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# Optimal Consumption-Investment and CAPMs under a Quadratic Model based on Homothetic Robust Epstein–Zin Utility*

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**Abstract**

This paper presents a theoretical analysis of the consumption-investment problem and CAPMs based on homothetic robust Epstein–Zin utility under a quadratic security market model in which interest rates, the market price of risk, the variances and covariances of asset returns, and inflation rates are stochastic. Robust investors first determine the "conditional worst-case probability" of minimizing utility for a given consumption and investment and then determine the optimal consumption and investment that maximizes utility under the conditional worst-case probability. The optimal consumption-investment decisions implicitly determine the worst-case probability. We clarify the theoretical structures of i) the budget constraint and market price of risk under the conditional worst-case probability; ii) the market price of risk under the worst-case probability; iii) the CAPMs; and iv) the CAPMs under the worst-case probability.

**Keywords** Homothetic robust utility, Consumption-investment problem, CAPM, Stochastic volatility, Stochastic inflation

**JEL classification** C61, D81, G11

## 1 Introduction

Two issues should be considered when studying consumption and investment problems. The first is to incorporate into security market models the stylized facts that interest rates, the market price of risk, the variances and covariances of asset returns, and inflation rates are stochastic and mean-reverting.

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The second is to assume utility that accounts for Knightian uncertainty, as recognized during the global financial crisis. Regarding the first issue, Batbold, Kikuchi, and Kusuda (2022) examine the consumption-investment problem for long-term investors with constant relative risk aversion (CRRA) utility under a quadratic security market model that satisfies the above stylized facts. The class of quadratic models, a generalization of affine models presented by Duffie and Kan (1996), has been independently developed by Ahn, Dittmar, and Gallant (2002) and Leippold and Wu (2002).[1] Batbold *et al.* (2022) derive an optimal portfolio decomposed into myopic demand, intertemporal hedging demand, and inflation-deflation hedging demand, and show that all three types of demand are nonlinear functions of the state vector. Their numerical analysis presents the nonlinearity and significance of market timing effects. Such nonlinearity is attributed to the stochastic variances and covariances of asset returns, while such significance is attributed to inflation-deflation hedging demand in addition to myopic demand.

Kikuchi and Kusuda (2023a) consider both these issues and study the consumption-investment problem for long-term investors with homothetic[2] robust utility under the quadratic security market model of Batbold *et al.* (2022). Investors with homothetic robust utility, introduced by Maenhout (2004) and theoretically justified by Skiadas (2003), regard the "base probability" as the most likely probability; however, they also consider other probabilities because the true probability is unknown. Homothetic robust utility is characterized by relative risk aversion and relative ambiguity aversion, which represents the investor's degree of distrust of the base probability. Since a nonlinear term appears in the partial differential equation (PDE) for the indirect utility function, Kikuchi and Kusuda (2023a) use a linear approximation method to derive an approximate optimal portfolio. Their numerical analysis confirms the nonlinearity and significance of market timing effects.

Investors with homothetic robust utility first determine the "conditional worst-case probability" of minimizing utility for a given consumption and investment and then determine the optimal consumption and optimal investment that maximize utility under the conditional worst-case probability. These optimal consumption-investment decisions implicitly determine the worst-case probability. Homothetic robust utility is used for robust portfolio studies such as Skiadas (2003), Maenhout (2006), Liu (2010), Branger, Larsen, and Munk (2013), Munk and Rubtsov (2014), Yi, Viens, Law, and Li (2015), Batbold, Kikuchi, and Kusuda (2019), and Kikuchi and Kusuda

[1] Quadratic models are adopted in security pricing studies (Chen, Filipović, and Poor (2004), Kim and Singleton (2012), Filipović, Gourier, and Mancini (2016)) and in optimal consumption-investment studies (Batbold *et al.* (2022), Kikuchi and Kusuda (2023b)).

[2] A utility function $U$ is homothetic if, for any consumption plan $c$ and $\tilde{c}$, and any scaler $\alpha > 0$, $U(\alpha\tilde{c}) \geq U(\alpha c) \Leftrightarrow U(\tilde{c}) \geq U(c)$.

(2023a)[3]. These studies have done little to elucidate the theoretical structures of i) the budget constraint equation and market price of risk under the conditional worst-case probability; ii) the market price of risk under the worst-case probability; iii) the capital asset pricing formula in general equilibrium (*i.e.*, the CAPMs); and iv) the CAPMs under the worst-case probability.

Homothetic robust utility can be interpreted as homothetic robust CRRA utility in the sense that homothetic robust utility converges to CRRA utility as ambiguity aversion approaches zero. CRRA utility does not separate relative risk aversion and the elasticity of intertemporal substitution (EIS). Epstein–Zin utility (Epstein and Zin (1989)) generalizes CRRA utility and separates these properties while retaining homotheticity. Kikuchi and Kusuda (2023b) present "homothetic robust Epstein–Zin (HREZ) utility" and study the finite-time consumption-investment problem under the quadratic security market model of Batbold *et al.* (2022). Their research aims to quantitatively evaluate the market timing effects of the optimal robust portfolio. By contrast, to clarify the above theoretical issues, we examine the consumption-investment problem under the quadratic security market model, assuming infinitely lived investors with HREZ utility.

The main results of this study are summarized as follows. First, we introduce the optimal consumption-investment problem based on HREZ utility and derive the conditional worst-case probability for a given consumption and "investment," which is the product of the volatility matrix of risky securities and the vector of the fractions of wealth invested in those risky securities. Comparing the budget constraint equation under the conditional worst-case probability with the budget constraint equation under the base probability, we find that the volatility of wealth is invariant, while the market price of risk in the return on wealth is replaced by the "investor price of risk under the conditional worst-case probability" discounted from the market price of risk. Since the discount from the market price of risk here is permanent, this implies that investors with HREZ utility assume long-term stagnation rather than increased volatility as the worst-case scenario.

Second, we derive the optimal consumption and investment, both of which depend on the unknown function that comprises the indirect utility function, as well as the nonlinear PDE for the unknown function. We show that the optimal investment is a weighted average of the market price of risk and the "investor hedging value of intertemporal uncertainty." The weights are relative risk tolerance and one minus relative risk tolerance, respectively. We also show that the "investor price of risk under the worst-case probability" is a weighted average of the market price of risk and the investor hedging value of intertemporal uncertainty. The weights are the ratio of risk aver-

[3]With the exception of Kikuchi and Kusuda (2023a), these ignore some of the above stylized facts in the securities market.

sion to uncertainty aversion and that of ambiguity aversion to uncertainty aversion, respectively. Interestingly, the optimal investment and investor price of risk under the worst-case probability are both weighted averages of the market price of risk and the investor hedging value of intertemporal uncertainty.

Third, we derive robust versions of the ICAPM based on Epstein–Zin utility and of the two-factor CAPM (Duffie and Epstein (1992), Fisher and Gilles (1999)), which is a linear combination of the consumption-based CAPM and market portfolio-based CAPM. We also show that the market price of risk under the worst-case probability in equilibrium is consistent with the market price of risk based on Epstein–Zin utility.

Finally, we derive the exact solution of the PDE for the unit EIS case and a loglinear approximate solution of the PDE for the general case. We then present the approximate optimal portfolio and approximate CAPM based on the loglinear approximate solution.

The remainder of this paper is organized as follows. In Section 2, we review the quadratic security market model and real budget constraint. In Section 3, we introduce the control problem based on HREZ utility and derive the conditional worst-case probability. In Section 4, we theoretically analyze the optimal investment and investor price of risk under the worst-case probability. In Section 5, we derive two types of CAPMs. In Section 6, we derive the optimal portfolio for the unit EIS case and an approximate optimal portfolio for the general case. In Section 7, we address future research directions.

# 2 Quadratic Security Market Model and Real Budget Constraint

In this section, we review the quadratic security market model and real budget constraint according to Batbold *et al.* (2022).

## 2.1 Quadratic Security Market Model

We consider frictionless US markets over the period $[0, \infty)$. Investors' common subjective probability and information structure are modeled by a complete filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathrm{P})$, where $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,\infty)}$ is the natural filtration generated by an $N$-dimensional standard Brownian motion $B_t$. We denote the expectation operator under P by E and the conditional expectation operator given $\mathcal{F}_t$ by $\mathrm{E}_t$.

There are markets for a consumption commodity and securities at every date $t \in [0, \infty)$, and the consumer price index $p_t$ is observed. The traded securities are the instantaneously nominal risk-free security called the money market account and a continuum of zero-coupon bonds and zero-coupon inflation-indexed bonds whose maturity dates are $(t, t+\tau^*]$. Each zero-coupon bond has a 1 US dollar payoff at maturity and each zero-coupon inflation-indexed bond has a $p_T$ US dollar payoff at maturity $T$. Moreover, $J$ types of non-bond indices (*e.g.*, stock indices and REIT indices) are traded.

At every date $t$, $P_t$, $P_t^T$, $Q_t^T$, and $S_t^j$ denote the USD prices of the money market account, zero-coupon bond with maturity date $T$, zero-coupon inflation-indexed bond with maturity date $T$, and $j$-th index, respectively. Let $A'$ and $I$ denote the transpose of $A$ and $N \times N$ identity matrix, respectively.

We assume the following quadratic latent factor security market model.

