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Group Completions I : An Observation from a Category-Theoretical Point of View on the Ordinary Group Completions of Commutative Monoids

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Summary

In this series on group completions various categorical constructions of universal group-like objects from symmetric monoidal objects will be studied. This paper is the first part in which we deal with the ordinary group completions of commutative monoids and observe some other aspects than the ordinary Grothendieck group construction. The relations of the ordinary construction with colimits and lax colimits over the translation category of a commutative monoid are established. Further we consider the commutativity of the group completion construction with limits and lax limits over a group (*i. e.* Galois descent).

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§ 1. The ordinary group completions of commutative monoids

Throughout this paper the composition law of every commutative monoid is written additively $+$ and the identity element is written 0. On the other side we denote the composition law of a (not neccesarily commutative) group multiplicatively and its identity element by 1.

Let M be a commutative monoid. The ordinary (algebraic) group completion $K(M)$ is associated with M. In detail we put

$$K(M) = M \times M/\sim$$

where $\sim$ is an equivalence relation on $M \times M$ defined as follows. For

$$(a, b), (c, d) \in M \times M$$
$$(a, b) \sim (c, d) \Leftrightarrow a + d + f = c + b + f, \quad \exists f \in M.$$

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$C(a, b)$ denotes the equivalence class of (a, b) in $K(M)$. The composition law of $K(M)$ is defined by

$$C(a, b) + C(c, d) = C(a+c, b+d).$$

Then $K(M)$ is a commutative monoid with $C(0, 0)$ as the identity element and further becomes a commutative group since every element $C(a, b)$ has an inverse element $C(b, a)$. When $i: M \to K(M)$ denotes the homomorphism defined by

$$i(a) = C(0, a) \quad \text{for} \quad a \in M,$$

$(K(M), i)$ is endowed with the following 'universal' property: If $f: M \to L$ is any homomorphism of M into a commutative group L, there is an unique homomorphism $g: K(M) \to L$ satisfying $f = g \cdot i$. That is to say f factors uniquely through $K(M)$.

A submonoid N of a monoid M is called *cofinal* if for any $a \in M$ there is an element b of M such that $a + b \in N$. Let N be a cofinal submonoid of a monoid M. We define in the same manner as above

$$K_N(M) = N \times M / \sim$$

where an equivalence relation $\sim$ and classes $C(\ ,\ )$ are as above. Then $K_N(M)$ is also a commutative group. Remark that the inverse elemnt of $C(a, b)$ for $a \in N, b \in M$ is given by $C(b+c, a+c)$ where $c \in M$ such that $b + c \in N$. Moreover if $C(a, b)$ is any element of $K(M)$, by taking $c \in M$ such that $a + c \in N$ one has $C(a, b) = C(a+c, b+c)$ in $K(M)$ and $C(a+c, b+c) \in K_N(M)$. These imply the following well-known fact.

Proposition 1.1. *If N is a cofinal submonoid of a monoid M there is an isomorphism of commutative groups*

$$K_N(M) \cong K(M).$$

Let $f: M \to L$ be a homomorphism of commutative monoids. Then by applying the universality of $(K(M), i_M)$ to the composed homomorphism $i_L \cdot f$ a homorphism $K(f): K(M) \to K(L)$ of their group completions is induced. If f is surjective, one deduces easily that $K(f)$ is also surjective. But we have to remark that though f is injective $K(f)$ is not neccesarily injective, because $C_L(f(a), f(b)) = 0$ does not imply $C_M(a, b) = 0$.

§ 2. Colimits and lax colimits over the translation categories of commutative monids

For a commutative monid M we will define its *translation category* $\hat{M}$. The objects of $\hat{M}$ are all elements of M and a morphism of $\hat{M}$ from u to v $(u, v \in M)$ is a pair (u, w) of elements of M such that $u + w = v$.

We need to investigate some properties of translation categories. We shall use mainly the terminology on category theory in MacLane [4]. The category $\hat{M}$ has an initial object 0, hence it becomes a cofiltered category. Further we can show it is filtered.

