Review of Fixed Point Theorems for a Certain Class of Nonlinear Mappings in Hilbert Spaces

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I Introduction

In this article, we denote by H a real Hilbert space with an inner product $\langle \cdot , \cdot \rangle$. Norm is defined as $\|x\| = \sqrt{\langle x, x \rangle}$ for an element x of H. The following is a simple version of the fixed point theorem by Kirk [10], Browder [3] and Göhde [4]:

Theorem 1 *Let C be a nonempty, closed, convex, and bounded subset of a real Hilbert space H. Let $T : C \to C$ be a nonexpansive mapping, that is,*

$$\|Tx - Ty\| \leq \|x - y\| \quad \textit{for all } x, y \in C. \tag{1}$$

Then, T has at least one fixed point.

A fixed point $v \in C$ of a mapping T indicates that $Tv = v$. Theorem 1 has been extended by many researchers. In particular, various types of mappings have been proposed. *Nonspreading mappings* [12] and *hybrid mappings* [24] are defined by the conditions

$$2\|Tx - Ty\|^2 \leq \|x - Ty\|^2 + \|Tx - y\|^2 \text{ and} \tag{2}$$
$$3\|Tx - Ty\|^2 \leq \|x - y\|^2 + \|x - Ty\|^2 + \|Tx - y\|^2, \tag{3}$$

respectively. Nonspreading mappings were proposed out of necessity in light of optimizing problems. Although a nonexpansive mapping is continuous, nonspreading and hybrid mappings are not necessarily continuous; for examples of such mappings, see Igarashi et al. [6] or recent articles by Kondo [14, 17]. In 2010, these types of mappings (1)-(3) were unified by Kocourek et al. [11] as a *generalized hybrid mapping*. If there exist $\alpha, \beta \in \mathbb{R}$ such that

$$\begin{aligned} &\alpha\|Tx - Ty\|^2 + (1-\alpha)\|x - Ty\|^2 \\ &\leq \beta\|Tx - y\|^2 + (1-\beta)\|x - y\|^2 \end{aligned} \tag{4}$$

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for all $x, y \in C$, then $T : C \to H$ is called generalized hybrid, where $\mathbb{R}$ is the set of real numbers. If $\alpha = 1$ and $\beta = 0$, then a generalized hybrid mapping is nonexpansive. Hence, a nonexpansive mapping is a special case of generalized hybrid mappings. Similarly, a class of generalized hybrid mappings contains nonspreading mappings (2) as a case of $\alpha = 2$ and $\beta = 1$ and hybrid mappings as a case of $\alpha = 3/2$ and $\beta = 1/2$. A class of generalized hybrid mappings also includes *λ-hybrid mappings*; see Aoyama et al. [2]. Kocourek et al. [11] established a fixed point theorem that asserts the existence of at least one fixed point of a generalized hybrid mapping. They also studied how to construct sequences that converge to a fixed point.

Theorem 1 is further studied for more general types of mappings. A mapping $T : C \to C$ is called 2*-generalized hybrid* [21] if there exist $\alpha_1, \beta_1, \alpha_2, \beta_2, \in \mathbb{R}$ such that

$$\begin{aligned} &\alpha_2 \left\|T^2x - Ty\right\|^2 + \alpha_1 \left\|Tx - Ty\right\|^2 + (1 - \alpha_1 - \alpha_2) \\ &\|x - Ty\|^2 \le \beta_2 \left\|T^2x - y\right\|^2 + \beta_1 \left\|Tx - y\right\|^2 \\ &+ (1 - \beta_1 - \beta_2) \left\|x - y\right\|^2 \end{aligned} \tag{5}$$

for all $x, y \in C$. Substituting $\alpha_2 = \beta_2 = 0$ into (5), we obtain the condition of generalized hybrid mappings (4) and thus, the class of 2-generalized hybrid mappings contains generalized hybrid mappings as special cases. Hojo et al. [5] presented examples of 2-generalized hybrid mappings that are not generalized hybrid; see also Kondo [14, 17]. Maruyama et al. [21] demonstrated a fixed point theorem and convergence theorems that approximate fixed points for this type of mappings in the framework of real Hilbert spaces.

