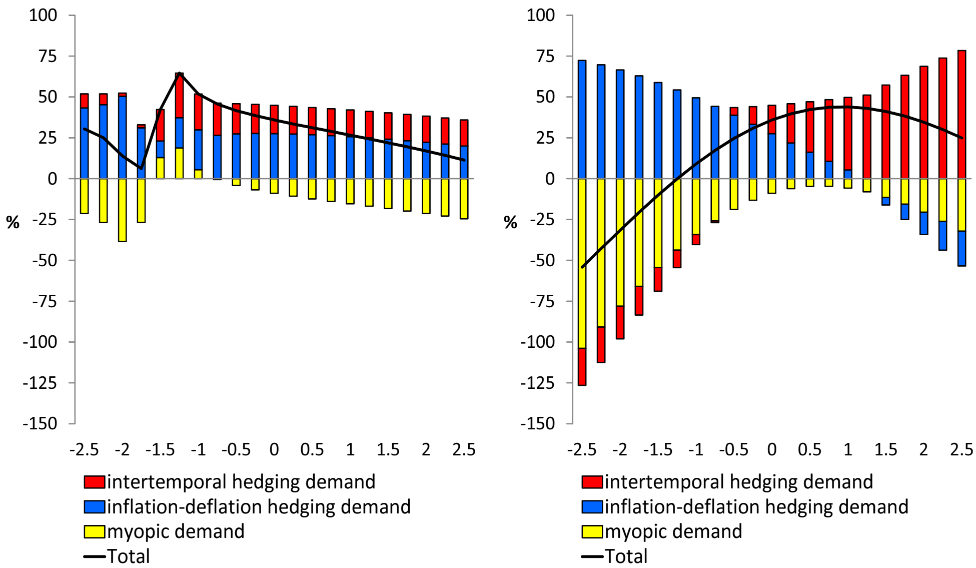


Figure 7: Factor decomposition of optimal allocation (%) to the TIPS plotted against the state vector ($k = 2, 3$).

As the state vector changes, all types of demand change significantly.

7.3.2 Standard Deviations and Correlations of Asset Returns and Investment Control

Next, we measure the contribution of asset return standard deviations and correlations, as well as the optimal investment control, to the magnitude of the market timing effects in the optimal asset allocation. Let $\hat{\varsigma}^*(X_t) = \tilde{\varsigma}_t^* + \sigma_t^p$ where $\tilde{\varsigma}_t^*$ is given by eq.(5.15). It follows from eq.(4.19) that the product decomposition of $\tilde{\Phi}_\varepsilon^*(X_t)$ is given by

$$\tilde{\Phi}_\varepsilon^*(X_t) = \Sigma_\varepsilon(X_t)'^{-1}\hat{\varsigma}^*(X_t) = D_\varepsilon(X_t)^{-1}\sqrt{R_\varepsilon(X_t)}'^{-1}\hat{\varsigma}^*(X_t). \tag{7.3}$$

Let $\Delta g(x) = g(x) - g(0)$. We use the following notations.

$$\begin{aligned}
\Delta_1 \tilde{\Phi}^*_\varepsilon(X_t) &= \Delta D_\varepsilon(X_t)^{-1} \sqrt{R_\varepsilon(0)}'^{-1} \hat{\varsigma}^*(0) \\
&\quad + \frac{1}{2}\Big(\Delta D_\varepsilon(X_t)^{-1} \Delta \sqrt{R_\varepsilon(X_t)}'^{-1} \hat{\varsigma}^*(0) + \Delta D_\varepsilon(X_t)^{-1} \sqrt{R_\varepsilon(0)}'^{-1} \Delta \hat{\varsigma}^*(X_t)\Big), \\
\Delta_2 \tilde{\Phi}^*_\varepsilon(X_t) &= D_\varepsilon(0)'^{-1} \Delta \sqrt{R_\varepsilon(X_t)}'^{-1} \hat{\varsigma}^*(0) \\
&\quad + \frac{1}{2}\Big(\Delta D_\varepsilon(X_t)^{-1} \Delta \sqrt{R_\varepsilon(X_t)}'^{-1} \hat{\varsigma}^*(0) + D_\varepsilon(0)^{-1} \Delta \sqrt{R_\varepsilon(X_t)}'^{-1} \Delta \hat{\varsigma}^*(X_t)\Big), \\
\Delta_3 \tilde{\Phi}^*_\varepsilon(X_t) &= D_\varepsilon(0)'^{-1} \sqrt{R_\varepsilon(0)}'^{-1} \Delta \hat{\varsigma}^*(X_t) \\
&\quad + \frac{1}{2}\Big(\Delta D_\varepsilon(X_t)^{-1} \sqrt{R_\varepsilon(0)}'^{-1} \Delta \hat{\varsigma}^*(X_t) + D_\varepsilon(0)^{-1} \Delta \sqrt{R_\varepsilon(X_t)}'^{-1} \Delta \hat{\varsigma}^*(X_t)\Big), \\
\Delta_4 \tilde{\Phi}^*_\varepsilon(X_t) &= \Delta D_\varepsilon(X_t)^{-1} \Delta \sqrt{R_\varepsilon(X_t)}'^{-1} \Delta \hat{\varsigma}^*(X_t).
\end{aligned}$$