Lemma 2.1. *The translation category $\hat{M}$ of a commutative monoid M is cofiltered and filtered.*

Proof. For any two objects u, v of $\hat{M}$ there are two morphisms (u, v), (v, u) of $\hat{M}$ with the same target $u+v$. And for two morphisms (u, v_1), (u, v_2) of $\hat{M}$ from the same souce u to the same target w there is a morphism (w, u) of $\hat{M}$ from w to $w+u$ such that two composed morphisms coincide;

$$(w, u) \cdot (u, v_1) = (u, w) = (w, u) \cdot (u, v_2).$$

These two facts show that $\hat{M}$ is a filtered category. □

A homomorphism $f: M \to L$ of commutative monoids induces in a natural manner a functor $\hat{f}: \hat{M} \to \hat{L}$ of their translation categories. A functor $F: I \to J$ of categories is called *final* (resp. *right cofinal*) if for any object j of J the right fiber of F at j is non-empty and connected (resp. contractible) as in [4] IX-3 (resp. as the dual notion of [1] XI-9. 1).

Lemma 2. 2. *Let N be a submonoid of a commutative monoid M. $\hat{\imath}: \hat{N} \to \hat{M}$ denotes the induced functor of the translation categories from the inclusion homomorphism $i: N \to M$. Then the following three statements are equivalent.*

(1) *N is a cofinal submonoid of M.*

(2) *$\hat{\imath}$ is a right cofinal functor.*

(3) *$\hat{\imath}$ is a final functor.*

Proof. It is trivial that (2) implies (3). For an object $u_0 (\in M)$ of $\hat{M}$ the right fiber of $\hat{\imath}$ at u_0 is the category with the pairs (u, w) of $u \in N$, $w \in M$ such that $u_0 + w = u$ as the objects. A morphism from (u_1, w_1) to (u_2, w_2) in the right fiber at u_0 is represented by an element $v \in N$ such that $v + w_1 = w_2$, which implies $v + u_1 = u_2$. Now suppose N is not cofinal in M. Then there is an object $u_0 (\in M)$ of $\hat{M}$ such that the right fiber at u_0 is empty, which shows $\hat{\imath}$ is not a final functor. It follows that (3) implies (1).

Finally assume that N is a cofinal submonoid of M. We mention that for every object u_0 $(\in M)$ of $\hat{M}$ the right fiber of $\hat{\imath}$ at u_0 is a filtered category, hence contractible.

(a) It follows immediately from the cofinality of N that all right fibers are non-empty.

(b) Take any two objects (u_1, w_1), (u_2, w_2) of the right fiber at u_0, which satisfy $u_1, u_2 \in N$, $u_0 + w_1 = u_1$ and $u_0 + w_2 = u_2$. Since $w_1 + u_2 = w_1 + u_0 + w_2 = u_1 + w_2$, we have two morphisms $u_2: (u_1, w_1) \to (u_1 + u_2, w_1 + w_2)$, $u_1: (u_2, w_2) \to (u_1 + u_2, w_1 + w_2)$ with the same target.

(c) Given two morphisms $v_1, v_2: (u_1, w_1) \to (u_2, w_2)$ one has $w_1 + v_1 = w_1 + v_2 = w_2$. Consider a morphism $u_1: (u_2, w_2) \to (u_1 + u_2, u_1 + w_2)$ of the right fiber at u_0. Since $u_1 + v_1 = u_0 + w_1 + v_1 = u_0 + w_1 + v_2 = u_1 + v_2$, the two compositions $u_1 \cdot v_1$ and $u_1 \cdot v_2$ coincide.

It follows from the above (a) ~ (c) that the right fiber at u_0 is filtered. □

(Set) denotes the category of sets as objects and maps as morphisms and *(Cat)* denotes the 2-category of small categories as objects, functors as 1-morphisms and natural transformations as 2-morphisms. A set can be considered as a category of which the objects are its elements and the morphisms consist only of the identity maps. Thus *(Set)* is contained in *(Cat)*.