**Assumption 1.** *Let $(\rho_0, \iota_0, \delta_{0j}, \sigma_{0j})$ and $(\lambda, \rho, \iota, \lambda_I, \delta_j, \sigma_j)$ denote the scalers and $N$-dimensional vectors, respectively.*

1. *The state vector process $X_t$ satisfies the following stochastic differential equation (SDE):*
$$dX_t = -\mathcal{K}X_t\,dt + I\,dB_t, \tag{1}$$
*where $\mathcal{K}$ is an $N \times N$ lower triangular matrix such that $\mathcal{K}+\mathcal{K}'$ is positive-definite.*

2. *The market price $\lambda_t$ of risk and instantaneous nominal risk-free rate $r_t$ are provided as*
$$\lambda_t = \lambda + \Lambda X_t, \tag{2}$$
$$r_t = \rho_0 + \rho' X_t + \frac{1}{2}X_t'\mathcal{R}X_t, \tag{3}$$
*where $\Lambda$ is an $N \times N$ matrix such that $\mathcal{K}+\Lambda$ is regular, $\mathcal{R}$ is a positive-definite symmetric matrix, and*[4]
$$\rho_0 \geq \frac{1}{2}\rho'\mathcal{R}^{-1}\rho. \tag{4}$$

3. *The consumer price $p_t$ satisfies*
$$\frac{dp_t}{p_t} = i_t\,dt + (\sigma_t^p)'dB_t, \qquad p_0 = 1, \tag{5}$$
*where $i_t$ and $\sigma_t^p$ are given by*
$$i_t = \iota_0 + \iota' X_t + \frac{1}{2}X_t'\mathcal{I}X_t, \tag{6}$$
$$\sigma_t^p = \sigma_p + \Sigma_p X_t. \tag{7}$$
*For eq.(6), $\mathcal{I}$ is a positive-definite symmetric matrix and a matrix $\bar{\mathcal{R}}$ defined by*
$$\bar{\mathcal{R}} = \mathcal{R} - \mathcal{I} + \Sigma_p'\Lambda + \Lambda'\Sigma_p \tag{8}$$
*is positive-definite.*

[4]Condition (4) ensures that the instantaneous nominal risk-free rate is non-negative.

4. *The dividend of the $j$-th index is given by*

$$D_t^j = \left(\delta_{0j} + \delta_j' X_t + \frac{1}{2} X_t' \Delta_j X_t\right) \exp\left(\sigma_{0j} t + \sigma_j' X_t + \frac{1}{2} X_t' \Sigma_j X_t\right), \tag{9}$$

*where $(\delta_{0j}, \delta_j, \Delta_j)$ is such that $\Delta_j$ is a positive-definite symmetric matrix and*[5]

$$\delta_{0j} \geq \frac{1}{2} \delta_j' \Delta_j^{-1} \delta_j. \tag{10}$$

*Note that $\delta_{0j} + \delta_j' X_t + \frac{1}{2} X_t' \Delta_j X_t$ is the instantaneous dividend.*

5. *Markets are complete and no-arbitrage.*

## 2.2 No-arbitrage Dynamics of Security Price Processes and Real Budget Constraint

We define the real market price $\bar{\lambda}_t$ of risk and real instantaneous interest rate $\bar{r}_t$ by

$$\begin{aligned} \bar{\lambda}_t &= \lambda_t - \sigma_t^p, & (11) \\ \bar{r}_t &= r_t - i_t + \lambda_t' \sigma_t^p. & (12) \end{aligned}$$

Note that the real market price of risk is an affine function of $X_t$, and $\bar{r}_t$ is a quadratic function of $X_t$:

$$\begin{aligned} \bar{\lambda}_t &= \bar{\lambda} + \bar{\Lambda} X_t, & (13) \\ \bar{r}_t &= \bar{\rho}_0 + \bar{\rho}' X_t + \frac{1}{2} X_t' \bar{\mathcal{R}} X_t, & (14) \end{aligned}$$

where $\bar{\mathcal{R}}$ is given by eq.(8) and

$$\begin{aligned} \bar{\lambda} &= \lambda - \sigma_p, & (15) \\ \bar{\Lambda} &= \Lambda - \Sigma_p, & (16) \\ \bar{\rho}_0 &= \rho_0 - \iota_0 + \lambda' \sigma^p, & (17) \\ \bar{\rho} &= \rho - \iota + \Lambda' \sigma_p + \Sigma_p' \lambda. & (18) \end{aligned}$$

Batbold *et al.* (2022) show the SDEs of no-arbitrage security price processes.

**Lemma 1.** *Let $\tau = T - t$ denote the time to maturity of bond $P_t^T$ or inflation-indexed bond $Q_t^T$. Under Assumption 1, the dynamics of security price processes satisfy the following:*

1. *The SDEs of security price processes:*

[5] Condition (10) ensures that dividends are non-negative processes.

*(i) The default-free bond with time $\tau$ to maturity:*

$$\frac{dP_t^T}{P_t^T} = \left(r_t + (\sigma(\tau) + \Sigma(\tau)X_t)'\lambda_t\right) dt + (\sigma(\tau)+\Sigma(\tau)X_t)' \, dB_t, \quad P_T^T = 1,$$

*where*

$$\frac{d\Sigma(\tau)}{d\tau} = \Sigma(\tau)^2 - (\mathcal{K}+\Lambda)'\Sigma(\tau) - \Sigma(\tau)(\mathcal{K}+\Lambda) - \mathcal{R}, \quad \Sigma(0) = 0,$$

$$\frac{d\sigma(\tau)}{d\tau} = -(\mathcal{K}+\Lambda-\Sigma(\tau))'\sigma(\tau) - (\Sigma(\tau)\lambda + \rho), \quad \sigma(0) = 0.$$

*(ii) The default-free inflation-indexed bond with time $\tau$ to maturity:*

$$\frac{dQ_t^T}{Q_t^T} = \left(r_t + \left(\sigma_Q(\tau) + \sigma_p + (\Sigma_Q(\tau) + \Sigma_p)X_t\right)'\lambda_t\right) dt + \left(\sigma_Q(\tau) + \sigma_p + (\Sigma_Q(\tau)+\Sigma_p)X_t\right)' dB_t, \quad Q_T^T = p_T,$$

*where*

$$\frac{d\Sigma_Q(\tau)}{d\tau} = \Sigma_Q(\tau)^2 - (\mathcal{K}+\bar{\Lambda})'\Sigma_Q(\tau) - \Sigma_Q(\tau)(\mathcal{K}+\bar{\Lambda}) - \bar{\mathcal{R}}, \quad \Sigma_Q(0) = 0,$$

$$\frac{d\sigma_Q(\tau)}{d\tau} = -(\mathcal{K}+\bar{\Lambda}-\Sigma_Q(\tau))'\sigma_Q(\tau) - (\Sigma_Q(\tau)\bar{\lambda} + \bar{\rho}), \quad \sigma_Q(0) = 0.$$

*(iii) The $j$-th index:*

$$\frac{dS_t^j + D_t^j dt}{S_t^j} = \left(r_t + (\sigma_j + \Sigma_j X_t)'\lambda_t\right) dt + (\sigma_j + \Sigma_j X_t)' \, dB_t,$$

*where*

$$\Sigma_j^2 - (\mathcal{K}+\Lambda)'\Sigma_j - \Sigma_j(\mathcal{K}+\Lambda) + \Delta_j - \mathcal{R}_j = 0,$$

$$\sigma_j = (\mathcal{K}+\Lambda-\Sigma_j)'^{-1}(\delta_j - \rho - \Sigma_j\lambda).$$

*Proof.* See Appendix A.1 in Batbold *et al.* (2022). □

## 2.3 Real Budget Constraint

We assume that the investor invests in $P_t(\tau_1), \cdots, P_t(\tau_J), Q_t(\tau_1^Q), \cdots, Q_t(\tau_K^Q)$, $S_t^1, \cdots, S_t^L$ where $J+K+L=N$. Let $\Phi(\tau)$ and $\varphi_t^Q(\tau^Q)$ denote the portfolio weight on the default-free bond with $\tau$-time to maturity and the default-free inflation-indexed bond with $\tau^Q$-time to maturity, respectively. Let $\Phi_t^l$ denote the portfolio weight on the $l$-th index. Let $\Phi_t$ and $\Sigma(X_t)$ denote the portfolio weight vector and volatility matrix, respectively. $\varPhi_t$ and $\Sigma(X_t)$ are expressed as

$$\varPhi_t = \begin{pmatrix} \varPhi_t(\tau_1) \\ \vdots \\ \varPhi_t(\tau_J) \\ \varPhi_t^Q(\tau_1^Q) \\ \vdots \\ \varPhi_t^Q(\tau_K^Q) \\ \varPhi_t^1 \\ \vdots \\ \varPhi_t^L \end{pmatrix}, \quad \Sigma(X_t) = \begin{pmatrix} (\sigma(\tau_1) + \Sigma(\tau_1)X_t)' \\ \vdots \\ (\sigma(\tau_J) + \Sigma(\tau_J)X_t)' \\ (\sigma_q(\tau_1^Q) + \Sigma_q(\tau_1^Q)X_t)' \\ \vdots \\ (\sigma_q(\tau_K^Q) + \Sigma_q(\tau_K^Q)X_t)' \\ (\sigma_1 + \Sigma_1 X_t)' \\ \vdots \\ (\sigma_L + \Sigma_L X_t)' \end{pmatrix}. \tag{19}$$

Let $c_t$ and $\bar{W}_t$ denote the consumption process and real wealth process, respectively. Batbold *et al.* (2022) show the real budget constraint.

**Lemma 2.** *The real budget constraint given* $(c_t, \bar{\varsigma}_t^W)$ *is expressed as*

$$\frac{d\bar{W}_t}{\bar{W}_t} = \left( \bar{r}_t + \bar{\varsigma}' \bar{\lambda}_t - \frac{c_t}{\bar{W}_t} \right) dt + \bar{\varsigma}'\, dB_t, \tag{20}$$

*where*

$$\bar{\varsigma}_t = \Sigma(X_t)' \varPhi_t - \sigma_t^p. \tag{21}$$

*Proof.* See Appendix A.2 in Batbold *et al.* (2022). □

**Remark 1.** *The real budget constraint stands for the instantaneous real rate of return on wealth. Eq.(20) shows that increasing the investment in the measure of* $\bar{\varsigma}_t$ *increases the volatility of wealth, while the real expected excess return on wealth increases in proportion to* $\bar{\varsigma}_t$*. Thus, the (real) market price* $\bar{\lambda}_t$ *of risk is interpreted as the price per unit of investment for all investors.*

We call $\bar{\varsigma}_t$ the investment control. Let $\mathbf{X}_t = (\bar{W}_t, X_t')'$ and let $\bar{W}_0 > 0$. We call the control satisfying budget constraint (20) with initial state $\mathbf{X}_0 = (\bar{W}_0, X_0')'$ the admissible control and denote the set of admissible controls by $\mathcal{B}(\mathbf{X}_0)$.