Recently, Kondo and Takahashi [19] introduced a class of mappings called *generic 2-generalized hybrid mappings* that is more general than the class of 2-generalized hybrid mappings. If there exist $\alpha_{ij}, \beta_i, \gamma_i \in \mathbb{R}$ $(i, j = 0, 1, 2)$ such that

$$\begin{aligned} &\alpha_{00}\|x - y\|^2 + \alpha_{01}\|x - Ty\|^2 + \alpha_{02}\|x - T^2y\|^2 \\ &+\alpha_{10} \left\|Tx - y\right\|^2 + \alpha_{11} \left\|Tx - Ty\right\|^2 + \alpha_{12} \left\|Tx - T^2y\right\|^2 \\ &+\alpha_{20} \left\|T^2x - y\right\|^2 + \alpha_{21} \left\|T^2x - Ty\right\|^2 + \alpha_{22} \left\|T^2x - T^2y\right\|^2 \\ &+\beta_0 \left\|x - Tx\right\|^2 + \beta_1 \left\|Tx - T^2x\right\|^2 + \beta_2 \left\|T^2x - x\right\|^2 \\ &+\gamma_0 \left\|y - Ty\right\|^2 + \gamma_1 \left\|Ty - T^2y\right\|^2 + \gamma_2 \left\|T^2y - y\right\|^2 \le 0 \end{aligned} \tag{6}$$

for all $x, y \in C$, then $T : C \to C$ is called an $(\alpha_{ij}, \beta_i, \gamma_i; i, j = 0, 1, 2)$-generic 2-generalized hybrid mapping; see also [13, 20]. They proved a fixed point theorem and convergence theorems to fixed points in real Hilbert spaces. In a very recent article by Kondo [15], a fixed point theorem is proved under more general conditions on the parameters $\alpha_{ij}, \beta_i, \gamma_i \in \mathbb{R}$ than in the previous article [19]. Other than 2-generalized hybrid mappings, the class of generic 2-generalized hybrid mappings contains normally generalized hybrid mappings [25], normally 2-generalized hybrid mappings [18], further generalized hybrid mappings [7], and further 2-generalized hybrid mappings [1] as special cases. For recent contributions to the fixed point theory, see also Kawasaki [8, 9], and Shukla [22].

In this elementary review article, we introduce simple examples of $(\alpha_{ij}, \beta_i, \gamma_i; i, j = 0, 1, 2)$-generic 2-generalized hybrid mappings and demonstrate proofs of fixed point theorems for undergraduate students or researchers working on various fields. Special focus is placed on cases such as

$$\lambda \left\|Tx - Ty\right\|^2 + (1 - \lambda) \left\|T^2x - T^2y\right\|^2 \le \|x - y\|^2, \tag{7}$$

$$\mu \left\|Tx - T^2y\right\|^2 + (1 - \mu) \left\|T^2x - T^2y\right\|^2 \le \left\|x - T^2y\right\|^2, \tag{8}$$

for all $x, y \in C$, where $\lambda, \mu \in (0, 1]$, in addition to the case of nonexpansive mappings (1). These types of mappings are generic 2-generalized hybrid within the class that is targeted in this article and the author's previous one [15]. For example, substituting $\alpha_{00} = -1$, $\alpha_{11} = 1$, and 0 for all the other parameters into (6) yields the condition of nonexpansive mappings (1). Furthermore, letting $\alpha_{00} = -1$, $\alpha_{11} = \lambda$, $\alpha_{22} = 1 - \lambda$, and 0 for all the other parameters, we obtain the condition (7). In Section 2, examples of mappings are provided. Although one of them is not continuous, they are within the class that satisfies (1), (7) or (8). In Section 3, proofs of fixed point theorems are demonstrated for types (1), (7) and a slightly general type of (8).

II Fixed point theorem and examples

In this section, we reproduce the fixed point theorem for generic 2-generalized hybrid mappings by Kondo [15] and present examples of mappings within the class of mappings that are addressed in this article. For an (α_{ij}, β_i, γ_i; $i, j = 0, 1, 2$)-generic 2-generalized hybrid mapping (6), we use the following notations:

$$\alpha_{i\bullet} \equiv \alpha_{i0} + \alpha_{i1} + \alpha_{i2} \quad \text{and} \quad \alpha_{\bullet\bullet} \equiv \sum_{i,j=0,1,2} \alpha_{ij}, \tag{9}$$

where $i = 1, 2$. The theorem of Kondo [15] is as follows:

Theorem 2 *Let C be a nonempty, closed, and convex subset of H and let $T : C \to C$ be an (α_{ij}, β_i, γ_i; $i, j = 0, 1, 2$)-generic 2-generalized hybrid mapping with the conditions*

$$\begin{aligned} &\alpha_{\bullet\bullet} \geq 0,\ \alpha_{1\bullet} + \beta_0 > 0,\ \beta_1 \geq 0,\ \alpha_{2\bullet} + \beta_2 \geq 0, \\ &\gamma_0 + \gamma_1 \geq 0,\ \gamma_2 \geq 0. \end{aligned} \tag{10}$$

Suppose there exists an element $z \in C$ such that the sequence $\{T^n z\}$ in C is bounded. Then, T has at least one fixed point.