It is known that if $f: I \to J$ is a final functor and $F: J \to$ *(Set)* is a functor then there is a bijection

$Colim\ F \bullet f \cong Colim\ F.$

(See [4] IX−9 Theorem 1.) It is also known that if $f: I \to J$ is a right cofinal functor and $F: J \to (Cat)$ a pseudo functor then there is a homotopy equivalence of small categories

$Lax\ Colim\ F \bullet f \simeq Lax\ Colim\ F.$

(Consider the dual statement of [1] X I−9. 2 Cofinality Theorem.)

Now let N be a cofinal submonoid of a commutative monoid M. We will consider a functor λ from the translation category $\hat{N}$ of N to (Set) defined by sending any object $u (\in N)$ of $\hat{N}$ to the same set M and a morphism (u, v) $(\in N \times M)$ of $\hat{N}$ to a map $M \to M$, $m \mapsto m + v$, and a pseudo functor μ from $\hat{N}$ to (Cat) is defined by composing of λ with the inclusion of (Set) into (Cat). From Lemma 2.2 and the above remarks one has

Proposition 2.3. *Let M, N, λ and μ be as above.*

(1) *Colim λ is independent of the choice of a cofinal submonoid N up to bijection.*

(2) *Lax Colim μ is independent of the choice of a cofinal submonoid N up to homotopy equivalence.*

Let's make *Colim* λ and *Lax Colim* μ in an explicit form. Take a map $f_u: \lambda(u) = M \to K_N(M)$ defined by sending w to (u, w). Since $(K_N(M), \{f_u\})$ satisfy an universal property with respect to the system over $\hat{N}$, we have the following

Proposition 2.4. *Let M, N and λ be as above. Then there is a bijection*

$Colim\ \lambda \cong K_N(M).$

In general lax colimits are known to be given by *Grothendieck construction.* (See for example [1], [2], [6].) *Lax Colim* μ is a cofibered category $\mu \int \hat{N}$ over $\hat{N}$. This category has pairs (u, w) of $u \in N$ and $w \in M$ as objects and a morphism from (u_1, w_1) to (u_2, w_2) is represented by $v \in N$ such $u_1 + v = u_2$, $w_1 + v = w_2$.

For a small category C the set $\pi_0(C)$ of connected components is defined as follows;

$\pi_0(C) =$ (the set of objects of C)$/\sim$

where for two objects a, b the condition on $a \sim b$ is that there are finitely many morphisms connecting a with b; *e. g.*

$a \to \bullet \leftarrow \bullet \to \bullet \quad \cdots\cdots \quad \bullet \leftarrow \bullet \to \bullet \leftarrow b.$

Then it is easy to see the following

Proposition 2.5. *Let M, N and μ be as above. There is a bijection*

$\pi_0(Lax\ Colim\ \mu) \cong K_N(M).$

§ 3. Limits and lax limits over groups

The general theory of this subject was studied in full detail in my paper [6]. Here a very special case is dealt with.

We regard a group G as a category G (used the same letter) with the only one object $\bullet$ and with the elements of G as morphisms. Given a right G-set X we define a functor σ of G into *(Set)* and a pseudo functor τ of G into *(Cat)*. σ is defined by

$\sigma(\bullet) = X$ on objects and
$\sigma(s): X \to X, \quad \sigma(s)(x) = x^s$ for $s \in G, \quad x \in X$ on morphisms.

τ is the composed (pseudo) functor of σ with the inclusion of *(Set)* into *(Cat)*.

First let's consider *Lim* σ. Putting

$$X^G = \{x \in X; x^s = x, \quad \forall s \in G\},$$

the inclusion map of X^G into X has an universal property for the functor σ, hence we have

Proposition 3.1. *Let G, X and σ be as above. Then there is a bijection*

$$\mathit{Lim}\ \sigma \cong X^G.$$

Next we investigate *Lax Colim* τ. This is given by the Grothendieck construction $\tau \int G$ which is a cofibered category over G. The objects of $\tau \int G$ are elements of X and there is the only morphism of grade $s \in G$ from x to y $(x, y \in X)$ if $x^s = y$, hence

Proposition 3.2. *Let G, X and τ be as above. Then there is a bijection*

$$\pi_0(\mathit{Lax\ Colim}\ \tau) \cong X/G \ (= \textit{the set of G-orbits of } X).$$

We consider finally *Lax Lim* τ. It is known from [6] that it is given by the category $\mathit{Cart}_G(G, \tau \int G)$ of cartesian section functors of the cofibered categories $\tau \int G$ over G. But since X is a discrete category in our case, it has a simple form.

