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A Linear Approximate Robust Strategic Asset Allocation with Inflation-Deflation Hedging Demand

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Abstract

This study considers a finite-time consumption-investment problem for investors with homothetic robust utility under the quadratic security market model with stochastic volatilities and inflation rates. This leads to a nonlinear nonhomogeneous partial differential equation for indirect utility. We propose a linear approximation method and derive the approximate optimal robust portfolio decomposed into myopic, intertemporal hedging, and inflation-deflation hedging demands. We also propose a method to estimate our quadratic security market model that achieves stability of optimal portfolio estimates. We then apply our estimation method to the two-factor quadratic security market model. Our numerical analysis shows that the market timing effects in the optimal robust allocation are significant and nonlinear and are mainly owing to inflation-deflation hedging demand.
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1 Introduction

To analyze dynamic consumption-investment problems, the establishment of a realistic security market model that captures actual asset price fluctuations is crucial. Prior empirical studies have shown that interest rates, the market price of risk, asset volatilities, and inflation rates are stochastic and

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mean-reverting, with such findings now considered stylized facts. Batbold, Kikuchi, and Kusuda (2022) consider a finite-time consumption-investment problem for investors with constant relative risk aversion (CRRA) utility under a quadratic security market model, wherein all the above-mentioned processes are stochastic and mean-reverting. This class of quadratic security market models is a generalization of affine models (Ahn, Dittmar, and Gallant (2002)) and is independently developed by Ahn *et al.* (2002) and Leippold and Wu (2002). Quadratic security market models are used in security pricing studies, such as Chen, Filipović, and Poor (2004), Kim and Singleton (2012), and Filipović, Gourier, and Mancini (2016). Since a stochastic state process assumed in the quadratic security market model makes the investment opportunity set stochastic, intermediate utility generates a nonhomogeneous term in the linear partial differential equation (PDE) for indirect utility.

Batbold *et al.* (2022) derive a semi-analytical solution of the PDE and obtain an optimal portfolio, which is decomposed into myopic, intertemporal hedging, and "inflation-deflation hedging"[1] demands. The optimal portfolio shows that all demands are nonlinear functions of the state vector and their numerical analysis highlights the nonlinearity and significance of market timing effects. Nonlinearity stems from stochastic volatility, while significance is attributed primarily to inflation-deflation hedging demand, in addition to myopic demand. This result highlights the importance of incorporating both stochastic volatility and stochastic inflation into the security market model in consumption-investment problems. Strategic asset allocation, proposed by Brennan, Schwartz, and Lagnado (1997) and later endorsed by Campbell and Viceira (2002), underscores the magnitude of market timing effects of intertemporal hedging demand, and Campbell and Viceira (1999, 2000) estimate that it is considerably large. However, to date, strategic asset allocation has not been viewed as an effective asset allocation strategy, partly because some empirical analyses including Brandt (1999) and Ang and Bekaert (2002) show that it is small. The result shown by Batbold *et al.* (2022) gives insights on the effectiveness of strategic asset allocation from a different perspective: the market timing effects of inflation-deflation hedging demand.

We should also consider the need for robust dynamic investment control, which has been highlighted by the global financial crisis. Robust utility is proposed by Hansen and Sargent (2001). Investors with robust utility regard the "base probability" as the most likely probability; however, they also consider other probabilities because the true probability is unknown. Robust utility does not possess homotheticity[2] , a property possessed by CRRA

[1] This is called "inflation hedging demand" in Batbold *et al.* (2022). However, we call it "inflation-deflation hedging demand" because the impact of deflation on security markets through unconventional monetary policy is comparable to inflation.

[2] A utility function U is homothetic if, for any consumption plan c and $\tilde{c}$, and any scaler

utility. Maenhout (2004) proposes homothetic robust utility, which is a generalization of CRRA utility.[3] Homothetic robust utility is characterized by relative risk aversion and "relative ambiguity aversion" which stands for the investor's degree of distrust of the base probability. Liu (2010) and Batbold *et al.* (2019) consider the infinite-time consumption-investment problem with homothetic robust utility under a stochastic opportunity set. Then, the nonhomogeneous term as well as the nonlinear term appears in the PDE.[4] Campbell and Viceira (2002) note that in the infinite-time problem, the nonhomogeneous term is equal to the stable optimal consumption-wealth ratio; thus, they use a loglinear approximation[5] of the nonhomogeneous term to derive an approximate solution. Liu (2010) apply the loglinear approximation method of Campbell and Viceira (2002), and Batbold *et al.* (2019) use another loglinear approximation method to derive another approximate solution. All the above-mentioned studies consider infinite-time consumption-investment problems.