Obviously, if C is a bounded subset of H, the condition in Theorem 2, i.e., there exists an element $z \in C$ such that the sequence $\{T^n z\}$ is bounded, is sufficiently guaranteed. We consider an (α_{ij}, β_i, γ_i; $i, j = 0, 1, 2$)-generic 2-generalized hybrid mapping T that is characterized by the conditions (6) and (10).

(i) Substitute $\alpha_{00} = -1$ and $\alpha_{11} = 1$ with all other parameters set to be 0 into (6). The parameter constellation satisfies the condition (10). Then, the mapping T is nonexpansive.

(ii) Setting $\alpha_{00} = -1$, $\alpha_{11} = \lambda$, $\alpha_{22} = 1 - \lambda$, and all the other parameters as 0 in (6), we have

$$\lambda \|Tx - Ty\|^2 + (1 - \lambda) \|T^2x - T^2y\|^2 \leq \|x - y\|^2 \tag{11}$$

for all $x, y \in C$, where $\lambda \in (0, 1]$.

(iii) Setting $\alpha_{02} = -1$, $\alpha_{12} = \mu$, $\alpha_{22} = 1 - \mu$, and all the other parameters as 0 yields

$$\mu \|Tx - T^2y\|^2 + (1 - \mu) \|T^2x - T^2y\|^2 \leq \|x - T^2y\|^2 \tag{12}$$

for all $x, y \in C$, where $\mu \in (0, 1]$.

(iii)′ Let $\alpha_{02} = -1$, $\alpha_{12} = \mu'$, $\alpha_{22} = 1 - \mu'$, $\beta_0 = -\varepsilon\mu'$, and all the other parameters be 0 in (6). Then,

$$\begin{aligned} &\mu' \|Tx - T^2y\|^2 + (1 - \mu') \|T^2x - T^2y\|^2 \\ &\leq \|x - T^2y\|^2 + \varepsilon\mu' \|x - Tx\|^2 \end{aligned} \tag{13}$$

for all $x, y \in C$, where $\mu' \in (0, 1]$ and $\varepsilon \in [0, 1)$. The case (13) with $\varepsilon = 0$ coincides with (12).

Therefore, a mapping with the condition (13) is more general than the case of (12).

(iv) Letting $\alpha_{11} = \alpha$, $\alpha_{01} = 1 - \alpha$, $\alpha_{10} = -\beta$, $\alpha_{00} = -(1 - \beta_3)$, and all the other parameters be 0, we obtain a generalized hybrid mapping (4), where $\alpha, \beta_3 \in \mathbb{R}$.

(v) Similarly, a 2-generalized hybrid mapping (5) is within the class of generic 2-generalized hybrid mappings (6) that satisfies the parameter condition (10).

We present some examples that satisfy (i)-(iii).

Example 1. Let $H = C = \mathbb{R}^2$. Using a matrix, define a mapping $T : \mathbb{R}^2 \to \mathbb{R}^2$ as below:

$$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \quad \text{for all } \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2.$$

This matrix represents a rotation and a (linear) nonexpansive transformation in $\mathbb{R}^2$ with a unique fixed point $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ in $\mathbb{R}^2$. □

Next, we deal with a mapping that satisfies the condition (11). Note that mappings of this type are continuous. Indeed, from (11),

$$\lambda \|Tx - Ty\|^2 \leq \|x - y\|^2 \quad \text{for all } x, y \in C,$$

where $\lambda \in (0, 1]$, which shows that T is continuous. Letting $\lambda = 1/3$ in (11) results in

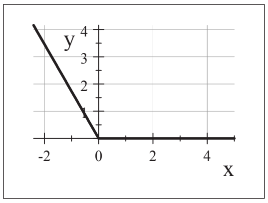
$$\frac{1}{3}\|Tx - Ty\|^2 + \frac{2}{3}\left\|T^2x - T^2y\right\|^2 \leq \|x - y\|^2 \tag{14}$$

for all $x, y \in C$. A mapping in Example 2 satisfies the condition (14).

Example 2. Consider the case $H = C = \mathbb{R}$. Define a mapping $T : \mathbb{R} \to \mathbb{R}$ as follows:

$$Tx = \begin{cases} 0 & \text{if } x \geq 0, \\ -\sqrt{3}x & \text{if } x < 0. \end{cases} \tag{15}$$

for all $x \in \mathbb{R}$. As seen in the following figure, T is not a "nonexpansive-type" mapping although it is continuous.