Proposition 3.3. *Let G, X and τ be as above. Then there is an equivalence of categories*

$$\mathit{Lax\ Lim}\ \tau \approx X^G \textit{ considered as a category.}$$

§ 4. The commutativity of group completions with (lax) limits over a group

The problem considered in this section is whether the group completions commute (lax) limits over a group *i. e.* the problem of Galois descent. (See [6].) In the case of commutative G-monoids M handled in this paper we obtain the affirmative results under the assumption that a group G is finite.

In what follows let G be a finite group and M a commutative G-monoid in which the action of G is from right as in §2.

We consider first the ordinary group completions and the limits over G. The ordinary group completion $K(M)$ of M becomes a commutative G-group by defining the natural action of G on $K(M)$ by

$$(u, w)^s = (u^s, w^s) \quad \text{for} \quad s \in G, \quad u, \quad w \in M.$$

Theorem 4.1. *Let G and M be as above. There is a bijection*

$$K(M^G) \cong K(M)^G.$$

Proof. The inclusion homomorphism of M^G into M induces a homomorhism $h: K(M^G) \to K(M)$ of commutative groups. Let's verify that h is injective and $Im\ h = K(M)^G$. $C(\ ,\ )$ (*resp.* $C'(\ ,\ )$) denotes the equivalence classes in $K(M)$ (*resp.* in $K(M^G)$). Suppose $h(C'(m, n)) = C(0, 0)$ for $m, n \in M^G$. Then $C(m, n) = C(0, 0)$ hence there is an element f of M such that $f + m = f + n$. Put $f^\circ = \Sigma_{s \in G} f^s$ then $f^\circ \in M^G$ and $f^\circ + m = f^\circ + n$, hence $C'(m, n) = C'(0, 0)$. Thus $Ker\ h$ consists of the only identity element, which shows h is injective.

Next since $C(m, n)^s = C(m^s, n^s) = C(m, n)$ for any $s \in G$, $m, n \in M^G$, *we have* $Im\ h \subset K(M)^G$.

Finally we must show $K(M)^G \subset Im\ h$. It is sufficient to verify that for any $C(u, v) \in K(M)^G$ there are $m, n \in M^G$ such that $C(u, v) = C(m, n)$. If $C(u, v) \in K(M)^G$, $C(u, v)^s = C(u, v)$ for any $s \in G$. This means that there is an element f_s of M such that $f_s + u^s + v = f_s + u + v^s$ for every $s \in G$. By taking $f^\circ = \Sigma_{t \in G}(\Sigma_{s \in G} f_s)^t$ we have $f^\circ \in M^G$ and $f^\circ + u^s + v = f^\circ + u + v^s$ for every $s \in G$. Put $u^\circ = \Sigma_{s \in G} u^s$, $v^\circ = \Sigma_{s \in G} v^s$, $w = \Sigma_{s \in G, s \neq 1} u^s$ and $z = \Sigma_{s \in G, s \neq 1} v^s$ then $u + w = u^\circ \in M^G$ and $v + z = v^\circ \in M^G$. Consider $x^\circ = f^\circ + u^\circ + v + w$ then for any $s \in G$

$$x^\circ = f^\circ + u^s + w^s + v + w = f^\circ + u + v^s + w + w^s = f^\circ + u^\circ + v^s + w^s = (x^\circ)^s,$$

hence $x^\circ \in M^G$. Further one obtains

$$x^\circ + u = f^\circ + u^\circ + v + w + u = f^\circ + 2u^\circ + v.$$

If we put $m = f^\circ + 2u^\circ$, $n = x^\circ$ then $m, n \in M^G$ and $C(u, v) = C(m, n)$ as required. □

For a commutative G–monoid M we define a functor ν of $\hat{M} \times G$ into (Set) as follows; on objects $\nu(u, \bullet) = M$ and on morphisms $\nu(v, s): M \to M$, $m \mapsto v + m^s$. And further we have a (pseudo) functor μ of $\hat{M} \times G$ into (Cat) by composing ν with the natural inclusion of (Set) into (Cat). It is known from [4] IX-2 Theorem 1 that finite limits commute with filtered colimits. This implies that there is a bijection

$$Colim_{(M^G)^\wedge} (Lim_G\, \nu) \cong Lim_G\, (Colim_{\hat{M}}\, \nu).$$

But this follows the above Theorem 4.1 from Proposition 2.4 and Proposition 3.1.