# 3 Robust Control Problem and Conditional Worst-case Probability

Following Kikuchi and Kusuda (2023b), we introduce the robust cosumption-investment problem based on HREZ utility. Then, we show the conditional worst-case probability for a given control.

### 3.1 HREZ Utility and Robust Control Problem

We begin with the following continuous-time version (Duffie and Epstein (1992)) of Epstein–Zin utility:

$$V_t = \mathrm{E}_t \left[ \int_t^{\infty} f(c_s, V_s) ds \right], \tag{22}$$

where $\tilde{f}$ denotes the normalized aggregator of the form:

$$f(c,v) = \begin{cases} \dfrac{\beta}{1-\psi^{-1}} c^{1-\psi^{-1}} \big((1-\gamma)v\big)^{1-\frac{1-\psi^{-1}}{1-\gamma}} - \dfrac{\beta(1-\gamma)}{1-\psi^{-1}} v, & \text{if} \quad \psi \neq 1, \\ \\ \beta(1-\gamma) v \log c - \beta v \log\big((1-\gamma)v\big), & \text{if} \quad \psi = 1, \end{cases} \tag{23}$$

where $\beta > 0$ is the subjective discount rate, $\gamma > 1$ is relative risk aversion, and $\psi > 0$ is the EIS.

While an investor with robust utility regards probability P ("base probability") as the most likely probability, they also consider other probabilities because the true probability is unknown. Thus, the investor assumes set $\mathbb{P}$ of all equivalent probability measures[6] as alternative probabilities. According to Girsanov's theorem, any equivalent probability measure is characterized by a measurable process $\xi_t$ with Novikov's integrable condition as the following Radon–Nikodým derivative:

$$\frac{d\mathrm{P}^{\xi}}{d\mathrm{P}} = \exp\bigg( \int_0^{\infty} \xi_t \, dB_t - \frac{1}{2} \int_0^{\infty} |\xi_t|^2 dt \bigg). \tag{24}$$

Therefore, the investor decides the worst-case probability, which minimizes their utility among $\mathbb{P}$ for every consumption plan.

**Definition 1.** *HREZ utility is defined by*

$$u(c) = \inf_{\mathrm{P}^{\xi} \in \mathbb{P}} \mathrm{E}^{\xi} \left[ \int_0^{\infty} \left( f(c_t, V_t) + \frac{(1-\gamma)V_t}{2\theta} |\xi_t|^2 \right) dt \right], \tag{25}$$

*where* $\mathrm{E}^{\xi}$ *is the expectation under* $\mathrm{P}^{\xi}$, $\theta$ *is relative ambiguity aversion, and* $V_t$ *is the utility process defined recursively as follows:*

$$V_t = \mathrm{E}_t^{\xi} \left[ \int_t^{\infty} \left( f(c_s, V_s) + \frac{(1-\gamma)V_s}{2\theta} |\xi_s|^2 \right) ds \right]. \tag{26}$$

**Assumption 2.** *An investor maximizes HREZ utility (25) under the real budget constraint (20).*

[6] A probability measure $\tilde{\mathrm{P}}$ is said to be an equivalent probability measure of P if and only if $\mathrm{P}(A) = 0 \Leftrightarrow \tilde{\mathrm{P}}(A) = 0$.

The investor's consumption-investment problem and indirect utility function are recursively defined by

$$V_t = \sup_{(c,\bar{\varsigma})\in\mathcal{B}(\mathbf{X}_t)} \inf_{\mathrm{P}^\xi\in\mathbb{P}} \mathrm{E}_t^\xi\left[\int_t^\infty \left(\bar{f}(c_s, V_s^\xi) + \frac{(1-\gamma)V_s^\xi}{2\theta}\,|\xi_s|^2\right) ds\right]. \tag{27}$$

## 3.2 Conditional Worst-case Probability

As the standard Brownian motion under $\mathrm{P}^\xi$ is given by $B_t^\xi = B_t - \displaystyle\int_0^t \xi_s\,ds$, the SDE for $\mathbf{X}_t$ under $\mathrm{P}^\xi$ is rewritten as

$$d = \left(\begin{pmatrix}\bar{W}_t(\bar{r}_t + \bar{\varsigma}_t'\bar{\lambda}_t) - c_t \\ -\mathcal{K}X_t\end{pmatrix} + \begin{pmatrix}\bar{W}_t\bar{\varsigma}_t' \\ I\end{pmatrix}\xi_t\right) dt + \begin{pmatrix}\bar{W}_t\bar{\varsigma}_t' \\ I\end{pmatrix} dB_t^\xi. \tag{28}$$

Let $J$ denote the indirect utility function. Then, the HJB equation for problem (27) is expressed as

$$\begin{aligned}\sup_{(c,\bar{\varsigma})\in\mathcal{B}(\mathbf{X}_0)} \inf_{\mathrm{P}^\xi\in\mathbb{P}} \Bigg\{ &\begin{pmatrix}\bar{W}_t\left(\bar{r}_t + \bar{\varsigma}_t'\bar{\lambda}_t\right) - c_t \\ -\mathcal{K}X_t\end{pmatrix}' \begin{pmatrix}J_W \\ J_X\end{pmatrix} + \xi_t' \begin{pmatrix}\bar{W}_t\bar{\varsigma}_t' \\ I\end{pmatrix}' \begin{pmatrix}J_W \\ J_X\end{pmatrix} \\ &+ \frac{1}{2}\,\mathrm{tr}\left[\begin{pmatrix}\bar{W}_t\bar{\varsigma}_t' \\ I\end{pmatrix}\begin{pmatrix}\bar{W}_t\bar{\varsigma}_t' \\ I\end{pmatrix}'\begin{pmatrix}J_{WW} & J_{WX} \\ J_{XW} & J_{XX}\end{pmatrix}\right] \\ &+ f(c_t, J) + \frac{(1-\gamma)J}{2\theta}|\xi_t|^2 \Bigg\} = 0. \end{aligned} \tag{29}$$

The conditional worst-case probability $\mathrm{P}^{\hat{\xi}}$ for a given control $(c_t, \bar{\varsigma}_t)$ satisfies

$$\hat{\xi}_t = -\frac{\theta}{(1-\gamma)J}\begin{pmatrix}\bar{W}_t\bar{\varsigma}_t' \\ I\end{pmatrix}'\begin{pmatrix}J_W \\ J_X\end{pmatrix}. \tag{30}$$

The real budget constraint eq.(20) under the conditional worst-case probability $\mathrm{P}^{\hat{\xi}}$ for the given control $(c_t, \bar{\varsigma}_t)$ is rewritten as

$$\frac{d\bar{W}_t}{\bar{W}_t} = \left\{\bar{r}_t + \bar{\varsigma}_t'\hat{\lambda}_t - \frac{c_t}{\bar{W}_t}\right\} dt + \bar{\varsigma}_t'\,dB_t^{\hat{\xi}}, \tag{31}$$

where

$$\hat{\lambda}_t = \bar{\lambda}_t + \hat{\xi}_t = \bar{\lambda}_t - \frac{\theta}{(1-\gamma)J}\begin{pmatrix}\bar{W}_t\bar{\varsigma}_t' \\ I\end{pmatrix}'\begin{pmatrix}J_W \\ J_X\end{pmatrix}. \tag{32}$$

**Remark 2.** *In eq.(31), the real market price $\bar{\lambda}_t$ of risk in the real budget constraint eq.(20) is replaced by $\hat{\lambda}$, which is the investor price per unit of investment under the conditional worst-case probability $\mathrm{P}^{\hat{\xi}}$ for a given control. As shown by Remark 1, when ambiguity is not considered, the price per* *unit of investment risk is the real market price $\bar{\lambda}_t$ of risk, which is common to all investors. By contrast, $\hat{\lambda}_t$ varies across investors. Eq.(31) shows that a more ambiguity averse investor values a lower price per unit of investment than the real market price of risk, under the conditional worst-case probability.*

Henceforth, we refer to the real market price of risk simply as the market price of risk.

**Remark 3.** *In eq.(31), under the conditional worst-case probability assumed by investors with HREZ utility, the investment control $\bar{\varsigma}_t$, which is the volatility of the wealth process, is as assumed under the base probability, but its price $\hat{\lambda}_t$ is permanently discounted from the market price of risk. This implies that investors with HREZ utility assume long-term stagnation rather than increased volatility as the worst-case scenario.*

Substituting $\mathrm{P}^{\hat{\xi}}$ into the HJB eq.(29) yields

$$\sup_{u \in \mathcal{B}(X_0)} \left[ \begin{pmatrix} \bar{W}_t \left( \bar{r}_t + \bar{\varsigma}_t' \bar{\lambda}_t \right) - c_t \\ -\mathcal{K} X_t \end{pmatrix}' \begin{pmatrix} J_W \\ J_X \end{pmatrix} + \frac{1}{2} \operatorname{tr} \left[ \begin{pmatrix} \bar{W}_t \bar{\varsigma}_t' \\ I \end{pmatrix} \begin{pmatrix} \bar{W}_t \bar{\varsigma}_t' \\ I \end{pmatrix}' \begin{pmatrix} J_{WW} & J_{WX} \\ J_{XW} & J_{XX} \end{pmatrix} \right] \right.$$
$$\left. + f(c_t, J) - \frac{\theta}{2(1-\gamma)J} \left| \begin{pmatrix} \bar{W}_t \bar{\varsigma}_t' \\ I \end{pmatrix}' \begin{pmatrix} J_W \\ J_X \end{pmatrix} \right|^2 \right] = 0. \quad (33)$$

Let $u^* = (c^*, \bar{\varsigma}^*)$ and $\bar{W}^*$ denote the optimal robust investment control and optimal wealth. Define the "worst-case probability" and "investor price of risk under the worst-case probability" by

$$\xi_t^* = -\frac{\theta}{(1-\gamma)J} \begin{pmatrix} \bar{W}_t^* (\bar{\varsigma}_t^*)' \\ I \end{pmatrix}' \begin{pmatrix} J_W \\ J_X \end{pmatrix}, \quad (34)$$

$$\bar{\lambda}_t^* = \bar{\lambda}_t - \theta \left( \frac{\bar{W}_t^* J_W}{(1-\gamma)J} \bar{\varsigma}_t^* + \frac{J_X}{(1-\gamma)J} \right). \quad (35)$$

# 4 Theoretical Analysis of the Optimal Robust Control

We theoretically analyze the optimal robust control and investor price of risk under the worst-case probability to solve the optimal consumption-investment problem.