It can be verified that the function T satisfies the condition (14) as follows: Choose $x, y \in \mathbb{R}$ arbitrarily. (i) If $x, y \geq 0$, then $Tx = Ty = T^2x = T^2y = 0$. Thus, the condition (14) is satisfied. (ii) If $x, y < 0$, then it holds that $Tx = -\sqrt{3}x$, $Ty = -\sqrt{3}y$, and $T^2x = T^2y = 0$. In this case, it is true that

$$\begin{aligned} &\text{LHS of (14)} - \text{RHS} \\ &= \frac{1}{3}\|Tx - Ty\|^2 + \frac{2}{3}\left\|T^2x - T^2y\right\|^2 - \|x - y\|^2 \\ &= \frac{1}{3}\left(-\sqrt{3}x + \sqrt{3}y\right)^2 - (x - y)^2 = 0. \end{aligned}$$

This indicates that the condition (14) is satisfied. (iii) If $y < 0 \leq x$, then $Ty = \sqrt{3}y$ and $Tx = T^2x = T^2y = 0$. Consequently, it follows that

$$\begin{aligned} &\text{LHS of (14)} - \text{RHS} \\ &= \frac{1}{3}\|Tx - Ty\|^2 + \frac{2}{3}\left\|T^2x - T^2y\right\|^2 - \|x - y\|^2 \\ &= \frac{1}{3}\|Ty\|^2 - \|x - y\|^2 \\ &= y^2 - (x - y)^2 \\ &= -x(x - 2y) \leq 0, \end{aligned}$$

which demonstrates that the condition (14) is satisfied. We have verified that the mapping defined by (15) satisfies the condition (14) as claimed. □

Now, an example of a mapping that satisfies (12) is presented. Setting $\mu = 1/3$ in (12), we have

$$\frac{1}{3}\left\|Tx - T^2y\right\|^2 + \frac{2}{3}\left\|T^2x - T^2y\right\|^2 \leq \left\|x - T^2y\right\|^2 \quad \text{for all } x, y \in C. \tag{16}$$

Although the mapping in Example 2 satisfies (16), we consider another example that is not continuous and satisfies the condition (16).

Example 3. Let $H = \mathbb{R}$ and $C = [0, \infty)$. Define a mapping $T : C \to \mathbb{R}$ as follows:

$$Tx = \begin{cases} 0 & \text{if } 0 \leq x \leq \dfrac{1}{\sqrt{3}} \text{ or } x = 1, \\ 1 & \text{if } \dfrac{1}{\sqrt{3}} < x \text{ and } x \neq 1, \end{cases} \tag{17}$$

for all $x \in C$. We demonstrate that the mapping T defined by (17) satisfies the condition (16). For $x, y \in C = [0, \infty)$, it holds that $T^2x = T^2y = 0$. Therefore, our aim is to show that $(Tx)^2 \leq 3x^2$. Consider two cases according to x. (i) If $0 \leq x \leq 1/\sqrt{3}$ or $x = 1$, then $Tx = 0$. In this case, the desired result follows. (ii) If $1/\sqrt{3} < x$ and $x \neq 1$, then $Tx = 1$. Therefore, it follows that

$$(Tx)^2 - 3x^2 < 1 - 3 \cdot \frac{1}{3} = 0.$$

Therefore, the mapping defined by (17) satisfies (16) as claimed. $\square$

Given that mappings with the condition (13) are more general than those with (12), the example (17) also satisfies the condition (13). For examples of (iv) generalized hybrid mappings and (v) 2-generalized hybrid mappings, see Kondo [14, 17] and articles cited therein.

Ⅲ Proofs for some special cases

In this section, we present proofs of fixed point theorems for cases corresponding to (i), (ii), and (iii)′, which simplify Theorem 2. Before proving the theorems, as prior knowledge, recall the following:

(a) For a bounded sequence $\{x_n\}$ in H, there exist a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ and $v \in H$ such that $x_{n_i} \rightharpoonup v$, where $x_{n_i} \rightharpoonup v$ indicates weak convergence, that is, $<x_{n_i}, z> \to <v, z>$ for all $z \in H$.

(b) A closed and convex subset C of H is weakly closed; in other words, for a sequence $\{x_n\}$ in C, if $x_n \rightharpoonup v \in H$, then $v \in C$.