Then, the approximate optimal asset allocation is decomposed as

$$\tilde{\Phi}^*_\varepsilon(X_t) = \tilde{\Phi}^*_\varepsilon(0) + \sum_{i=1}^{4} \Delta_i \tilde{\Phi}^*_\varepsilon(X_t), \tag{7.4}$$

where $\Delta_1 \tilde{\Phi}^*_\varepsilon(X_t)$ represents the contribution of the standard deviation factor to market timing effects in optimal asset allocation, $\Delta_2 \tilde{\Phi}^*_\varepsilon(X_t)$ represents the contribution of the correlation factor, $\Delta_3 \tilde{\Phi}^*_\varepsilon(X_t)$ represents the contribution of the investment control factor, and $\Delta_4 \tilde{\Phi}^*_\varepsilon(X_t)$ represents the contribution of the compound factor.

Figs.8 and 9 show the factor decomposition of optimal allocation to S&P 500 plotted against the state vector based on the product decomposition.

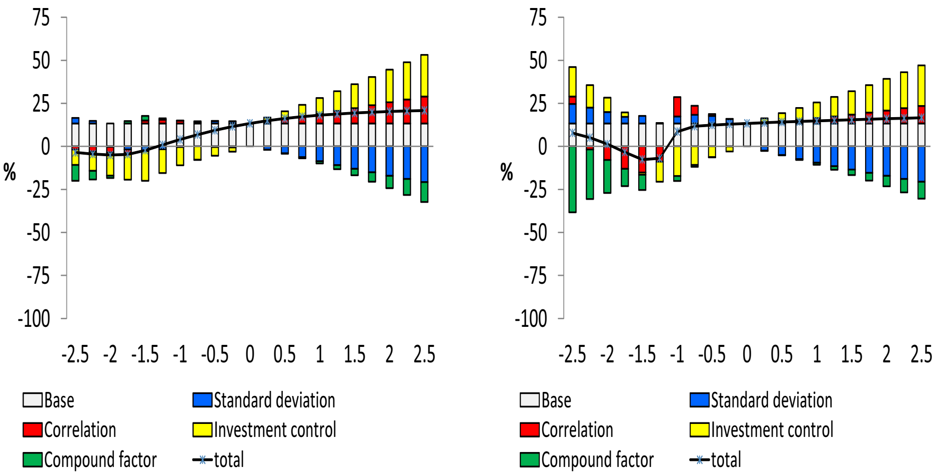


Figure 8: Factor decomposition of optimal allocation (%) to S&P 500 plotted against the state vector based on the product decomposition ($k = 0, 1$).

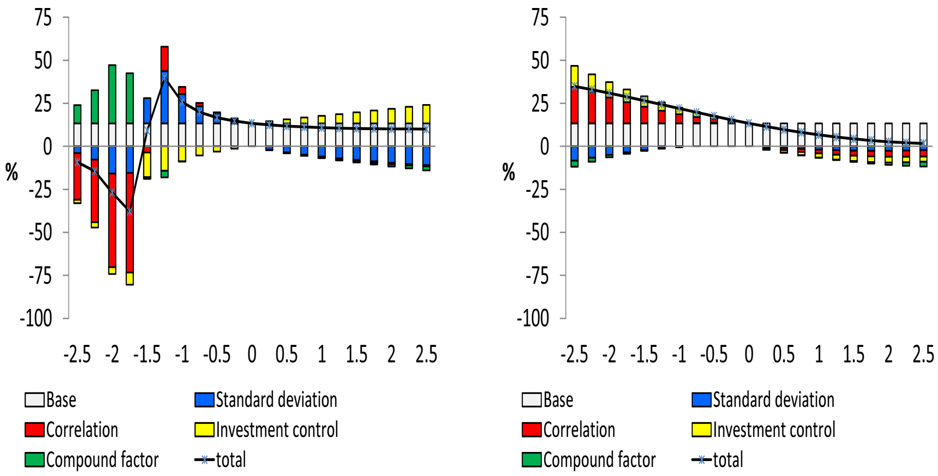


Figure 9: Factor decomposition of optimal allocation (%) to S&P 500 plotted against the state vector based on the product decomposition ($k = 2, 3$).

As the state vector changes, all factors change significantly. Figs. 10 and 11 show the factor decomposition of optimal allocation to the TIPS plotted against the state vector.