This study considers the finite-time consumption-investment problem for investors with homothetic robust utility under the quadratic security market model of Batbold *et al.* (2022). Under the finite-time setting, the nonhomogeneous term is time-dependent and unstable. Therefore, we propose a time-dependent linear approximation method. This study aims to derive a time-dependent linear approximate solution and analyze important properties such as the market timing effects of approximate optimal robust portfolios. The main results of this study are summarized as follows.

First, we propose a time-dependent linear approximation method, which linearly approximates the nonlinear term in the nonlinear nonhomogeneous PDE using a linear function with time-dependent coefficients. We apply the method to the nonlinear PDE and derive ordinary differential equations (ODEs) for the unknown coefficients constituting the approximate solution. Then, we derive the approximate optimal robust portfolio decomposed into myopic, intertemporal hedging, and inflation-deflation hedging demands. The optimal robust portfolio is proportional to the inverse of the volatility matrix which is a nonlinear function of the state process. Thus, as the determinant of the volatility matrix approaches zero, the inverse of the volatility

$\alpha > 0$, $U(\alpha\tilde{c}) \geq U(\alpha c) \Leftrightarrow U(\tilde{c}) \geq U(c)$.

[3] Homothetic robust utility has been applied to robust control studies including Skiadas (2003), Maenhout (2006), Liu (2010), Branger, Larsen, and Munk (2013), Munk and Rubtsov (2014), Yi, Viens, Law, and Li (2015), and Batbold, Kikuchi, and Kusuda (2019).

[4] In contrast, Maenhout (2006), Branger *et al.* (2013), Munk and Rubtsov (2014), and Yi *et al.* (2015) study dynamic portfolio choice problems under both homothetic robust utility and stochastic investment opportunity set. In this case, the nonlinear term appears in the PDE. However, given that there is no intermediate consumption in the portfolio selection problem, the nonhomogeneous term does not appear. Therefore, the PDE is homogeneous, and an exact solution is obtained.

[5] Their loglinear approximation method is a continuous-time version of the method proposed by Campbell (1993) in the discrete-time model.

matrix diverges and the estimates of the optimal robust portfolio become unstable.

Second, we propose an estimation method of our quadratic security market model, which realizes a considerably large likelihood with stable optimal portfolio estimates. We add, to the negative quasi-loglikelihood, the absolute value of the determinant of the inverse of the volatility matrix as a regularization term. Then, we estimate the two-factor quadratic security market model by minimizing the negative quasi-likelihood with the regularization term based on a nonlinear Kalman filter.

Third, we assume a long-term investor who plans to invest in the S&P 500 and 10-year Treasury Inflation-Protected Securities (TIPS), in addition to the money market account. Our numerical analysis shows that the market timing effects in optimal robust allocation to the S&P 500 are significant and nonlinear and are primarily due to inflation-deflation hedging demand. Moreover, the market timing effects in optimal allocation to the TIPS are substantially large and highly nonlinear. All demands contribute to the market timing effect, with inflation-deflation hedging demand contributing the most. Inflation-deflation hedging demand is interpreted as being amplified by monetary policy, especially the quantitative easing implemented by the Fed against the backdrop of deflationary concerns after the global financial crisis. In the market timing effects claimed by strategic asset allocation, we have focused on intertemporal hedging demand and ignored inflation-deflation hedging demand; however, this numerical analysis suggests that inflation-deflation hedging demand does matter in strategic asset allocation.

The remainder of this paper is organized as follows. In Section 2, we explain the quadratic security market model and introduce the investor's robust consumption-investment control problem to derive the PDE for indirect utility. In Section 3, we propose the time-dependent linear approximation method and derive the approximate optimal robust portfolio. In Section 4, we propose the estimation method for the quadratic security market model and show the estimation results. In Section 5, we present the results of the numerical analysis. In Section 6, we conclude this study and address future research issues. Finally, the Appendix shows proofs of the lemmas and propositions.