Finally we show that for the (pseudo) functor μ lax colimits over the translation categories of commutative monoids commute with lax limits over a group (considered as a category).

Theorem 4.2. *Let G be a finite group and M a commutative G–monoid.* μ *denotes the (pseudo) functor of* $\hat{M} \times G$ *into (Cat) as above. Then there is a homotopy equivalence*

$$Lax\ Colim_{(M^G)^\wedge} (Lax\ Lim_G\, \mu) \simeq Lax\ Lim_G(Lax\ Colim_{\hat{M}}\, \mu).$$

Proof. It follows from Proposition 3.3 that $Lax\ Lim_G\, \mu$ is a set M^G considered as a category, therefore the category on the left-hand side is nothing but the lax colimit over the translation category $(M^G)^\wedge$ of M^G for the commutative monoid M^G. (See § 2.) $Lax\ Colim_{\hat{M}}\, \mu$ becomes a G–category in the sense of [6] by taking $(u, w)^s = (u^s, w^s)$ on objects. So the category on the right-hand side is the category of cartesian section functors from G to the fibered category over G associated with the G–category *Lax Colim*$_{\tilde{M}}$ μ. A denotes this category and let us make A in an explicit form. The objects of A are represented by the triples $(u, w, (a_s)_{s\in G})$ for $u, w, a_s \in M$, satisfying $u + a_s = u^s$, $w + a_s = w^s$ for any $s \in G$, which implies the 1–cocycle conditions $a_1 = 0$, $a_{st} = (a_s)^t + a_t$ for $s, t \in G$. The morphisms of A from $(u, w, (a_s))$ to $(u', w', (b_s))$ are represented by $v \in M$ satisfying $u + v = u'$, $w + v = w'$ and $a_s + v^s = v + b_s$ for any $s \in G$. We shall consider two subcategories of A.

B denotes the full subcategory of A consisting of objects $(u, w, (a_s))$ satisfying $u \in M^G$. We must remark that under our assumption that G is a finite group the submonoid M^G is cofinal in M since for any $u \in M$ $u + \Sigma_{s\in G, s\neq 1} u^s \in M^G$. It is easy to see that the category B is nothing but the category of cartesin section functors from G to the fibered category over G associated with the lax colimit over the cofinal submonoid M^G of μ. It follows from Proposition 2.3 (2) that B is homotopy equivalent to A.

C denotes the full subcategory of A consisting of objects $(u, w, (a_s))$ satisfying $a_s = 0$ for any $s \in G$. Since this condition implies $u, w \in M^G$ C is contained in B. Also $v(\in M)$ representing a morphism of C belongs to M^G. Thus C is equivalent to the category on the left-hand side. Therefore we must show that the inclusion functor j of C into B is a homotopy equivalence. To see this it is sufficient to verify that for any object $(u_0, w_0, (a_s))$ of B the right fiber D of j at $(u_0, w_0, (a_s))$ is a cofiltered or filtered category, hence contractible.

If $(u_0, w_0, (a_s))$ is an object of C *i. e.* $a_s = 0$ for any $s \in G$ then the right fiber D becomes the category of objects of C under $(u_0, w_0, (0))$, consequently it has an initial object and so it is cofiltered.