(c) It hold that $\|x + y\|^2 = \|x\|^2 + 2\langle x, y\rangle + \|y\|^2$ for any elements x, y of H from the relation between the inner product and the norm.

The author referred to the textbook of Takahashi [23] for the following proof of Theorem 3.

Theorem 3 *Let C be a nonempty, closed, and convex subset of H. Let T be a nonexpansive mapping from C into itself. Suppose there exists an element $z \in C$ such that $\{T^n z\}$ is a bounded sequence in C. Then, T has at least one fixed point.*

Proof. Define

$$A_n^0 = \frac{1}{n} \sum_{k=0}^{n-1} T^k z.$$

As $T : C \to C$ and C is convex, $\{A_n^0\}$ is a sequence in C. As $\{T^n z\}$ is bounded, so is $\{A_n^0\}$. Based on (a), there exists a subsequence $\{A_{n_i}^0\}$ of $\{A_n^0\}$ such that $A_{n_i} \rightharpoonup v$ for some $v \in H$. As C is a closed and convex subset of H, based on (b), it is weakly closed. As $\{A_{n_i}^0\} \subset C$ and $A_{n_i}^0 \rightharpoonup v$, we have $v \in C$. As $T : C \to C$, an element Tv ($\in C$) exists. Our goal is to prove that $Tv = v$, in other words, v is a fixed point of T.

As T is nonexpansive, it holds that

$$\left\|T^{k+1}z - Tv\right\|^2 \leq \left\|T^k z - v\right\|^2$$

for any $k = 0, 1, 2, \ldots$. Using (c), we obtaın

$$\left\|T^{k+1}z - Tv\right\|^2 = \left\|T^k z - Tv + Tv - v\right\|^2$$
$$= \left\|T^k z - Tv\right\|^2 + 2\left\langle T^k z - Tv,\ Tv - v\right\rangle + \|Tv - v\|^2 .$$

More concretely, it holds true that

$$\|Tz - Tv\|^2 \le \|z - Tv\|^2 + 2\langle z - Tv,\ Tv - v\rangle + \|Tv - v\|^2,$$
$$\left\|T^2 z - Tv\right\|^2 \le \|Tz - Tv\|^2 + 2\langle Tz - Tv,\ Tv - v\rangle + \|Tv - v\|^2,$$
$$\left\|T^3 z - Tv\right\|^2 \le \left\|T^2 z - Tv\right\|^2 + 2\left\langle T^2 z - Tv,\ Tv - v\right\rangle + \|Tv - v\|^2,$$
$$\cdots\cdots\cdots$$
$$\left\|T^{n-1}z - Tv\right\|^2 \le \left\|T^{n-2}z - Tv\right\|^2 + 2\left\langle T^{n-2}z - Tv,\ Tv - v\right\rangle + \|Tv - v\|^2,$$
$$\|T^n z - Tv\|^2 \le \left\|T^{n-1}z - Tv\right\|^2 + 2\left\langle T^{n-1}z - Tv,\ Tv - v\right\rangle + \|Tv - v\|^2,$$

where $n \in \mathbb{N}$. Summing these inequalities results in

$$\|T^n z - Tv\|^2 \le \|z - Tv\|^2 + 2\left\langle \sum_{k=0}^{n-1} T^k z - nTv,\ Tv - v\right\rangle + n\|Tv - v\|^2 .$$

As $\|T^n z - Tv\|^2 \ge 0$, it follows that

$$0 \le \|z - Tv\|^2 + 2\left\langle \sum_{k=0}^{n-1} T^k z - nTv,\ Tv - v\right\rangle + n\|Tv - v\|^2 .$$

Dividing by n, we have

$$0 \le \frac{1}{n}\|z - Tv\|^2 + 2\left\langle A_n^0 - Tv,\ Tv - v\right\rangle + \|Tv - v\|^2 .$$

As n is any natural number, it holds that

$$0 \le \frac{1}{n_i}\|z - Tv\|^2 + 2\left\langle A_{n_i}^0 - Tv,\ Tv - v\right\rangle + \|Tv - v\|^2$$

for all $i \in \mathbb{N}$. As $A_{n_i}^0 \to v$, we obtain in the limit as $i \to \infty$ that

$$0 \le 2\langle v - Tv,\ Tv - v\rangle + \|Tv - v\|^2$$
$$= -2\|Tv - v\|^2 + \|Tv - v\|^2$$
$$= -\|Tv - v\|^2,$$

which implies that $Tv = v$. This ends the proof. ■

Theorem 1 immediately follows form Theorem 3.