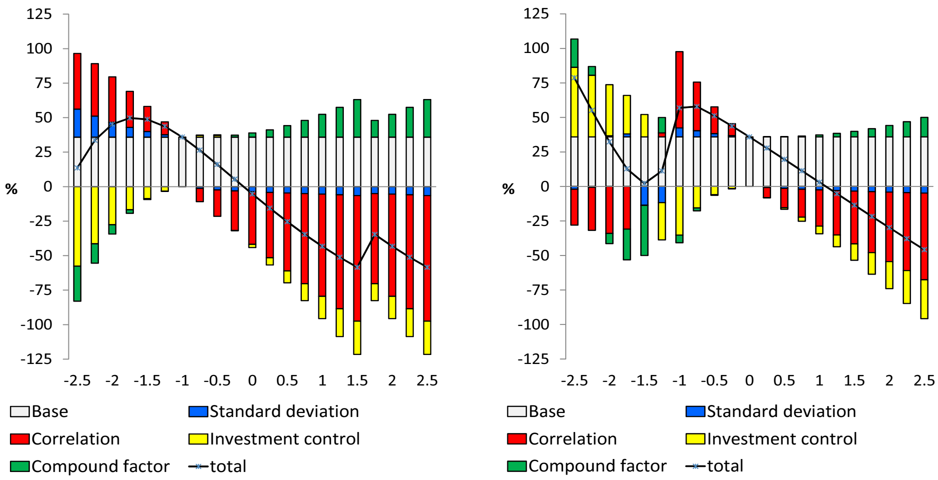
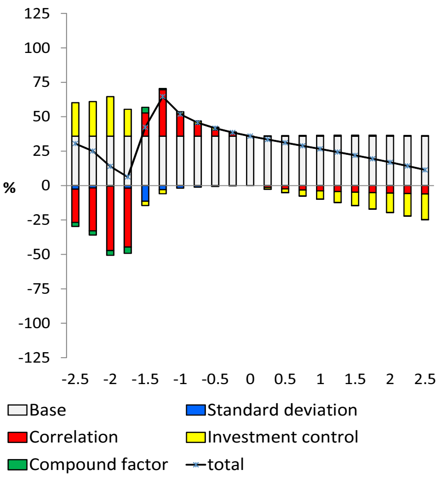


Figure 10: Factor decomposition of optimal allocation (%) to the TIPS plotted against the state vector based on the product decomposition ($k = 0, 1$).



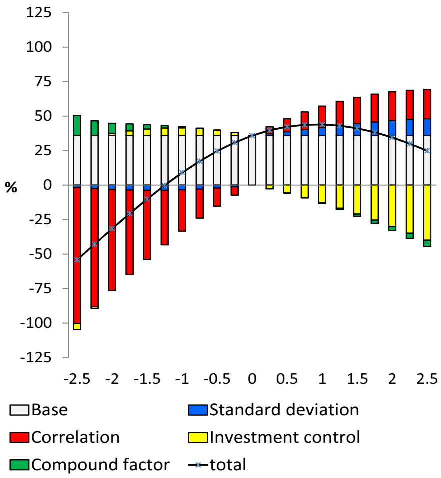


Figure 11: Factor decomposition of optimal allocation (%) to the TIPS plotted against the state vector based on the product decomposition ($k = 2, 3$).

As the state vector changes, the correlation factor changes the most, followed by the investment control factor.

8 Conclusion

We considered a finite-time consumption-investment problem for investors with homothetic robust Epstein-Zin utility under a quadratic security market model in which interest rates, market price of risk, variances and covariances of asset returns, and inflation rates are stochastic and mean-reverting. First, we showed that since the optimal robust portfolio is proportional to the inverse of the stochastic variance-covariance matrix, it becomes unstable as the near-singularity of $\Sigma(X_t)\Sigma(X_t)'$ occurs. Therefore, we proposed a regularized consumption-investment problem in which the near-singularity risk is added as a regularization term to the volatility of the asset returns. We derived the optimal regularized robust portfolio and showed that the maximum eigenvalue of the inverse of the regularized variance-covariance matrix in the optimal portfolio is controlled below a certain value, which ensures portfolio stability, even in the presence of the near-singularity of $\Sigma(X_t)\Sigma(X_t)'$. Subsequently, we decomposed the optimal portfolio into the product of the standard deviation, correlation, and investment control factors. Since the optimal regularized robust portfolio contains an unknown function that is a solution of a nonlinear nonhomogeneous PDE, we applied the linear approximation method of Kikuchi and Kusuda (2023) to the PDE and derived an approximate optimal regularized robust portfolio.