2 Quadratic Security Market Model and the PDE for Indirect Utility

In this section, we first introduce the quadratic security market model assumed by Batbold *et al.* (2022) and the robust consumption-investment problem. Then, we derive the PDE for indirect utility.

2.1 Quadratic Security Market Model

We consider frictionless U.S. markets over the period $[0, T^*]$. Investors' common subjective probability and information structures are modeled by a complete filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathrm{P})$, where $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,\infty)}$ is the natural filtration generated by an N-dimensional standard Brownian motion B_t. We denote the expectation operator under P by E and the conditional expectation operator given $\mathcal{F}_t$ by E_t.

There are markets for a consumption commodity and securities at every date $t \in [0, \infty)$, and the consumer price index p_t is observed. The traded securities comprise the instantaneously nominal risk-free security called the money market account, and a continuum of zero-coupon bonds and zero-coupon inflation-indexed bonds whose maturity dates are $(t, t + \tau^*]$. Each zero-coupon bond has a 1 USD payoff at maturity, and each zero-coupon inflation-indexed bond has a p_T USD payoff at maturity T. Moreover, J types of non-bond indices (stock indices, REIT indices, *etc.*) are also traded.

At every date t, P_t, P_t^T, Q_t^T, and S_t^j denote the USD prices of the money market account, zero-coupon bond with maturity date T, zero-coupon inflation-indexed bond with maturity date T, and the j-th index, respectively. Let A' and I_N denote the transposition of A and $N \times N$ identity matrix, respectively.

We assume the following quadratic security market model introduced by Batbold *et al.* (2022).

Assumption 1. *Let $(\rho_0, \iota_0, \delta_{0j}, \sigma_{0j})$ and $(\lambda, \rho, \iota, \sigma_p, \delta_j, \sigma_j)$ denote scalers and N-dimensional vectors, respectively.*

1. *State vector process X_t satisfies the following stochastic differential equation (SDE):*
$$dX_t = -\mathcal{K}X_t\,dt + I_N\,dB_t, \tag{2.1}$$
where $\mathcal{K}$ is an $N \times N$ positive lower triangular matrix.

2. *The market price λ_t of risk and the instantaneous nominal risk-free rate r_t are provided as*
$$\begin{aligned} \lambda_t &= \lambda + \Lambda X_t, & (2.2)\\ r_t &= \rho_0 + \rho' X_t + \frac{1}{2}X_t'\mathcal{R}X_t, & (2.3)\end{aligned}$$
where Λ is an $N \times N$ matrix such that $\mathcal{K} + \Lambda$ is regular, $\mathcal{R}$ is a positive-definite symmetric matrix, and[6]
$$\rho_0 \geq \frac{1}{2}\rho'\mathcal{R}^{-1}\rho. \tag{2.4}$$

[6]Condition (2.4) ensures that the instantaneous nominal risk-free rate is non-negative.

3. *The consumer price index p_t satisfies*

$$\frac{dp_t}{p_t} = i_t\,dt + (\sigma_t^p)' dB_t, \qquad p_0 = 1, \tag{2.5}$$

where i_t and σ_t^p are given by

$$\begin{aligned} i_t &= \iota_0 + \iota' X_t + \frac{1}{2} X_t' \mathcal{I} X_t, & (2.6)\\ \sigma_t^p &= \sigma_p + \Sigma_p X_t. & (2.7) \end{aligned}$$

For eq.(2.6), $\mathcal{I}$ is a positive-definite symmetric matrix and matrix $\bar{\mathcal{R}}$ defined by

$$\bar{\mathcal{R}} = \mathcal{R} - \mathcal{I} + \Sigma_p' \Lambda + \Lambda' \Sigma_p \tag{2.8}$$

is positive-definite.

4. *The dividend rate of the j-th index is given by*

$$D_t^j = \left(\delta_{0j} + \delta_j' X_t + \frac{1}{2} X_t' \Delta_j X_t\right) \exp\left(\sigma_{0j} t + \sigma_j' X_t + \frac{1}{2} X_t' \Sigma_j X_t\right), \tag{2.9}$$

where $(\delta_{0j}, \delta_j, \Delta_j)$ is such that Δ_j is a positive definite symmetric matrix and[7]

$$\delta_{0j} \geq \frac{1}{2} \delta_j' \Delta_j^{-1} \delta_j. \tag{2.10}$$

Note that $\delta_{0j} + \delta_j' X_t + \frac{1}{2} X_t' \Delta_j X_t$ is the instantaneous dividend rate.