Next, we proceed with a theorem concerning the condition (11). As an additional prior knowledge, note the following:

(d) If $x_n \rightharpoonup v$ and $x_n - y_n \to 0$, then $y_n \rightharpoonup v$, where $\{x_n\}$ and $\{y_n\}$ are sequences in H and $v \in H$.

Theorem 4 *Let C be a nonempty, closed, and convex subset of H. Let T be a mapping from C into itself such that*

$$\lambda\|Tx - Ty\|^2 + (1-\lambda)\left\|T^2x - T^2y\right\|^2 \le \|x - y\|^2 \quad (18)$$

for all $x, y \in C$, where $\lambda \in (0, 1]$. Suppose there exists an element $z \in C$ such that $\{T^n z\}$ is a bounded sequence in C. Then, T has at least one fixed point.

Proof. Define

$$A_n^0 = \frac{1}{n}\sum_{k=0}^{n-1} T^k z, \quad A_n^1 = \frac{1}{n}\sum_{k=1}^{n} T^k z, \quad \text{and} \quad A_n^2 = \frac{1}{n}\sum_{k=2}^{n+1} T^k z.$$

As C is convex, by their definitions, $\{A_n^0\}$, $\{A_n^1\}$ and $\{A_n^2\}$ are sequences in C. As $\{T^n z\}$ is bounded, so is $\{A_n^0\}$. Therefore, there exists a subsequence $\{A_{n_i}^0\}$ of $\{A_n^0\}$ such that $A_{n_i}^0 \rightharpoonup v$ for some $v \in H$. As C is closed and convex, it is weakly closed. From $\{A_{n_i}^0\} \subset C$ and $A_{n_i}^0 \rightharpoonup v$, it follows that $v \in C$. As T is a mapping from C into itself, Tv and T^2v $(\in C)$ exist. Our aim is to prove that $Tv = v$.

Next, let us show that

$$A_{n_i}^1 \rightharpoonup v \quad \text{and} \quad A_{n_i}^2 \rightharpoonup v \quad (19)$$

as $i \to \infty$. Notice that

$$A_n^0 - A_n^1 \to 0 \quad \text{and} \quad A_n^0 - A_n^2 \to 0. \quad (20)$$

Indeed, it holds that

$$
\begin{aligned}
&\|A_n^0 - A_n^2\| \\
&= \left\| \frac{1}{n}\sum_{k=0}^{n-1} T^k z - \frac{1}{n}\sum_{k=2}^{n+1} T^k z \right\| \\
&= \frac{1}{n}\left\| z + Tz + T^2 z + \cdots + T^{n-1} z \right. \\
&\quad \left. - \left(T^2 z + \cdots + T^n z + T^{n+1} z\right) \right\| \\
&= \frac{1}{n}\left\| z + Tz - T^n z - T^{n+1} z \right\|.
\end{aligned}
$$

As $\{T^n z\}$ is bounded, we obtain $A_n^0 - A_n^2 \to 0$ as $n \to \infty$. Similarly, the part $A_n^0 - A_n^1 \to 0$ also follows. As $A_{n_i}^0 \rightharpoonup v$, from (20) and (d), we have $A_{n_i}^1 \rightharpoonup v$ and $A_{n_i}^2 \rightharpoonup v$ as claimed.

From the condition (18), it holds that

$$
\lambda \left\| T^{k+1} z - Tv \right\|^2 + (1-\lambda) \left\| T^{k+2} z - T^2 v \right\|^2 \le \left\| T^k z - v \right\|^2
$$

for all $k = 0, 1, 2, \ldots$ From this,

$$
\begin{aligned}
&\lambda \left(\left\| T^{k+1} z - v \right\|^2 + 2\left\langle T^{k+1} z - v,\ v - Tv \right\rangle + \|v - Tv\|^2 \right) \\
&+ (1-\lambda) \left(\left\| T^{k+2} z - v \right\|^2 + 2\left\langle T^{k+2} z - v,\ v - T^2 v \right\rangle \right. \\
&\left. + \left\| v - T^2 v \right\|^2 \right) \le \left\| T^k z - v \right\|^2 .
\end{aligned}
$$

Consequently, we have

$$
\begin{aligned}
&\lambda \left(\left\| T^{k+1} z - v \right\|^2 - \left\| T^k z - v \right\|^2 \right) \\
&+ (1-\lambda) \left(\left\| T^{k+2} z - v \right\|^2 - \left\| T^k z - v \right\|^2 \right) \\
&+ \lambda \|v - Tv\|^2 + (1-\lambda) \left\| v - T^2 v \right\|^2 \\
&+ 2\lambda \left\langle T^{k+1} z - v,\ v - Tv \right\rangle \\
&+ 2(1-\lambda) \left\langle T^{k+2} z - v,\ v - T^2 v \right\rangle \le 0.
\end{aligned}
$$