Subsequently, we removed the regularization term for portfolio stabilization from the loss function of Kikuchi and Kusuda (2023) and re-estimated the quadratic security market model. Using the re-estimated security market model, we examined the market timing effects in the regularized approximate optimal robust portfolio. Our numerical analysis showed that the market timing effects were nonlinear and significant. Then, we analyzed the contribution of each demand to the magnitude of the market timing effects based on the sum decomposition. The results confirmed that in addition to myopic demand, inflation-deflation hedging demand is a significant contributor to market timing effects, thereby consistent with the results shown by Batbold *et al.* (2022) and Kikuchi and Kusuda (2023). Finally, we analyzed the contribution of each factor to the magnitude of the market timing effects based on the above product decomposition. The results indicated that all factors contributed to the magnitude of the market timing effects. These results suggest that incorporating stochastic variance-covariance of asset returns in addition to stochastic inflation-deflation risk into security market models is essential for the analysis of dynamic asset allocation.

A Proofs

A.1 Proof of Lemma 3

It is clear that the worst-case probability P^{ξ^*} is obtained as follows:

$$\xi_t^* = -\frac{\theta}{(1-\gamma)J}\begin{pmatrix}\bar{W}_t\bar{\sigma}_t' \\ I\end{pmatrix}'\begin{pmatrix}J_W \\ J_X\end{pmatrix}. \tag{A.1}$$

Substituting ξ^* into the HJB eq.(3.6) yields

$$\begin{aligned}\sup_{(c,\bar{\sigma})\in\mathcal{B}(X_0)}\Bigg[J_t &+ \begin{pmatrix}\bar{W}_t\left(\bar{r}_t+\bar{\sigma}_t'\bar{\lambda}_t\right)-c_t \\ -\mathcal{K}X_t\end{pmatrix}'\begin{pmatrix}J_W \\ J_X\end{pmatrix} \\ &+\frac{1}{2}\operatorname{tr}\left[\begin{pmatrix}\bar{W}_t\bar{\sigma}_t' \\ I\end{pmatrix}\begin{pmatrix}\bar{W}_t\bar{\sigma}_t' \\ I\end{pmatrix}'\begin{pmatrix}J_{WW} & J_{WX} \\ J_{XW} & J_{XX}\end{pmatrix}\right] \\ &+f(c_t,J)-\frac{\theta}{2(1-\gamma)J}\left|\begin{pmatrix}\bar{W}_t\bar{\sigma}_t' \\ I\end{pmatrix}'\begin{pmatrix}J_W \\ J_X\end{pmatrix}\right|^2\Bigg]=0. \end{aligned}\tag{A.2}$$

It is easy to see that optimal control $u_t^* = (c_t^*, \bar{\sigma}_t^*)$ in the HJB equation (A.2) satisfies

$$c_t^* = \beta^{\psi}J_W^{-\psi}\left((1-\gamma)J\right)^{\frac{\gamma\psi-1}{\gamma-1}}, \tag{A.3}$$

$$\bar{\sigma}_t^* = \mathcal{T}_t\left(\bar{\lambda}_t+\frac{J_{XW}}{J_W}+\frac{\theta}{\gamma-1}\frac{J_X}{J}\right), \tag{A.4}$$

where $\mathcal{T}_t$ is given by

$$\mathcal{T}_t = \left(-\frac{\bar{W}_t^*J_{WW}}{J_W}+\theta\frac{\bar{W}_t^*J_W}{(1-\gamma)J}\right)^{-1}. \tag{A.5}$$

The consumption-related terms in the HJB eq.(A.2) are computed as

$$-c_t^* J_W + f(c_t^*, J) = c_t^* \left(-J_W + \frac{1}{1-\psi^{-1}} J_W \right) - \frac{\beta(1-\gamma)}{1-\psi^{-1}} J = \frac{1}{\psi - 1} c_t^* J_W - \frac{\beta(1-\gamma)}{1-\psi^{-1}} J. \tag{A.6}$$

The investment-related terms in the HJB eq.(A.2) are computed as

$$\begin{aligned} \bar{W}_t^* J_W \bar{\lambda}_t' \bar{\sigma}_t^* + \frac{1}{2} \operatorname{tr} \left[\begin{pmatrix} \bar{W}_t^* (\bar{\sigma}_t^*)' \\ I \end{pmatrix} \begin{pmatrix} \bar{W}_t^* (\bar{\sigma}_t^*)' \\ I \end{pmatrix}' \begin{pmatrix} J_{WW} & J_{WX} \\ J_{XW} & J_{XX} \end{pmatrix} \right] \\ - \frac{\theta}{2(1-\gamma)J} \left| \begin{pmatrix} \bar{W}_t \bar{\sigma}_t' \\ I \end{pmatrix}' \begin{pmatrix} J_W \\ J_X \end{pmatrix} \right|^2 \\ = \frac{1}{2} \operatorname{tr} [J_{XX}] - \frac{\theta}{2(1-\gamma)J} |J_X|^2 - \left(\bar{W}_t^{*2} J_{WW} - \frac{\theta (W_t^* J_W)^2}{(1-\gamma)J} \right)^{-1} |\zeta_t|^2, \end{aligned} \tag{A.7}$$

where

$$\zeta_t = -\bar{W}_t^* J_W \left(\bar{\lambda}_t + \frac{J_{XW}}{J_W} + \frac{\theta}{\gamma - 1} \frac{J_X}{J} \right). \tag{A.8}$$