## 4.1 A First Expression of the Optimal Robust Consumption-Investment

Let

$$\mathcal{U} = -\frac{\bar{W}_t J_{WW}}{J_W} + \theta \frac{\bar{W}_t J_W}{(1-\gamma)J}. \quad (36)$$

We obtain the following lemma.

**Lemma 3.** *Under Assumptions 1 and 2, the optimal control is given by*

$$c_t^* = \begin{cases} \beta J_W^{-1}(1-\gamma)J, & \text{if} \quad \psi = 1, \\ \beta^{\psi}\left(\dfrac{(1-\gamma)J}{\bar{W}_t^*}\right)^{-\psi}\big((1-\gamma)J\big)^{\frac{\gamma\psi-1}{\gamma-1}}, & \text{if} \quad \psi \neq 1, \end{cases} \tag{37}$$

$$\varsigma_t^* = \frac{1}{\mathcal{U}}\left(\bar{\lambda}_t + \frac{J_{XW}}{J_W} - \frac{\theta J_X}{(1-\gamma)J}\right), \tag{38}$$

*where $J$ is a solution of the following PDE:*

$$\begin{aligned} 0 = {} & \frac{1}{2}\mathrm{tr}\left[J_{XX}\right] - \frac{\theta}{2(1-\gamma)J}|J_X|^2 \\ & - \frac{1}{2}\left(\bar{W}_t^{*2}J_{WW} - \frac{\theta(W_t^* J_W)^2}{(1-\gamma)J}\right)^{-1}|\pi_t|^2 + \bar{r}_t\bar{W}_t^* J_W - (\mathcal{K}X_t)' J_X \\ & + \begin{cases} \beta\big\{(1-\gamma)(\log c_t^* - 1) - \log\big((1-\gamma)\big)\big\}J, & \text{if} \quad \psi = 1, \\ \dfrac{1}{\psi-1}c_t^* J_W - \dfrac{\beta(1-\gamma)}{1-\psi^{-1}}J, & \text{if} \quad \psi \neq 1, \end{cases} \end{aligned} \tag{39}$$

*where*

$$\pi_t = -\bar{W}_t J_W\left(\bar{\lambda}_t + \frac{J_{XW}}{J_W} - \frac{\theta J_X}{(1-\gamma)J}\right). \tag{40}$$

*Proof.* See Appendix A. □

It follows from eqs.(21) and (38) that the optimal robust portfolio $\varPhi_t^*$ satisfies

$$\Sigma(X_t)'\varPhi_t^* - \sigma_t^p = \frac{1}{\mathcal{U}}\left(\bar{\lambda}_t + \frac{J_{XW}}{J_W} - \theta\frac{J_X}{(1-\gamma)J}\right). \tag{41}$$

Thus, from eq.(11), the optimal robust portfolio is decomposed into the following four terms.

$$\begin{aligned} \varPhi_t^* = {} & \frac{1}{\mathcal{U}}\Sigma(X_t)'^{-1}\lambda_t + \frac{1}{\mathcal{U}}\Sigma(X_t)'^{-1}\frac{J_{XW}}{J_W} \\ & - \frac{1}{\mathcal{U}}\Sigma(X_t)'^{-1}\frac{\theta J_X}{(1-\gamma)J} + \left(1-\frac{1}{\mathcal{U}}\right)\Sigma(X_t)'^{-1}\sigma_t^p. \end{aligned} \tag{42}$$

The first term is myopic demand. The fourth term insures inflation-deflation risk. Following Kikuchi and Kusuda (2023a), we call the fourth term "inflation-deflation hedging demand," as presented by Brennan and Xia (2002), Sangvinatsos and Wachter (2005), Batbold *et al.* (2022), and Kikuchi and Kusuda (2023a).

**Remark 4.** *Since the PDE (39) depends not only on relative risk aversion but also on relative ambiguity aversion, the indirect utility function depends on both relative risk aversion and ambiguity aversion. Thus, the second and third terms in eq.(42) are related to the intertemporal uncertainty on marginal indirect utility and on indirect utility, respectively.*

From the PDE (39), we infer that the indirect utility function takes the form in eq.(43):

$$J(\mathbf{X}_t) = \begin{cases} \dfrac{\bar{W}_t^{1-\gamma}}{1-\gamma} G(X_t), & \text{if} \quad \psi = 1, \\ \\ \dfrac{\bar{W}_t^{1-\gamma}}{1-\gamma} \big(G(X_t)\big)^{\frac{1-\gamma}{\psi-1}}, & \text{if} \quad \psi \neq 1. \end{cases} \tag{43}$$

Thus, the partial derivatives of $J$ with respect to $\bar{W}$ are given by

$$\begin{aligned} \bar{W} J_W &= (1-\gamma) J, \\ \bar{W}^2 J_{WW} &= -\gamma(1-\gamma) J. \end{aligned} \tag{44}$$

**Remark 5.** *Eq.(32) is rewritten as follows:*

$$\hat{\lambda}_t = \bar{\lambda}_t - \theta \left( \frac{\bar{W}_t J_W}{(1-\gamma)J} \bar{\varsigma}_t + \frac{J_X}{(1-\gamma)J} \right). \tag{45}$$

*The second term represents the discount from the market price of risk to* $\hat{\lambda}_t$*. Eq.(45) shows that the discount is proportional to the investor's ambiguity aversion. Substituting eq.(44) into eq.(45) yields*

$$\hat{\lambda}_t = \bar{\lambda}_t - \theta \left( \bar{\varsigma}_t + \frac{J_X}{(1-\gamma)J} \right). \tag{46}$$

*Eq.(46) shows that the discount from the market price of risk to* $\hat{\lambda}_t$ *increases as the investment* $\bar{\varsigma}_t$ *increases. Since the discount from the market price of risk increases with relative ambiguity aversion and investment, these combined effects suppress the optimal investment of the ambiguity averse investor.*

Substituting eq.(44) into eq.(36), we obtain

$$\mathcal{U} = \gamma + \theta. \tag{47}$$

Batbold *et al.* (2022) call the sum of relative risk aversion and relative ambiguity aversion "relative uncertainty aversion." Since the reciprocal of relative risk aversion is called relative risk tolerance, they also call the reciprocal of relative uncertainty aversion "relative uncertainty tolerance."

**Remark 6.** *Eq.(47) shows that* $\mathcal{U}^{-1}$ *is relative uncertainty tolerance. Eq.(38) shows that the optimal investment is proportional to relative uncertainty tolerance.*

The partial derivatives of $J$ with respect to $X$ are given by

$$J_X = \begin{cases} J\dfrac{G_X}{G}, & \text{if} \quad \psi = 1, \\ \dfrac{1-\gamma}{\psi-1} J\dfrac{G_X}{G}, & \text{if} \quad \psi \neq 1, \end{cases}$$

$$\bar{W} J_{XW} = \begin{cases} (1-\gamma) J\dfrac{G_X}{G}, & \text{if} \quad \psi = 1, \\ \dfrac{(1-\gamma)^2}{\psi-1} J\dfrac{G_X}{G}, & \text{if} \quad \psi \neq 1, \end{cases} \tag{48}$$

$$J_{XX} = \begin{cases} J\left(\dfrac{G_X}{G}\dfrac{G'_X}{G} + \dfrac{G_{XX}}{G}\right), & \text{if} \quad \psi \neq 1, \\ \dfrac{1-\gamma}{\psi-1} J\left(\dfrac{2-\gamma-\psi}{\psi-1}\dfrac{G_X}{G}\dfrac{G'_X}{G} + \dfrac{G_{XX}}{G}\right), & \text{if} \quad \psi \neq 1. \end{cases}$$

Substituting the derivatives of $J$ into eq.(35), the investor price of risk under the worst-case probability is given by

$$\bar{\lambda}_t^* = \frac{\gamma}{\gamma+\theta}\bar{\lambda}_t + \frac{\theta}{\gamma+\theta}\eta_t^*, \tag{49}$$

where

$$\eta_t^* = \begin{cases} \dfrac{1}{\gamma-1}\dfrac{G_X}{G}, & \text{if} \quad \psi = 1, \\ \\ -\dfrac{1}{\psi-1}\dfrac{G_X}{G}, & \text{if} \quad \psi \neq 1. \end{cases} \tag{50}$$

We refer to $\eta_t^*$ as the "investor hedging value of intertemporal uncertainty."

**Remark 7.** *Eq.(49) shows that the investor price of risk under the worst-case probability is a weighted average of the market price of risk and investor hedging value of intertemporal uncertainty. The weights are the ratio of risk aversion to uncertainty aversion and that of ambiguity aversion to uncertainty aversion, respectively. The optimal investor price of uncertainty converges to the market price of risk in the case of Epstein–Zin utility* ($\theta \searrow 0$) *and to the investor hedging value of intertemporal uncertainty as relative ambiguity aversion diverges to infinity.*

### 4.2 A Second Expression of the Optimal Robust Control

We obtain the following proposition.