5. *Markets are complete and arbitrage-free.*

2.2 No-arbitrage Dynamics of Security Prices and Real Budget Constraint

We define the real market price $\bar{\lambda}_t$ of risk and the real instantaneous interest rate $\bar{r}_t$ by

$$\begin{aligned} \bar{\lambda}_t &= \lambda_t - \sigma_t^p, & (2.11)\\ \bar{r}_t &= r_t - i_t + \lambda_t' \sigma_t^p. & (2.12) \end{aligned}$$

Note that the real market price of risk is an affine function of X_t, whereas $\bar{r}_t$ is a quadratic function of X_t:

$$\begin{aligned} \bar{\lambda}_t &= \bar{\lambda} + \bar{\Lambda} X_t, & (2.13)\\ \bar{r}_t &= \bar{\rho}_0 + \bar{\rho}' X_t + \frac{1}{2} X_t' \bar{\mathcal{R}} X_t, & (2.14) \end{aligned}$$

[7] Condition (2.10) ensures that dividend rates are non-negative processes.

where $\bar{\mathcal{R}}$ is given by eq. (2.8) and

$$\begin{aligned}
\bar{\lambda} &= \lambda - \sigma_p, && (2.15)\\
\bar{\Lambda} &= \Lambda - \Sigma_p, && (2.16)\\
\bar{\rho}_0 &= \rho_0 - \iota_0 + \lambda'\sigma^p, && (2.17)\\
\bar{\rho} &= \rho - \iota + \Lambda'\sigma_p + \Sigma_p'\lambda. && (2.18)
\end{aligned}$$

Let Φ_t^j denote the portfolio weight on the j-th index. Regarding the default-free bond, let $\varphi_t(\tau)$ and $\varphi_t^Q(\tau)$ denote the densities of the portfolio weights of the default-free and inflation-indexed bonds with τ-time to maturity. We assume that the functional space of the portfolio weights' densities includes the set of distributions, such as the Dirac delta function.

Let c_t and $\bar{W}_t$ denote the consumption rate process and the real wealth process, respectively. Batbold *et al.* (2022) show the SDEs of no-arbitrage security price processes and the real budget constraint.

Lemma 1. *Let $\tau = T - t$ denote the time to maturity of bond P_t^T or inflation-indexed bond Q_t^T. Under Assumption 1, the dynamics of security price processes and the real budget constraint satisfy the following:*

1. *The SDEs of security price processes:*

   (i) The default-free bond with time τ to maturity:

$$P_t^T = \exp\left(\sigma_0(\tau) + \sigma(\tau)'X_t + \frac{1}{2}X_t'\Sigma(\tau)X_t\right), \qquad (2.19)$$

where

$$\begin{aligned}
\frac{d\Sigma(\tau)}{d\tau} &= \Sigma(\tau)^2 - (\mathcal{K} + \Lambda)'\Sigma(\tau) - \Sigma(\tau)(\mathcal{K} + \Lambda) - \mathcal{R},(2.20)\\
\frac{d\sigma(\tau)}{d\tau} &= -(\mathcal{K} + \Lambda - \Sigma(\tau))'\sigma(\tau) - (\Sigma(\tau)\lambda + \rho), \quad (2.21)\\
\frac{d\sigma_0(\tau)}{d\tau} &= -\lambda'\sigma(\tau) + \frac{1}{2}\big(\sigma(\tau)'\sigma(\tau) + \mathrm{tr}\big[\Sigma(\tau)\big]\big) - \rho_0,(2.22)
\end{aligned}$$

with $(\Sigma, \sigma, \sigma_0)(0) = (0, 0, 0)$.