By substituting optimal control (A.3) and (A.4) into the HJB equation (A.2), and using eqs.(A.6) and (A.7), the following PDE for J is obtained:

$$\begin{aligned} J_t + \frac{1}{2} \operatorname{tr} [J_{XX}] - \frac{\theta}{2(1-\gamma)J} |J_X|^2 - \frac{1}{2} \left(\bar{W}_t^{*2} J_{WW} - \frac{\theta (W_t^* J_W)^2}{(1-\gamma)J} \right)^{-1} |\zeta_t|^2 \\ + \bar{r}_t \bar{W}_t^* J_W - (\mathcal{K} X_t)' J_X + \frac{1}{\psi - 1} c_t^* J_W - \frac{\beta(1-\gamma)}{1-\psi^{-1}} J = 0. \end{aligned} \tag{A.9}$$

From the above PDE, we conjecture that the indirect utility function takes the form of (3.7). The derivatives of J are given by

$$J_t = -\frac{1-\gamma}{\psi - 1} J \frac{G_\tau}{G}, \quad \bar{W} J_W = (1-\gamma)J, \quad J_X = \frac{1-\gamma}{\psi - 1} J \frac{G_X}{G}, \quad \bar{W}^2 J_{WW} = -\gamma(1-\gamma)J,$$

$$\bar{W} J_{XW} = \frac{(1-\gamma)^2}{\psi - 1} J \frac{G_X}{G}, \quad J_{XX} = \frac{1-\gamma}{\psi - 1} J \left(\frac{2-\gamma-\psi}{\psi - 1} \frac{G_X}{G} \frac{G_X'}{G} + \frac{G_{XX}}{G} \right).$$

The optimal consumption control (3.9) follows from eq.(A.3):

$$c_t^* = \beta^\psi \left(\frac{(1-\gamma)J}{\bar{W}_t^*} \right)^{-\psi} ((1-\gamma)J)^{\frac{\gamma\psi - 1}{\gamma - 1}} = \beta^\psi \bar{W}_t^{*\psi} \left(\bar{W}_t^{*1-\gamma} G^{\frac{1-\gamma}{\psi - 1}} \right)^{\frac{\psi - 1}{\gamma - 1}} = \beta^\psi \frac{\bar{W}_t^*}{G}. \tag{A.10}$$

$\mathcal{T}_t$ in eq.(A.5) and ζ_t in eq.(A.8) are expressed as

$$\mathcal{T}_t = (\gamma + \theta)^{-1}, \tag{A.11}$$

$$\zeta_t = (\gamma - 1)J \left(\bar{\lambda}_t + \frac{\gamma + \theta - 1}{1 - \psi} \frac{G_X}{G} \right). \tag{A.12}$$

Therefore, by inserting eq.(A.12) and derivatives of J into eq.(A.4), we obtain the optimal investment control (3.10). The second to fourth terms in the PDE (A.9) are calculated from eqs.(A.12) as follows:

$$
\begin{aligned}
&\frac{1}{2}\operatorname{tr}\left[J_{XX}\right]-\frac{\theta}{2(1-\gamma)J}|J_X|^2-\frac{1}{2}\left(\bar{W}_t^{*2}J_{WW}-\frac{\theta(W_t^*J_W)^2}{(1-\gamma)J}\right)^{-1}|\zeta_t|^2\\
=&J\left\{\frac{1-\gamma}{2(\psi-1)}\operatorname{tr}\left[\frac{2-\gamma-\psi}{\psi-1}\frac{G_X}{G}\frac{G_X'}{G}+\frac{G_{XX}}{G}\right]-\frac{(1-\gamma)\theta}{2(\psi-1)^2}\left|\frac{G_X}{G}\right|^2\right.\\
&\left.+\frac{1-\gamma}{2(\psi-1)^2(\gamma+\theta)}\left|(\psi-1)\bar{\lambda}_t-(\gamma+\theta-1)\frac{G_X}{G}\right|^2\right\}\\
=&\frac{1-\gamma}{\psi-1}J\left\{\frac{1}{2}\operatorname{tr}\left[\frac{2-\gamma-\psi}{\psi-1}\frac{G_X}{G}\frac{G_X'}{G}+\frac{G_{XX}}{G}\right]-\frac{\theta}{2(\psi-1)}\left|\frac{G_X}{G}\right|^2\right.\\
&\left.+\frac{1}{2(\psi-1)(\gamma+\theta)}\left|(\psi-1)\bar{\lambda}_t-(\gamma+\theta-1)\frac{G_X}{G}\right|^2\right\} \qquad \text{(A.13)}\\
=&\frac{1-\gamma}{\psi-1}J\left\{\frac{1}{2}\operatorname{tr}\left[\frac{G_{XX}}{G}\right]+\frac{\psi-1}{2(\gamma+\theta)}|\bar{\lambda}_t|^2-\left(1-(\gamma+\theta)^{-1}\right)\bar{\lambda}_t'\frac{G_X}{G}\right.\\
&\left.-\frac{1}{2(\psi-1)}\Big(\gamma+\psi-2+\theta-\left(1-(\gamma+\theta)^{-1}\right)(\gamma+\theta-1)\Big)\left|\frac{G_X}{G}\right|^2\right\}\\
=&\frac{1-\gamma}{\psi-1}J\left\{\frac{1}{2}\operatorname{tr}\left[\frac{G_{XX}}{G}\right]+\frac{\psi-1}{2}(\gamma+\theta)^{-1}|\bar{\lambda}_t|^2-\left(1-(\gamma+\theta)^{-1}\right)\bar{\lambda}_t'\frac{G_X}{G}\right.\\
&\left.-\frac{1}{2(\psi-1)}\left(\psi-(\gamma+\theta)^{-1}\right)\left|\frac{G_X}{G}\right|^2\right\}
\end{aligned}
$$