Summing these expressions with respect to $k = 0, 1, \cdots, n-1$ and dividing by n, we obtain

$$
\begin{aligned}
&\frac{\lambda}{n} \left(\|T^n z - v\|^2 - \|z - v\|^2 \right) \\
&+ \frac{1-\lambda}{n} \left(\left\| T^{n+1} z - v \right\|^2 + \|T^n z - v\|^2 \right. \\
&\left. - \|Tz - v\|^2 - \|z - v\|^2 \right) \\
&+ \lambda \|v - Tv\|^2 + (1-\lambda) \left\| v - T^2 v \right\|^2 \\
&+ 2\lambda \left\langle A_n^1 - v,\ v - Tv \right\rangle + 2(1-\lambda) \left\langle A_n^2 - v,\ v - T^2 v \right\rangle \le 0,
\end{aligned}
$$

where $n \in \mathbb{N}$. As $\{T^n z\}$ is bounded,

$$
\frac{1}{n} \|T^n z - v\|^2 \to 0 \ \text{ and } \ \frac{1}{n} \left\| T^{n+1} z - v \right\|^2 \to 0.
$$

Replacing n by n_i and taking the limit as $i \to \infty$, we have from (19) that

$$
\lambda \|v - Tv\|^2 + (1-\lambda) \left\| v - T^2 v \right\|^2 \le 0.
$$

Subtracting $(1 - \lambda) \|v - T^2 v\|^2 (\ge 0)$ from the left-hand side results in

$$
\lambda \|v - Tv\|^2 \le 0.
$$

Dividing by $\lambda (> 0)$, we obtain $\|v - Tv\|^2 \le 0$, which implies that $v = Tv$. This completes the proof. ■

In a similar way, the following theorem that is a generalized version of Theorem 4 can be established.

Theorem 5 *Let C be a nonempty, closed, and convex subset of H. Let $L \in N$ and let $\lambda_1, \cdots, \lambda_L \in [0, 1]$ that satisfies $\Sigma_{l=1}^{L} \lambda_l = 1$. Let T be a mapping from C into itself such that*

$$
\sum_{l=1}^{L} \lambda_l \left\| T^l x - T^l y \right\|^2 \le \|x - y\|^2
$$

for all $x, y \in C$. Suppose there exists an element $z \in C$ such that $\{T^n z\}$ is a bounded sequence in C. If $\lambda_1 > 0$, then T has at least one fixed point.

Theorem 5 includes Theorem 4 as the case of $L = 2$. For a proof of the case $L = 3$, see Kondo [15]. Theorem 5 includes Theorem 1 as a special case of $\lambda_1 = 1$ (or $L = 1$), while Theorem 4 also includes Theorem 1 as a case of $\lambda = 1$.

In a recent work Kondo [16], the common fixed point problem was studied for two nonlinear mappings S and T that jointly satisfy

$$
\lambda \|Sx - Sy\|^2 + (1-\lambda) \|Tx - Ty\|^2 \le \|x - y\|^2 \quad (21)
$$

by supposing $ST = TS$. Setting $S = T$ and $T = T^2$ in (21), we obtain the condition (18).

The following is a theorem for mappings characterized by the condition (13). The theorem includes the case for (12) as a special case of $\varepsilon = 0$.

Theorem 6 *Let C be a nonempty, closed, and convex subset of H. Let $\mu \in (0, 1]$ and $\varepsilon \in [0, 1)$. Let T be a mapping from C into itself such that*

$$\begin{aligned}&\mu \left\|Tx - T^2y\right\|^2 + (1-\mu)\left\|T^2x - T^2y\right\|^2 \\ &\le \left\|x - T^2y\right\|^2 + \varepsilon\mu\left\|x - Tx\right\|^2\end{aligned} \tag{22}$$

for all $x, y \in C$. Suppose there exists an element $z \in C$ such that $\{T^n z\}$ is a bounded sequence in C. Then, T has at least one fixed point.

Proof. Define

$$A_n^2 = \frac{1}{n}\sum_{k=2}^{n+1} T^k z.$$

As C is convex, $\{A_n^2\}$ is a sequence in C. As $\{T^n z\}$ is bounded, so is $\{A_n^2\}$. Hence, there exists a subsequence $\{A_{n_i}^2\}$ of $\{A_n^2\}$ such that $A_{n_i}^2 \rightharpoonup v$ for some $v \in H$. As C is closed and convex, it is weakly closed. As $\{A_{n_i}^2\} \subset C$ and $A_{n_i}^2 \rightharpoonup v$, we have $v \in C$. Hence, Tv and T^2v ($\in$ C) exist. Our aim is to show that $Tv = v$.