**Proposition 1.** *Under Assumptions 1 and 2, the optimal wealth, optimal consumption, and optimal investment for the problem (27) satisfy eqs.(51), (52), and (53), respectively:*

$$\bar{W}_t^* = \begin{cases} \bar{W}_0 \exp\left(\int_0^t \left(\bar{r}_s + (\bar{\varsigma}_s^*)'\bar{\lambda}_s - \frac{1}{2}|\bar{\varsigma}_s^*|^2 - \beta\right) ds + \int_0^t (\bar{\varsigma}_s^*)'\, dB_s\right), & \text{if} \quad \psi = 1, \\ \bar{W}_0 \exp\left(\int_0^t \left(\bar{r}_s + (\bar{\varsigma}_s^*)'\bar{\lambda}_s - \frac{1}{2}|\bar{\varsigma}_s^*|^2 - \frac{\beta^\psi}{G(X_t)}\right) ds + \int_0^t (\bar{\varsigma}_s^*)'\, dB_s\right), & \text{if} \quad \psi \neq 1, \end{cases} \tag{51}$$

$$c_t^* = \begin{cases} \beta \bar{W}_t^*, & \text{if} \quad \psi = 1, \\ \frac{\beta^\psi}{G(X_t)}\bar{W}_t^*, & \text{if} \quad \psi \neq 1, \end{cases} \tag{52}$$

$$\bar{\varsigma}_t^* = \frac{1}{\gamma+\theta}\bar{\lambda}_t + \left(1 - \frac{1}{\gamma+\theta}\right)\eta_t^*, \tag{53}$$

*where $G(X_t)$ is a solution of the following PDE:*

1. *The unit EIS case:*

$$\frac{1}{2}\text{tr}\left[\frac{G_{XX}}{G}\right] + \frac{\theta}{2(\gamma-1)(\gamma+\theta)}\left|\frac{G_X}{G}\right|^2 - \left(\mathcal{K}X_t + \frac{\gamma+\theta-1}{\gamma+\theta}\bar{\lambda}_t\right)'\frac{G_X}{G} - \beta\log G - \left(\frac{\gamma-1}{2(\gamma+\theta)}|\bar{\lambda}_t|^2 + (\gamma-1)\bar{r}_t + \beta(\log\beta - 1)(\gamma-1)\right) = 0. \tag{54}$$

2. *The general case:*

$$\frac{1}{2}\,\text{tr}\left[\frac{G_{XX}}{G}\right] - \frac{\psi-(\gamma+\theta)^{-1}}{2(\psi-1)}\left|\frac{G_X}{G}\right|^2 - \left(\mathcal{K}X_t + \left(1-(\gamma+\theta)^{-1}\right)\bar{\lambda}_t\right)'\frac{G_X}{G} + \frac{\beta^\psi}{G} + \left(\frac{(\psi-1)(\gamma+\theta)^{-1}}{2}|\bar{\lambda}_t|^2 + (\psi-1)\,\bar{r}_t - \beta\psi\right) = 0. \tag{55}$$

*Proof.* See Appendix B. □

**Remark 8.** *Eq.(53) shows that the optimal investment is a weighted average of the market price of risk and investor hedging value of intertemporal uncertainty. The weights are relative risk tolerance and one minus relative risk tolerance, respectively. Interestingly, the optimal investment and investor price of risk under the worst-case probability are both weighted averages of the market price of risk and investor hedging value of intertemporal uncertainty.*

We obtain the optimal robust portfolio from eq.(21). The optimal robust portfolio is given by

$$\Phi_t^* = \frac{1}{\gamma+\theta}\Sigma(X_t)'^{-1}\lambda_t + \left(1 - \frac{1}{\gamma+\theta}\right)\Sigma(X_t)'^{-1}\eta_t^* + \left(1 - \frac{1}{\gamma+\theta}\right)\Sigma(X_t)'^{-1}\sigma_t^p. \quad (56)$$

**Remark 9.** *The second and third terms related to intertemporal uncertainty in eq.(42) are integrated into the second term in eq.(56). Hereafter, we refer to the second term in eq.(56) as "intertemporal uncertainty hedging demand."*

# 5 CAPMs

In this section, we assume the general case ($\psi \neq 1$) and derive the CAPMs. Here, the consumption-investment problem (27) for investors is replaced by that for the representative agent. Then, the optimal consumption becomes the aggregated consumption and the optimal portfolio becomes the market portfolio. In addition, the investor price of risk under the worst-case probability becomes the market price of risk under the worst-case probability, which represents the price of risk for the representative agent under the worst-case probability.

Let $\bar{\sigma}_t^c$ and $\sigma_t^M$ denote the volatility of aggregated consumption and of the market portfolio, respectively. Let $\bar{\sigma}_t^M = \sigma_t^M - \sigma_t^p$. Note that $\bar{\sigma}_t^M = \bar{\varsigma}_t^*$. We obtain the following proposition.

**Proposition 2.** *In addition to Assumption 1, assume that the representative agent's problem is given by (27). Then, in equilibrium, the market price of risk is expressed as eqs.(57) and (58):*

$$\begin{aligned} \bar{\lambda}_t &= (\gamma+\theta)\bar{\varsigma}_t^* + \big(1-(\gamma+\theta)\big)\left(-\frac{1}{\psi-1}\frac{G_X}{G}\right), && (57) \\ &= -\frac{\gamma+\theta-1}{\psi-1}\bar{\sigma}_t^c + \frac{\psi(\gamma+\theta)-1}{\psi-1}\bar{\sigma}_t^M. && (58) \end{aligned}$$

*The market price of risk under the worst-case probability is expressed as*

$$\begin{aligned} \bar{\lambda}_t^* &= \gamma\bar{\varsigma}_t^* + (1-\gamma)\left(-\frac{1}{\psi-1}\frac{G_X}{G}\right), && (59) \\ &= -\frac{\gamma-1}{\psi-1}\bar{\sigma}_t^c + \frac{\gamma\psi-1}{\psi-1}\bar{\sigma}_t^M. && (60) \end{aligned}$$

*Proof.* Eq.(57) immediately follows from eq.(53). The derivative of eq.(52) leads to

$$\frac{G_X}{G} = \bar{\varsigma}_t^* - \bar{\sigma}_t^c. \quad (61)$$

Inserting the above equation and $\bar{\varsigma}_t^* = \bar{\sigma}_t^M$ into eq.(57), we obtain eq.(58). Substituting eq.(57) into eq.(49) yields (59). Substituting eq.(61) and $\bar{\varsigma}_t^* = \bar{\sigma}_t^M$ into eqs.(57) and (59) yields eqs.(58) and (60), respectively. □

**Remark 10.** *In the case of Epstein–Zin utility ($\theta = 0$), eqs.(57) and (58) are simplified to*

$$\begin{aligned} \bar{\lambda}_t &= \gamma\bar{\varsigma}_t^* + (1-\gamma)\left(-\frac{1}{\psi-1}\frac{G_X}{G}\right), && (62) \\ &= -\frac{\gamma-1}{\psi-1}\bar{\sigma}_t^c + \frac{\gamma\psi-1}{\psi-1}\bar{\sigma}_t^M. && (63) \end{aligned}$$

*Eq.(62) is the ICAPM based on Epstein–Zin utility, and eq.(63) is the two-factor CAPM shown by Duffie and Epstein (1992) and Fisher and Gilles (1999), which is a linear combination of the consumption-based CAPM and market portfolio-based CAPM. Eqs.(57) and .(58) are robust versions of the ICAPM (62) and of the two-factor CAPM. Note that eqs. (59) and (60), which represent the market price of risk under the worst-case probability based on HREZ utility, are identical to eqs.(62) and (63), which represent the market price of risk based on Epstein–Zin utility.*

As noted in the Introduction, we assume a quadratic security market model to incorporate the stylized facts of the securities market. Estimating the quadratic security market model requires at least monthly data; the consumption-based CAPM (58) is difficult to test because reliable monthly aggregate consumption data are unavailable. The ICAPM (57) is untestable because it contains the unknown function $G$. In the next section, we derive an approximate solution to the PDE (55) and present a testable ICAPM.

# 6 Analytical Expression of the Optimal Robust Consumption-Investment

First, for the unit EIS case, that is, $\psi = 1$, we derive the optimal solution. Second, for the general case, we derive an approximate optimal solution. Finally, we derive a testable CAPM.

## 6.1 Optimal Solution for the Unit EIS Case

An analytical solution of the PDE (54) is expressed as:

$$G(X_t) = \exp\left(a_0 + a'X_t + \frac{1}{2}X_t'AX_t\right), \tag{64}$$

where $A$ is a symmetric matrix.

Substituting $G$ and its derivatives into the PDE (54) and noting $A' = A$ and $X'\big(\mathcal{K} + \big(1 - (\gamma+\theta)^{-1}\big)\bar{\Lambda}\big)' AX = X'A\big(\mathcal{K} + \big(1 - (\gamma+\theta)^{-1}\big)\bar{\Lambda}\big)X$, we obtain

$$\begin{aligned}
&\frac{1}{2}\mathrm{tr}[A] + \frac{1}{2}\left(1 + \frac{\theta}{(\gamma-1)(\gamma+\theta)}\right)\left(|a|^2 + 2a'AX_t + X_t'A^2X_t\right) \\
&\quad - \left\{\frac{\gamma+\theta-1}{\gamma+\theta}\bar{\lambda} + \left(\mathcal{K} + \frac{\gamma+\theta-1}{\gamma+\theta}\bar{\Lambda}\right)X_t\right\}' a - \frac{\gamma+\theta-1}{\gamma+\theta}\bar{\lambda}'AX_t \\
&\quad - \frac{1}{2}X_t'\left(\mathcal{K} + \frac{\gamma+\theta-1}{\gamma+\theta}\bar{\Lambda}\right)' AX_t - \frac{1}{2}X_t'A\left(\mathcal{K} + \frac{\gamma+\theta-1}{\gamma+\theta}\bar{\Lambda}\right)X_t \\
&\quad - \beta\left(a_0 + a'X_t + \frac{1}{2}X_t'AX_t\right) - \frac{\gamma-1}{2(\gamma+\theta)}\left(|\bar{\lambda}|^2 + 2\bar{\lambda}'\bar{\Lambda}X_t + X_t'\bar{\Lambda}'\bar{\Lambda}X_t\right) \\
&\quad - (\gamma-1)\left(\bar{\rho}_0 + \bar{\rho}'X_t + \frac{1}{2}X_t'\bar{\mathcal{R}}X_t\right) - \beta(\log\beta - 1)(\gamma-1) = 0. \quad (65)
\end{aligned}$$

As eq.(65) is identical on $X$, we obtain the following proposition.