The first, fifth, and sixth terms in the PDE (A.9) are computed as follows:

$$J_t+\bar{r}_t\bar{W}_t^*J_W-(\mathcal{K}X_t)'J_X=\frac{1-\gamma}{\psi-1}J\left(-\frac{G_\tau}{G}+(\psi-1)\bar{r}_t-(\mathcal{K}X_t)'\frac{G_X}{G}\right). \qquad \text{(A.14)}$$

The seventh and eighth terms in the PDE (A.9) are calculated from eq.(A.10) as follows:

$$\frac{1}{\psi-1}c_t^*J_W-\frac{\beta(1-\gamma)}{1-\psi^{-1}}J=\frac{1}{\psi-1}\left(\beta^\psi\frac{\bar{W}_t}{G}\frac{(1-\gamma)J}{\bar{W}_t}+\beta(\gamma-1)\psi J\right)=\frac{1-\gamma}{\psi-1}J\left(\frac{\beta^\psi}{G}-\beta\psi\right). \qquad \text{(A.15)}$$

Substituting eqs.(A.13)–(A.15) into eq.(A.9) and dividing by $\frac{1-\gamma}{\psi-1}J$ yields PDE (3.11).

A.2 Proof of Proposition 1

Eqs.(3.7), (4.11)-(4.13) and the PDE (3.11) immediately follow from Lemma 3. Substituting eq.(2.11) for $\bar{\lambda}_t$ in eq.(4.13), we obtain

$$\varsigma_t^*=\frac{1}{\gamma+\theta}(\lambda_t-\sigma_t^p)+\left(1-\frac{1}{\gamma+\theta}\right)\left(-\frac{1}{\psi-1}\frac{G_X(\tau,X_t)}{G(\tau,X_t)}\right). \qquad \text{(A.16)}$$

It follows from eqs.(4.4) and (4.9) that $\Phi^*_\varepsilon(X_t)$ is computed as

$$\begin{aligned}
\Phi^*_\varepsilon(X_t) &= \Sigma_\varepsilon(X_t)'^{-1}(\bar{\varsigma}^*_t + \sigma^p_t) \\
&= \Big(\Big(I + \varepsilon(X_t)\big(\Sigma(X_t)\Sigma(X_t)'\big)^{-1}\Big)\Sigma(X_t)\Big)'^{-1}(\bar{\varsigma}^*_t + \sigma^p_t) \\
&= \big(\Sigma(X_t) + \varepsilon(X_t)\Sigma(X_t)'^{-1}\big)'^{-1}(\bar{\varsigma}^*_t + \sigma^p_t) \\
&= \big(\Sigma(X_t)' + \varepsilon(X_t)\Sigma(X_t)^{-1}\big)^{-1}(\bar{\varsigma}^*_t + \sigma^p_t) \\
&= \Big(\Sigma(X_t)\Big(\Sigma(X_t)' + \varepsilon(X_t)\Sigma(X_t)^{-1}\Big)\Big)^{-1}\Sigma(X_t)(\bar{\varsigma}^*_t + \sigma^p_t) \\
&= \Big(\Sigma(X_t)\Sigma(X_t)' + \varepsilon(X_t)I\Big)^{-1}\Sigma(X_t)(\bar{\varsigma}^*_t + \sigma^p_t).
\end{aligned} \tag{A.17}$$

Substituting eq.(A.16) for $\bar{\varsigma}^*_t$ in the equation shown above yields eqs.(4.14) and (4.15).