From (22), it holds that

$$\begin{aligned}&\mu \left\|Tv - T^{k+2}z\right\|^2 + (1-\mu)\left\|T^2v - T^{k+2}z\right\|^2 \\ &\le \left\|v - T^{k+2}z\right\|^2 + \varepsilon\mu\left\|v - Tv\right\|^2\end{aligned}$$

for all $k = 0, 1, 2, \cdots$. This yields

$$\begin{aligned}&\mu\left(\left\|T^{k+2}z - v\right\|^2 + 2\left\langle T^{k+2}z - v,\ v - Tv\right\rangle + \|v - Tv\|^2\right) \\ &+ (1-\mu)\left(\left\|T^{k+2}z - v\right\|^2 + 2\left\langle T^{k+2}z - v,\ v - T^2v\right\rangle\right. \\ &\left.+ \left\|v - T^2v\right\|^2\right) \\ &\le \left\|T^{k+2}z - v\right\|^2 + \varepsilon\mu\|v - Tv\|^2.\end{aligned}$$

It follows that

$$\begin{aligned}&2\mu\left\langle T^{k+2}z - v,\ v - Tv\right\rangle + 2(1-\mu)\left\langle T^{k+2}z - v,\ v - T^2v\right\rangle \\ &+\mu\|v - Tv\|^2 + (1-\mu)\left\|v - T^2v\right\|^2 \\ &\le \varepsilon\mu\|v - Tv\|^2.\end{aligned}$$

Let $n \in \mathbb{N}$. Summing these expressions with respect to $k = 0, 1, \ldots, n-1$ and dividing by n, we obtain

$$\begin{aligned}&2\mu\left\langle A_n^2 - v,\ v - Tv\right\rangle + 2(1-\mu)\left\langle A_n^2 - v,\ v - T^2v\right\rangle \\ &+\mu\|v - Tv\|^2 + (1-\mu)\left\|v - T^2v\right\|^2 \\ &\le \varepsilon\mu\|v - Tv\|^2.\end{aligned}$$

Given that $A_{n_i}^2 \rightharpoonup v$, replacing n by n_i, we have in the limit as $i \to \infty$ that

$$\mu\|v - Tv\|^2 + (1-\mu)\left\|v - T^2v\right\|^2 \le \varepsilon\mu\|v - Tv\|^2.$$

Therefore,

$$\mu(1-\varepsilon)\|v - Tv\|^2 + (1-\mu)\left\|v - T^2v\right\|^2 \le 0.$$

Subtracting $(1-\mu)\ \|v - T^2v\|^2 (\ge 0)$ yields

$$\mu(1-\varepsilon)\|v - Tv\|^2 \le 0.$$

As $\mu \in (0, 1]$ and $\varepsilon \in [0, 1)$, dividing by $\mu(1-\varepsilon)\ (> 0)$, we obtain $\|v - Tv\|^2 \le 0$. This means $v = Tv$, which completes the proof. ■

As stated in the Introduction, fixed point theorems for generalized hybrid mappings (4) and 2-generalized hybrid mappings (5) are provided in Kocourek et al. [11] and Maruyama et al. [21], respectively. The author hopes that readers enjoy discovering various types of fixed point theorems and examples of mappings that have (or do not have) fixed points by referring to the conditions (6) and (10).

Acknowledgements. The author would like to thank the Institute for Economics and Business Research of Shiga University for financial support.

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Review of Fixed Point Theorems for a Certain Class of Nonlinear Mappings in Hilbert Spaces

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In this elementary review article, we discuss a previous study [A. Kondo, Fixed Point Theorem for Generic 2-Generalized Hybrid Mappings in Hilbert Spaces, Topol. Methods Nonlinear Anal. **59**(2B) (2022), 833-849], which deals with a fixed point theorem for a general type of nonlinear mappings called generic 2-generalized hybrid mappings. While presenting some examples of this class of mappings, we demonstrate proofs of fixed point theorems for some simple cases as illustrations. One example presented in this article is not continuous. Some results extend a well-known fixed point theorem for a nonexpansive mapping.

Key Words Fixed point theorem, generic 2-generalized hybrid mappings, nonexpansive mappings, Hilbert space

MSC-Classification 2020: 47H10