**Theorem 1.** *Under Assumptions 1 and 2, the indirect utility function, optimal consumption, and optimal investment for problem (27) satisfy eqs.(66), (52), and (67), respectively:*

$$\begin{aligned}
J(\mathbf{X}_t) &= \frac{\bar{W}_t^{1-\gamma}}{1-\gamma}\exp\left(a_0 + a'X_t + \frac{1}{2}X_t'AX_t\right), \quad (66) \\
\varsigma_t^* &= \frac{1}{\gamma+\theta}\left(\bar{\lambda} + \bar{\Lambda}X_t\right) + \left(1 - \frac{1}{\gamma+\theta}\right)\left(\frac{1}{\gamma-1}(a + AX_t)\right), \quad (67)
\end{aligned}$$

*where $(A, a, a_0)$ is a solution of the simultaneous eqs.(68)–(70):*

$$\begin{aligned}
&\frac{\gamma(\gamma+\theta-1)}{(\gamma-1)(\gamma+\theta)}A^2 - \left(\mathcal{K} + \frac{\gamma+\theta-1}{\gamma+\theta}\bar{\Lambda}\right)' A - A\left(\mathcal{K} + \frac{\gamma+\theta-1}{\gamma+\theta}\bar{\Lambda}\right) \\
&\qquad - (\gamma-1)\left(\frac{1}{\gamma+\theta}\bar{\Lambda}'\bar{\Lambda} + \bar{\mathcal{R}}\right) = 0, \quad (68)
\end{aligned}$$

$$\begin{aligned}
&\left(\frac{\gamma(\gamma+\theta-1)}{(\gamma-1)(\gamma+\theta)}A - \left(\mathcal{K} + \frac{\gamma+\theta-1}{\gamma+\theta}\bar{\Lambda}\right)\right)' a \\
&\qquad - \left(\frac{\gamma+\theta-1}{\gamma+\theta}A\bar{\lambda} + \frac{\gamma-1}{\gamma+\theta}\bar{\Lambda}'\bar{\lambda} + (\gamma-1)\bar{\rho}\right) = 0, \quad (69)
\end{aligned}$$

$$\begin{aligned}
\beta a_0 &= \frac{1}{2}\,\mathrm{tr}[A] + \frac{\gamma(\gamma+\theta-1)}{2(\gamma-1)(\gamma+\theta)}|a|^2 - \frac{\gamma+\theta-1}{\gamma+\theta}\bar{\lambda}'a \\
&\qquad - \frac{\gamma-1}{2\gamma(\gamma+\theta)}|\bar{\lambda}|^2 - \frac{\gamma-1}{\gamma}\bar{\rho}_0 - \frac{\beta(\gamma-1)}{\gamma}(\log\beta - 1). \quad (70)
\end{aligned}$$

**Remark 11.** *Eq.(52) indicates that the optimal consumption-wealth ratio is constant and independent of the state process, which is unrealistic. However, this result implies that the optimal consumption-wealth ratio is stable if the EIS does not deviate from one and the state process does not deviate significantly from zero. In the next section, we consider the general case, in which a nonhomogeneous term appears in the PDE for indirect utility. We then use a loglinear approximation based on the stability of the optimal consumption-wealth ratio to derive an approximate solution.*

## 6.2 Approximate Optimal Solution for the General Case

Next, for the general case, that is, $\psi \neq 1$, we derive an approximate optimal solution by applying the loglinear approximation method presented by Campbell and Viceira (2002) to our quadratic security market model.

### 6.2.1 Loglinear Approximation

In the PDE (55), both the nonlinear term and the nonhomogeneous term appear. From eq.(52), the nonhomogeneous term $\dfrac{\beta^{\psi}}{G}$ is expressed as

$$\frac{\beta^{\psi}}{G} = \frac{c_t^*}{\bar{W}_t^*}. \tag{71}$$

Considering that the optimal consumption-wealth ratio is stable, Campbell and Viceira (2002) make a loglinear approximation to the nonhomogeneous term and derive an approximate analytical solution. We apply the loglinear approximation to the nonhomogeneous term as follows:

$$\frac{1}{G(X_t)} \approx g_0 - g_1 \log G(X_t), \tag{72}$$

where

$$g_0 = g_1(1 - \log g_1), \tag{73}$$

$$g_1 = \exp\Big(\mathrm{E}\Big[\log\Big(\lim_{t\to\infty} \frac{c_t^*}{W_t^*}\Big)\Big] - \psi \log \beta\Big). \tag{74}$$

In PDE (55), approximating the nonhomogeneous term by eq.(72) and inserting eqs.(11) and (12) into $\bar{\Lambda}_t$ and $\bar{r}_t$, respectively, yields the following approximate PDE:

$$
\begin{aligned}
&\frac{1}{2}\,\mathrm{tr}\left[\frac{G_{XX}}{G}\right] - \frac{\psi-(\gamma+\theta)^{-1}}{2(\psi-1)}\left|\frac{G_X}{G}\right|^2 \\
&\qquad - \Big(\mathcal{K}X_t + \big(1-(\gamma+\theta)^{-1}\big)\big(\bar\lambda+\bar\Lambda X_t\big)\Big)'\frac{G_X}{G} - \beta^{\psi} g_1 \log G \\
&+\beta^{\psi} g_0 + \frac{(\psi-1)(\gamma+\theta)^{-1}}{2}\left|\bar\lambda+\bar\Lambda X_t\right|^2 + (\psi-1)\left(\bar\rho_0 + \bar\rho' X_t + \frac{1}{2}X_t'\bar{\mathcal{R}}X_t\right) - \beta\psi = 0.
\end{aligned} \tag{75}
$$

An analytical solution of the PDE (75) is expressed as eq.(64). Substituting the optimal consumption control (52) into eq.(74), we obtain

$$
g_1 = \exp\left(-\mathrm{E}\left[\lim_{t\to\infty}\log G(X_t)\right]\right) = \exp\left(\left[-a_0 - a'\mathrm{E}[\lim_{t\to\infty} X_t] - \frac{1}{2}\mathrm{E}[\lim_{t\to\infty} X_t' A X_t]\right]\right). \tag{76}
$$

Since eq.(1) is transformed as $d\big(e^{t\mathcal{K}}X_t\big) = e^{t\mathcal{K}}\,dB_t$, $X_t$ is solved as

$$
X_t = e^{-t\mathcal{K}}X_0 + \int_0^t e^{(s-t)\mathcal{K}}\,dB_s. \tag{77}
$$

Hence, the stationary distribution of the state vector is $N(0,(\mathcal{K}+\mathcal{K}')^{-1})$. Thus, $g_1$ in eq.(76) is calculated as

$$
g_1 = \exp\left(-a_0 - \frac{1}{2}\mathrm{tr}\big[\big(\mathcal{K}+\mathcal{K}'\big)^{-1}A\big]\right). \tag{78}
$$

### 6.2.2 Approximate Optimal Solution

Substituting $G$ and its derivatives into the PDE (75) yields

$$
\begin{aligned}
&\frac{1}{2}\,\mathrm{tr}\left[aa' + A + aX_t'A + AX_ta' + AX_tX_t'A\right] - \frac{\psi-(\gamma+\theta)^{-1}}{2(\psi-1)}\left(a' + X_t'A\right)\left(a + AX_t\right) \\
&-\left\{\big(1-(\gamma+\theta)^{-1}\big)\bar\lambda + \left(\mathcal{K} + \big(1-(\gamma+\theta)^{-1}\big)\bar\Lambda\right)X_t\right\}' a - \big(1-(\gamma+\theta)^{-1}\big)\bar\lambda' A X_t \\
&\quad - \frac{1}{2}X_t'\left(\mathcal{K} + \big(1-(\gamma+\theta)^{-1}\big)\bar\Lambda\right)' AX_t - \frac{1}{2}X_t'A\left(\mathcal{K} + \big(1-(\gamma+\theta)^{-1}\big)\bar\Lambda\right)X_t \\
&\qquad\qquad - \beta^{\psi} g_1\left(\log g_1 - 1 + a_0 + a'X_t + \frac{1}{2}X_t'AX_t\right) \\
&\qquad\qquad + \frac{(\psi-1)(\gamma+\theta)^{-1}}{2}\left(|\bar\lambda|^2 + 2\bar\lambda'\bar\Lambda X_t + X_t'\bar\Lambda'\bar\Lambda X_t\right) \\
&\qquad\qquad\qquad + (\psi-1)\left(\bar\rho_0 + \bar\rho' X_t + \frac{1}{2}X_t'\bar{\mathcal{R}}X_t\right) - \beta\psi = 0.
\end{aligned} \tag{79}
$$

When the solution of the PDE (55) is approximated by the solution of the approximate PDE (75), the optimal control is called the approximate optimal control, denoted by $(\tilde c^*, \tilde\varsigma^*)$. We obtain the following proposition.