A.3 Proof of Proposition 2

Substituting eqs.(5.7) and $G_X = \big(b^*(\tau,0) + B^*(\tau,0)X_t\big)G$ into eqs.(4.12) and (4.13) yields the approximate optimal consumption (5.14) and investment (5.15). Substituting $\tilde{g}$ and its derivatives into the PDE (5.5), we obtain the following:

$$\begin{aligned}
&\frac{db_0}{d\tau} + X'\frac{db}{d\tau} + \frac{1}{2}X'\frac{dB}{d\tau}X \\
=& m(X_t;(B,b)) - \frac{\psi - (\gamma+\theta)^{-1}}{2(\psi-1)}\Big(\tilde{b}(\tau,0) + \tilde{B}(\tau,0)X_t\big)'\big(b(\tau) + B(\tau)X_t\big)\Big) \\
=& m(X_t;(B,b)) - \frac{\psi - (\gamma+\theta)^{-1}}{2(\psi-1)}\Big\{\tilde{b}(\tau,0)'b(\tau) \\
&+ X_t'\Big(\tilde{B}(\tau,0)'b(\tau) + \big(\psi - (\gamma+\theta)^{-1}\big)B(\tau)'\tilde{b}(\tau,0)\Big) + \big(\psi - (\gamma+\theta)^{-1}\big)X_t'\tilde{B}(\tau,0)'B(\tau)X_t\Big\},
\end{aligned} \tag{A.18}$$

where $m(X_t;(B,b))$ is given by

$$\begin{aligned}
m(X_t;(B,b)) =& \frac{1}{2}\mathrm{tr}[B] + \frac{1}{2}\left(\|b\|^2 + 2X_t'Bb + X_t'B^2X_t\right) \\
&- \Big\{\big(1-(\gamma+\theta)^{-1}\big)\bar{\lambda} + \big(\mathcal{K} + \big(1-(\gamma+\theta)^{-1}\big)\bar{\Lambda}\big)X_t\Big\}'b - \big(1-(\gamma+\theta)^{-1}\big)\bar{\lambda}'BX_t \\
&- \frac{1}{2}X_t'\left(\mathcal{K} + \big(1-(\gamma+\theta)^{-1}\big)\bar{\Lambda}\right)'BX_t - \frac{1}{2}X_t'B\left(\mathcal{K} + \big(1-(\gamma+\theta)^{-1}\big)\bar{\Lambda}\right)X_t \\
&+ \frac{\psi-1}{2}\Big((\gamma+\theta)^{-1}|\bar{\lambda}|^2 + 2(\gamma+\theta)^{-1}\bar{\lambda}'\bar{\Lambda}X_t + (\gamma+\theta)^{-1}X_t'\bar{\Lambda}'\bar{\Lambda}X_t\Big) \\
&+ (\psi-1)\left(\bar{\rho}_0 + \bar{\rho}'X_t + \frac{1}{2}X_t'\bar{\mathcal{R}}X_t\right) - \beta\psi.
\end{aligned} \tag{A.19}$$

As eq.(A.18) is identical on X, we obtain the system of ODEs (5.16).

B Estimation Methods and Results

B.1 State-space Model Representation

Eq.(2.1) can be transformed as $d\big(e^{t\mathcal{K}}X_t\big) = e^{t\mathcal{K}}\,dB_t$. Integrating this equation over the interval $[nh,(n+1)h]$ yields

$$e^{(n+1)h\mathcal{K}}X_{(n+1)h} - e^{nh\mathcal{K}}X_{nh} = \int_{nh}^{(n+1)h} e^{s\mathcal{K}}\,dB_s. \tag{B.1}$$

Dividing both sides of the above equation by $e^{(n+1)h\mathcal{K}}$, we get

$$X_{(n+1)h} = e^{-h\mathcal{K}}X_{nh} + \int_{nh}^{(n+1)h} e^{\{s-(n+1)h\}\mathcal{K}}\,dB_s. \tag{B.2}$$

From the above equation, we obtain eq.(6.3). By definitions of spot rate and TIPS real spot rate, the following holds:

$$s_t(\tau) = -\frac{1}{\tau}\log P_t(\tau), \tag{B.3}$$

$$\bar{s}_t^Q(\tau) = -\frac{1}{\tau}(\log Q_t(\tau) - \log p_t). \tag{B.4}$$

Thus, from eqs.(2.19), (2.24), (B.3), (B.4) (2.9), and (2.29), we obtain

$$Y_{nh} = H_2(X_{nh}) + H_1X_{nh} + H_0. \tag{B.5}$$

Adding the observation error term ε_n to the right-hand side in the above equation, we obtain eq.(6.4).