Otherwise we shall show that the right fiber D is filtered. Note that the objects of the right fiber D of j at $(u_0, w_0, (a_s))$ consist of morphisms of B from $(u_0, w_0, (a_s))$ to objects $(u_1, w_1, (0))$ of C which is represented by $v_1 \in M$ satisfying $u_0 + v_1 = u_1$, $w_0 + v_1 = w_1$ and $a_s + v_1{}^s = v_1$ for any $s \in G$, and that a morphism of D from $v_1 : (u_0, w_0, (a_s)) \to (u_1, w_1, (0))$ to $v_2 : (u_0, w_0, (a_s)) \to (u_2, w_2, (0))$ is a morphism $z : (u_1, w_1, (0)) \to (u_2, w_2, (0))$ of C such that $z \in M^G$, $z + v_1 = v_2$.

(a) Since $u_0 + a_s = u_0$, $w_0 + a_s = w_0^s$ imply $(w_0 + u_0)^s = w_0^s + u_0^s = w_0 + a_s + u_0 = w_0 + u_0$, $a_s + u_0^s = a_s + u_0 = u_0$ for any $s \in G$, there is an object $u_0 : (u_0, w_0, (a_s)) \to (2u_0, w_0 + u_0, (0))$. Hence the right fiber D is non-empty.

(b) Given any two objects $v_1 : (u_0, w_0, (a_s)) \to (u_1, w_1, (0))$, $v_2 : (u_0, w_0, (a_s)) \to (u_2, w_2, (0))$ of D, we can define two morphisms $u_0 + u_2 : (u_1, w_1, (0)) \to (u_1 + u_0 + u_2, w_1 + u_0 + u_2, (0))$, $u_0 + u_1 : (u_2, w_2, (0)) \to (u_1 + u_0 + u_2, w_2 + u_0 + u_1, (0))$ of D. Since $w_1 + u_0 + u_2 = w_0 + v_1 + u_0 + u_2 = w_0 + u_1 + u_2 = w_0 + v_2 + u_0 + u_1 = w_2 + u_0 + u_1$, the above two morphisms $u_0 + u_2$, $u_0 + u_1$ have the same target. Further since $(u_0 + u_2) \cdot v_1 = v_1 + u_0 + u_2 = u_1 + u_2 = u_1 + u_0 + v_2 = (u_0 + u_1) \cdot v_2$, the morphism $u_1 + u_2 : (u_0, w_0, (a_s)) \to (u_1 + u_0 + u_2, w_1 + u_0 + u_2, (0))$ of B becomes an object of D. Thus for any two objects of D we obtain two morphisms from those to some object of D.

(c) Given two objects $v_1 : (u_0, w_0, (a_s)) \to (u_1, w_1, (0))$, $v_2 : (u_0, w_0, (a_s)) \to (u_2, w_2, (0))$ of D and two morphisms $z_1, z_2 : (u_1, w_1, (0)) \to (u_2, w_2, (0))$ of D which satisfy $v_i + u_0 = u_i$, $v_i + w_0 = w_i$, $u_1 + z_i = u_2$, $w_1 + z_i = w_2$ for $i = 1, 2$, we can define an object $u_0 + v_1 + v_2 : (u_0, w_0, (a_s)) \to (u_1 + u_2, u_1 + w_2 = u_2 + w_1, (0))$ of D and a morphism $u_1 : (u_2, w_2, (0)) \to (u_1 + u_2, u_1 + w_2, (0))$ of D. Then since $u_1 + z_1 = u_0 + v_1 + z_1 = u_0 + v_2 = u_0 + v_1 + z_2 = u_1 + z_2$,

the two compositions $u_1 \cdot z_1$, $u_1 \cdot z_2$: $(u_1, w_1, (0)) \to (u_1+u_2, u_1+w_2, (0))$ coincide.
It follows from (a) $\sim$ (c) that the right fiber D is a filtered category as required. □

Remark 4.3. The assumption of finiteness of G in Theorem 4.2 can be improved, for we only used the fact that M^G is cofinal in M in the proof of Theorem 4.2. As an improved assumption we can take the following condition;
a group G acts finitely on a commutative monoid M i. e.

$$[G: G_u] < \infty \quad \textit{for any} \quad u \in M,$$

where G_u denotes the isotropic subgroup of u in G. We note also that since Theorem 4.2 implies Theorem 4.1 by applying π_0, the above improvement can be applied to Theorem 4.1.

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