**Theorem 2.** *Under Assumptions 1 and 2, the approximate optimal consumption and optimal approximate investment for problem (27) satisfy eqs.(80) and (81), respectively:*

$$\tilde{c}_t^* = \tilde{W}_t^* \exp\left[-\left(a_0 + a'X_t + \frac{1}{2}X_t'AX_t\right)\right], \tag{80}$$

$$\tilde{\varsigma}_t^* = \frac{1}{\gamma+\theta}(\bar{\lambda} + \bar{\Lambda}X_t) + \left(1 - \frac{1}{\gamma+\theta}\right)\left(-\frac{1}{\psi-1}(a + AX_t)\right), \tag{81}$$

*where $(A, a, a_0)$ is a solution of the simultaneous equations (82)–(84):*

$$-\frac{1-(\gamma+\theta)^{-1}}{\psi-1}A^2 - \big(\mathcal{K} + \big(1-(\gamma+\theta)^{-1}\big)\bar{\Lambda}\big)' A - A\left(\mathcal{K} + \big(1-(\gamma+\theta)^{-1}\big)\bar{\Lambda}\right) - \beta^{\psi}g_1 A + (\psi-1)\Big((\gamma+\theta)^{-1}\bar{\Lambda}'\bar{\Lambda} + \bar{\mathcal{R}}\Big) = 0, \tag{82}$$

$$-\frac{1-(\gamma+\theta)^{-1}}{\psi-1}Aa - \mathcal{K}'a - \big(1-(\gamma+\theta)^{-1}\big)(A\bar{\lambda} + \bar{\Lambda}'a) - \beta^{\psi}g_1 a + (\psi-1)\Big((\gamma+\theta)^{-1}\bar{\Lambda}'\bar{\lambda} + \bar{\rho}\Big) = 0, \tag{83}$$

$$\frac{1}{2}\,\mathrm{tr}[A] - \frac{1-(\gamma+\theta)^{-1}}{2(\psi-1)}|a|^2 - \big(1-(\gamma+\theta)^{-1}\big)\bar{\lambda}'a + \beta^{\psi}g_1(1 - a_0 - \log g_1) + (\psi-1)\left(\frac{(\gamma+\theta)^{-1}}{2}|\bar{\lambda}|^2 + \bar{\rho}_0\right) - \beta\psi = 0, \tag{84}$$

*where $g_1$ is expressed by eq.(78). Furthermore, the approximate optimal portfolio $\tilde{\Phi}_t^*$ is given by*

$$\tilde{\Phi}_t^* = \frac{1}{\gamma+\theta}\Sigma(X_t)'^{-1}\Big(\lambda + \Lambda X_t\Big) + \left(1-\frac{1}{\gamma+\theta}\right)\Sigma(X_t)'^{-1}\left(-\frac{1}{\psi-1}\right)\Big(a + AX_t\Big) + \left(1-\frac{1}{\gamma+\theta}\right)\Sigma(X_t)'^{-1}\Big(\sigma_p + \Sigma_p X_t\Big). \tag{85}$$

#### 6.2.3 Testable ICAPM

It follows from eq.(81) and $\bar{\varsigma}_t^* = \bar{\sigma}_t^M$ that a testable ICAPM is given by

$$\bar{\lambda}_t = (\gamma+\theta)\bar{\sigma}_t^M + \big(1-(\gamma+\theta)\big)\left(-\frac{1}{\psi-1}(a + AX_t)\right). \tag{86}$$

# 7 Future Research Directions

We studied the consumption-investment problem for infinitely lived investors with HREZ utility under the quadratic security market model. We clarified the theoretical structures of i) the budget constraint and market price of risk under the conditional worst-case probability; ii) the market price of risk under the worst-case probability; iii) the CAPMs; and iv) the CAPMs under the worst-case probability.

The next step is an empirical analysis of the derived approximate optimal portfolio and testable CAPM. However, since our quadratic security market model omits the foreign sector, these are only the optimal domestic portfolio and one-country CAPM, respectively, which limits these empirical analyses. Nevertheless, the optimal domestic portfolio can be interpreted as the optimal portfolio of an investor who invests only in domestic securities, which is meaningful for the empirical analysis[7], but the one-country CAPM is unlikely to be accepted. Our future research will aim to construct an international security market model, derive the optimal international portfolio and testable international CAPM, and conduct empirical analyses.

# A Proof of Lemma 3

## A.1 Proof for the Unit EIS Case

Substituting $f(c_t, J) = \beta(1-\gamma)J\log c_t - \beta v \log\big((1-\gamma)J\big)$ into the HJB eq.(33) yields

$$\begin{aligned}
&\sup_{(c,\varsigma)\in\mathcal{B}(X_0)} \left[ \begin{pmatrix} \bar{W}_t\left(\bar{r}_t + \varsigma_t'\bar{\lambda}_t\right) - c_t \\ -\mathcal{K}X_t \end{pmatrix}' \begin{pmatrix} J_W \\ J_X \end{pmatrix} \right.\\
&+ \frac{1}{2}\operatorname{tr}\left[ \begin{pmatrix} \bar{W}_t\varsigma_t' \\ I \end{pmatrix} \begin{pmatrix} \bar{W}_t\varsigma_t' \\ I \end{pmatrix}' \begin{pmatrix} J_{WW} & J_{WX} \\ J_{XW} & J_{XX} \end{pmatrix} \right] - \frac{\theta}{2(1-\gamma)J} \left| \begin{pmatrix} \bar{W}_t\varsigma_t' \\ I \end{pmatrix}' \begin{pmatrix} J_W \\ J_X \end{pmatrix} \right|^2 \\
&\left. + \beta(1-\gamma)J\log c_t - \beta J \log\big((1-\gamma)J\big) \right] = 0. \qquad (87)
\end{aligned}$$

It is evident that the optimal control $(c_t^*, \varsigma_t^*)$ in the HJB eq.(87) satisfies eqs.(37) and (38). The consumption-related terms in the HJB eq.(87) are computed as

$$-c_t^* J_W + \beta(1-\gamma)J\log c_t^* - \beta J \log\big((1-\gamma)J\big) = \beta J\big\{(1-\gamma)(\log c_t^* - 1) - \log\big((1-\gamma)J\big)\big\}. \qquad (88)$$

[7]See Kikuchi and Kusuda (2023b) for an empirical analysis of optimal domestic portfolios under the quadratic security market model based on HREZ utility.

The investment-related terms in the HJB eq.(87) are computed as

$$\bar{W}_t^* J_W \bar{\lambda}_t' \bar{\varsigma}_t^* + \frac{1}{2}\operatorname{tr}\left[\begin{pmatrix}\bar{W}_t^*(\bar{\varsigma}_t^*)'\\ I\end{pmatrix}\begin{pmatrix}\bar{W}_t^*(\bar{\varsigma}_t^*)'\\ I\end{pmatrix}'\begin{pmatrix}J_{WW} & J_{WX}\\ J_{XW} & J_{XX}\end{pmatrix}\right]$$
$$-\frac{\theta}{2(1-\gamma)J}\left|\begin{pmatrix}\bar{W}_t^*(\bar{\varsigma}_t^*)'\\ I\end{pmatrix}'\begin{pmatrix}J_W\\ J_X\end{pmatrix}\right|^2$$
$$=\frac{1}{2}\operatorname{tr}[J_{XX}] - \frac{\theta}{2(1-\gamma)J}|J_X|^2 - \frac{|\pi_t|^2}{2\bar{W}_t^{*2}\left(J_{WW} - \frac{\theta J_W^2}{(1-\gamma)J}\right)}, \quad (89)$$

where $\pi_t$ is given by eq.(3).

Substituting the optimal control (37) and (38) into the HJB eq.(87) and using eqs.(88) and (89), we obtain the PDE (39).

## A.2 Proof for the General Case

Substituting $f(c_t, J) = \frac{\beta}{1-\psi^{-1}} c_t^{1-\psi^{-1}} \left((1-\gamma)J\right)^{1-\frac{1-\psi^{-1}}{1-\gamma}} - \frac{\beta(1-\gamma)}{1-\psi^{-1}} J$ into the HJB eq.(33) yields

$$\sup_{(c,\bar{\varsigma})\in\mathcal{B}(X_0)}\left[\begin{pmatrix}\bar{W}_t\left(\bar{r}_t + \bar{\varsigma}_t'\bar{\lambda}_t\right) - c_t\\ -\mathcal{K}X_t\end{pmatrix}'\begin{pmatrix}J_W\\ J_X\end{pmatrix}\right.$$
$$+\frac{1}{2}\operatorname{tr}\left[\begin{pmatrix}\bar{W}_t\bar{\varsigma}_t'\\ I\end{pmatrix}\begin{pmatrix}\bar{W}_t\bar{\varsigma}_t'\\ I\end{pmatrix}'\begin{pmatrix}J_{WW} & J_{WX}\\ J_{XW} & J_{XX}\end{pmatrix}\right] - \frac{\theta}{2(1-\gamma)J}\left|\begin{pmatrix}\bar{W}_t\bar{\varsigma}_t'\\ I\end{pmatrix}'\begin{pmatrix}J_W\\ J_X\end{pmatrix}\right|^2$$
$$\left.+\frac{\beta}{1-\psi^{-1}} c_t^{1-\psi^{-1}}\left((1-\gamma)J\right)^{1-\frac{1-\psi^{-1}}{1-\gamma}} - \frac{\beta(1-\gamma)}{1-\psi^{-1}}J\right] = 0. \quad (90)$$

The optimal control $(c_t^*, \bar{\varsigma}_t^*)$ in the HJB eq.(90) satisfies eqs.(37) and (38). The consumption-related terms in the HJB eq.(90) are computed as

$$-c_t^* J_W + f(c_t^*, J) = c_t^*\left(-J_W + \frac{1}{1-\psi^{-1}}J_W\right) - \frac{\beta(1-\gamma)}{1-\psi^{-1}}J = \frac{1}{\psi-1}c_t^* J_W - \frac{\beta(1-\gamma)}{1-\psi^{-1}}J. \quad (91)$$

The investment-related terms in the HJB eq.(90) are computed as

$$\bar{W}_t^* J_W \bar{\lambda}_t' \bar{\varsigma}_t^* + \frac{1}{2}\operatorname{tr}\left[\begin{pmatrix}\bar{W}_t^*(\bar{\varsigma}_t^*)'\\ I\end{pmatrix}\begin{pmatrix}\bar{W}_t^*(\bar{\varsigma}_t^*)'\\ I\end{pmatrix}'\begin{pmatrix}J_{WW} & J_{WX}\\ J_{XW} & J_{XX}\end{pmatrix}\right]$$
$$-\frac{\theta}{2(1-\gamma)J}\left|\begin{pmatrix}\bar{W}_t\bar{\varsigma}_t'\\ I\end{pmatrix}'\begin{pmatrix}J_W\\ J_X\end{pmatrix}\right|^2$$
$$=\frac{1}{2}\operatorname{tr}[J_{XX}] - \frac{\theta}{2(1-\gamma)J}|J_X|^2 - \left(\bar{W}_t^{*2}J_{WW} - \frac{\theta(W_t^* J_W)^2}{(1-\gamma)J}\right)^{-1}|\pi_t|^2, \quad (92)$$

where $\pi_t$ is given by eq.(3).