B.2 Parameter Estimates

$$\begin{aligned} d\begin{pmatrix} X_{1t} \\ X_{2t} \end{pmatrix} &= -\begin{pmatrix} \mathcal{K}_{11} & 0 \\ \mathcal{K}_{21} & \mathcal{K}_{22} \end{pmatrix}\begin{pmatrix} X_{1t} \\ X_{2t} \end{pmatrix} dt + I_2 dB_t \\ &= -\begin{pmatrix} 0.08049 & 0 \\ -0.005062 & 0.1066 \end{pmatrix}\begin{pmatrix} X_{1t} \\ X_{2t} \end{pmatrix} dt + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} dB_t, \end{aligned} \tag{B.6}$$

$$\begin{pmatrix} \lambda_{1t} \\ \lambda_{2t} \end{pmatrix} = \lambda + \Lambda\begin{pmatrix} X_{1t} \\ X_{2t} \end{pmatrix} = \begin{pmatrix} -0.03605 \\ 0.3500 \end{pmatrix} + \begin{pmatrix} 0.03388 & 0 \\ 0.1296 & 0.01928 \end{pmatrix}\begin{pmatrix} X_{1t} \\ X_{2t} \end{pmatrix}, \tag{B.7}$$

$$\begin{aligned} r_t = \rho_0 + \rho'\begin{pmatrix} X_{1t} \\ X_{2t} \end{pmatrix} + \frac{1}{2}\begin{pmatrix} X_{1t} \\ X_{2t} \end{pmatrix}'\mathcal{R}\begin{pmatrix} X_{1t} \\ X_{2t} \end{pmatrix} &= 0.06964 + \begin{pmatrix} -0.02395 \\ 0.06803 \end{pmatrix}'\begin{pmatrix} X_{1t} \\ X_{2t} \end{pmatrix} \\ &+ \frac{1}{2}\begin{pmatrix} X_{1t} \\ X_{2t} \end{pmatrix}'\begin{pmatrix} 0.004145 & -0.001112 \\ -0.001112 & 0.0004623 \end{pmatrix}\begin{pmatrix} X_{1t} \\ X_{2t} \end{pmatrix}, \end{aligned} \tag{B.8}$$

$$i_t = \iota_0 + \iota'\begin{pmatrix} X_{1t} \\ X_{2t} \end{pmatrix} = 0.02445 + \begin{pmatrix} 0.006011 \\ 0.01658 \end{pmatrix}'\begin{pmatrix} X_{1t} \\ X_{2t} \end{pmatrix}, \tag{B.9}$$

$$\begin{pmatrix} \sigma^p_{1t} \\ \sigma^p_{2t} \end{pmatrix} = \sigma_p + \Sigma_p\begin{pmatrix} X_{1t} \\ X_{2t} \end{pmatrix} = \begin{pmatrix} 0.09348 \\ 0.01345 \end{pmatrix} + \begin{pmatrix} 0.05115 & 0 \\ 0 & 0.02112 \end{pmatrix}\begin{pmatrix} X_{1t} \\ X_{2t} \end{pmatrix}, \tag{B.10}$$

$$\begin{pmatrix}\bar{\lambda}_1\\ \bar{\lambda}_2\end{pmatrix} = \lambda - \sigma_p = \begin{pmatrix}-0.1295\\ 0.3366\end{pmatrix}, \tag{B.11}$$

$$\bar{\Lambda} = \Lambda - \Sigma_p = \begin{pmatrix}-0.01727 & 0\\ 0.1296 & -0.001843\end{pmatrix}, \tag{B.12}$$

$$\bar{\rho}_0 = \rho_0 - \iota_0 + \sigma_p'\lambda = 0.04653, \tag{B.13}$$

$$\begin{pmatrix}\bar{\rho}_1\\ \bar{\rho}_2\end{pmatrix} = \rho - \iota + \Lambda'\sigma_p + \Sigma_p'\lambda = \begin{pmatrix}-0.02690\\ -0.002124\end{pmatrix}, \tag{B.14}$$

$$\frac{D_t}{S_t} = \delta_0 + \begin{pmatrix}\delta_1\\ \delta_2\end{pmatrix}' \begin{pmatrix}X_{1t}\\ X_{2t}\end{pmatrix} + \frac{1}{2}\begin{pmatrix}X_{1t}\\ X_{2t}\end{pmatrix}' \Delta \begin{pmatrix}X_{1t}\\ X_{2t}\end{pmatrix} = 0.01482 + \begin{pmatrix}0.0004861\\ 0.001912\end{pmatrix}' \begin{pmatrix}X_{1t}\\ X_{2t}\end{pmatrix} + \frac{1}{2}\begin{pmatrix}X_{1t}\\ X_{2t}\end{pmatrix}' \begin{pmatrix}2.917\times 10^{-4} & 4.569\times 10^{-6}\\ 4.569\times 10^{-6} & 1.258\times 10^{-4}\end{pmatrix} \begin{pmatrix}X_{1t}\\ X_{2t}\end{pmatrix}. \tag{B.15}$$

$$\Omega_\varepsilon = 10^{-5} \times \mathrm{diag}(2.052\times 10^{-4}, 6.825, 7.595, 1.490\times 10^{-4}). \tag{B.16}$$

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