Substituting the optimal control (37) and (38) into the HJB eq.(90) and using eqs.(91) and (92), we obtain the PDE (39).

# B Proof of Proposition 1

## B.1 Proof for the Unit EIS Case

The optimal consumption (52) immediately follows from eq.(37). Eq.(3) is rewritten as

$$\pi_t = J\left((\gamma-1)\bar{\lambda}_t + (\gamma+\theta-1)\frac{G_X}{G}\right). \tag{93}$$

Inserting eqs.(47) and the derivatives of $J$ into eq.(38), we obtain the optimal investment (53). The first to third terms in the PDE (39) are calculated from eq.(93) and the derivatives of $J$ as follows:

$$\begin{aligned}
&\frac{1}{2}\operatorname{tr}[J_{XX}] - \frac{\theta}{2(1-\gamma)J}|J_X|^2 - \frac{|\pi_t|^2}{2\bar{W}_t^{*2}\left(J_{WW} - \frac{\theta J_W^2}{(1-\gamma)J}\right)} \\
=&\frac{1}{2}J\left(\operatorname{tr}\left[\frac{G_X}{G}\frac{G_X'}{G} + \frac{G_{XX}}{G}\right] + \frac{\theta}{\gamma-1}\left|\frac{G_X}{G}\right|^2 - \frac{1}{(\gamma-1)(\gamma+\theta)}\left|(\gamma-1)\bar{\lambda}_t + (\gamma+\theta-1)\frac{G_X}{G}\right|^2\right) \\
=&\frac{1}{2}J\left(\operatorname{tr}\left[\frac{G_{XX}}{G}\right] - \frac{\gamma-1}{\gamma+\theta}|\bar{\lambda}_t|^2 - \frac{2(\gamma+\theta-1)}{\gamma+\theta}\bar{\lambda}_t'\frac{G_X}{G} + \left(1 + \frac{\theta}{\gamma-1} - \frac{(\gamma+\theta-1)^2}{(\gamma-1)(\gamma+\theta)}\right)\left|\frac{G_X}{G}\right|^2\right) \\
=&J\left(\frac{1}{2}\operatorname{tr}\left[\frac{G_{XX}}{G}\right] - \frac{\gamma-1}{2(\gamma+\theta)}|\bar{\lambda}_t|^2 - \frac{\gamma+\theta-1}{\gamma+\theta}\bar{\lambda}_t'\frac{G_X}{G} + \frac{\theta}{2(\gamma-1)(\gamma+\theta)}\left|\frac{G_X}{G}\right|^2\right).
\end{aligned} \tag{94}$$

The fourth and fifth terms in the PDE (39) is computed as

$$\bar{r}_t\bar{W}_t^* J_W - (\mathcal{K}X_t)'J_X = J\left(-(\gamma-1)\bar{r}_t - (\mathcal{K}X_t)'\frac{G_X}{G}\right). \tag{95}$$

The sixth term in the PDE (39) is calculated from eq.(52) as follows:

$$\begin{aligned}
&\beta J\big\{(1-\gamma)(\log c_t^* - 1) - \log\big((1-\gamma)J\big)\big\} \\
=&\beta J\big\{(1-\gamma)(\log\beta + \log\bar{W}_t^* - 1) - \big((1-\gamma)\log\bar{W}_t^* + \log G\big)\big\} \\
=&\beta J\big((1-\gamma)(\log\beta - 1) - \log G\big).
\end{aligned} \tag{96}$$

Substituting eqs.(94)–(96) into eq.(39) and dividing by $J$ yields the PDE (54).

## B.2 Proof for the General Case

The optimal consumption (52) follows from eq.(37):

$$c_t^* = \beta^\psi \left(\frac{(1-\gamma)J}{\bar{W}_t^*}\right)^{-\psi} \left((1-\gamma)J\right)^{\frac{\gamma\psi-1}{\gamma-1}} = \beta^\psi \bar{W}_t^{*\psi} \left(\bar{W}_t^{*1-\gamma} G^{\frac{1-\gamma}{\psi-1}}\right)^{\frac{\psi-1}{\gamma-1}} = \beta^\psi \frac{\bar{W}_t^*}{G}. \tag{97}$$

Eq.(3) is rewritten as

$$\pi_t = (\gamma-1)J\left(\bar{\lambda}_t + \frac{\gamma+\theta-1}{1-\psi}\frac{G_X}{G}\right). \tag{98}$$

Inserting eq.(47) and the derivatives of $J$ into eq.(38), we obtain the optimal investment (53). The first to third terms in the PDE (39) are calculated from eq.(98) and the derivatives of $J$ as follows:

$$\begin{aligned}
&\frac{1}{2}\operatorname{tr}\left[J_{XX}\right] - \frac{\theta}{2(1-\gamma)J}|J_X|^2 - \frac{1}{2}\left(\bar{W}_t^{*2}J_{WW} - \frac{\theta(W_t^* J_W)^2}{(1-\gamma)J}\right)^{-1}|\pi_t|^2 \\
=& J\left\{\frac{1-\gamma}{2(\psi-1)}\operatorname{tr}\left[\frac{2-\gamma-\psi}{\psi-1}\frac{G_X}{G}\frac{G_X'}{G} + \frac{G_{XX}}{G}\right] - \frac{(1-\gamma)\theta}{2(\psi-1)^2}\left|\frac{G_X}{G}\right|^2\right. \\
&\left. + \frac{1-\gamma}{2(\psi-1)^2(\gamma+\theta)}\left|(\psi-1)\bar{\lambda}_t - (\gamma+\theta-1)\frac{G_X}{G}\right|^2\right\} \\
=& \frac{1-\gamma}{\psi-1}J\left\{\frac{1}{2}\operatorname{tr}\left[\frac{2-\gamma-\psi}{\psi-1}\frac{G_X}{G}\frac{G_X'}{G} + \frac{G_{XX}}{G}\right] - \frac{\theta}{2(\psi-1)}\left|\frac{G_X}{G}\right|^2\right. \\
&\left. + \frac{1}{2(\psi-1)(\gamma+\theta)}\left|(\psi-1)\bar{\lambda}_t - (\gamma+\theta-1)\frac{G_X}{G}\right|^2\right\} \\
=& \frac{1-\gamma}{\psi-1}J\left\{\frac{1}{2}\operatorname{tr}\left[\frac{G_{XX}}{G}\right] + \frac{\psi-1}{2(\gamma+\theta)}|\bar{\lambda}_t|^2 - \left(1-(\gamma+\theta)^{-1}\right)\bar{\lambda}_t'\frac{G_X}{G}\right. \\
&\left. - \frac{1}{2(\psi-1)}\Big(\gamma+\psi-2+\theta - \left(1-(\gamma+\theta)^{-1}\right)(\gamma+\theta-1)\Big)\left|\frac{G_X}{G}\right|^2\right\} \\
=& \frac{1-\gamma}{\psi-1}J\left\{\frac{1}{2}\operatorname{tr}\left[\frac{G_{XX}}{G}\right] + \frac{\psi-1}{2}(\gamma+\theta)^{-1}|\bar{\lambda}_t|^2 - \left(1-(\gamma+\theta)^{-1}\right)\bar{\lambda}_t'\frac{G_X}{G}\right. \\
&\left. - \frac{1}{2(\psi-1)}\left(\psi-(\gamma+\theta)^{-1}\right)\left|\frac{G_X}{G}\right|^2\right\}.
\end{aligned} \tag{99}$$

The fourth and fifth terms in the PDE (39) are computed as follows:

$$\bar{r}_t\bar{W}_t^* J_W - (\mathcal{K}X_t)'J_X = \frac{1-\gamma}{\psi-1}J\left(-\frac{G_\tau}{G} + (\psi-1)\bar{r}_t - (\mathcal{K}X_t)'\frac{G_X}{G}\right). \tag{100}$$

The sixth and seventh terms in the PDE (39) are calculated from eq.(52) as follows:

$$\frac{1}{\psi-1}c_t^* J_W - \frac{\beta(1-\gamma)}{1-\psi^{-1}}J = \frac{1}{\psi-1}\left(\beta^\psi\frac{\bar{W}_t}{G}\frac{(1-\gamma)J}{\bar{W}_t} + \beta(\gamma-1)\psi J\right) = \frac{1-\gamma}{\psi-1}J\left(\frac{\beta^\psi}{G} - \beta\psi\right). \tag{101}$$

Substituting eqs.(99)–(101) into eq.(39) and dividing by $\dfrac{1-\gamma}{\psi-1}J$ yields the PDE (55).


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## Statements and Declarations

### Competing Interests

The authors have no competing interests to declare that are relevant to the content of this article.


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