   (ii) The default-free inflation-indexed bond with time τ to maturity:

$$Q_t^T = \exp\left(\sigma_{0q}(\tau) + \sigma_q(\tau)'X_t + \frac{1}{2}X_t'\Sigma_q(\tau)X_t\right), \qquad (2.23)$$

where

$$\begin{aligned}
\frac{d\Sigma_q(\tau)}{d\tau} &= \Sigma_q(\tau)^2 - (\mathcal{K} + \bar{\Lambda})'\Sigma_q(\tau) - \Sigma_q(\tau)(\mathcal{K} + \bar{\Lambda}) - \bar{\mathcal{R}},(2.24)\\
\frac{d\sigma_q(\tau)}{d\tau} &= -(\mathcal{K} + \bar{\Lambda} - \Sigma_q(\tau))'\sigma_q(\tau) - (\Sigma_q(\tau)\bar{\lambda} + \bar{\rho}), \quad (2.25)\\
\frac{d\sigma_{0q}(\tau)}{d\tau} &= -\bar{\lambda}'\sigma_q(\tau) + \frac{1}{2}\big(\sigma_q(\tau)'\sigma_q(\tau) + \mathrm{tr}\big[\Sigma_q(\tau)\big]\big) - \bar{\rho}_0(2.26)
\end{aligned}$$

with $(\Sigma_q, \sigma_q, \sigma_{0q})(0) = (0, 0, 0)$.

(iii) The j-th index:

$$S_t^j = \exp\left(\sigma_{0j} t + \sigma_j' X_t + \frac{1}{2} X_t' \Sigma_j X_t\right), \tag{2.27}$$

where

$$\Sigma_j^2 - (\mathcal{K} + \Lambda)' \Sigma_j - \Sigma_j (\mathcal{K} + \Lambda) + \Delta_j - \mathcal{R}_j = 0, \tag{2.28}$$

$$\sigma_j = (\mathcal{K} + \Lambda - \Sigma_j)'^{-1} (\delta_j - \rho - \Sigma_j \lambda), \tag{2.29}$$

$$\sigma_{0j} = \lambda' \sigma_j - \frac{1}{2}\big(\sigma_j' \sigma_j + \mathrm{tr}\big[\Sigma_j\big]\big) + \rho_0 - \delta_{0j}. \tag{2.30}$$

2. The real budget constraint given $(c_t, \bar{\sigma}_t)$:

$$\frac{d\bar{W}_t}{\bar{W}_t} = \left(\bar{r}_t + \bar{\sigma}_t' \bar{\lambda}_t - \frac{c_t}{\bar{W}_t}\right) dt + \bar{\sigma}_t' \, dB_t, \tag{2.31}$$

where

$$\bar{\sigma}_t = \int_0^{\tau^*} \big\{ \varphi_t(\tau)(\sigma(\tau) + \Sigma(\tau) X_t) + \varphi_t^Q(\tau)(\sigma_q(\tau) + \Sigma_q(\tau) X_t) \big\} \, d\tau + \sum_{j=1}^{J} \Phi_t^j (\sigma_j + \Sigma_j X_t) - \sigma_t^p. \tag{2.32}$$

Proof. See Appendix A.1 and A.2 in Batbold *et al.* (2022). □

The real budget constraint (2.31) indicates that $(c_t, \bar{\sigma}_t)$ is the control in the optimal consumption-investment problem. Let $\mathbf{X}_t = (\bar{W}_t, X_t')'$ and let $\bar{W}_0 > 0$. We call the control satisfying the budget constraint (2.31) with initial state $\mathbf{X}_0 = (\bar{W}_0, X_0')'$ the admissible control and denote the set of admissible controls by $\mathcal{B}(\mathbf{X}_0)$.

2.3 Homothetic Robust Utility and Robust Control Problem

An investor with homothetic robust utility regards probability P ("base probability") as the most likely probability, but they also consider other probabilities, because the true probability is unknown. Thus, the investor assumes set $\mathbb{P}$ of all equivalent probability measures[8] as alternative probabilities. According to Girsanov's theorem, any equivalent probability measure is characterized by a measurable process ξ_t with Novikov's integrable condition as the following Radon–Nikodym derivative:

$$\frac{d\mathrm{P}^\xi}{d\mathrm{P}} = \exp\left(\int_0^{T^*} \xi_t \, dB_t - \frac{1}{2} \int_0^{T^*} |\xi_t|^2 dt\right). \tag{2.33}$$

[8] A probability measure $\tilde{\mathrm{P}}$ is said to be an equivalent probability measure of P if and only if $\mathrm{P}(A) = 0 \Leftrightarrow \tilde{\mathrm{P}}(A) = 0$.

Therefore, the investor decides the worst-case probability, which minimizes their utility among $\mathbb{P}$ for every consumption plan. In other words, the investor rationally determines the worst-case probability, considering deviations from P, as follows:

$$U(c) = \inf_{\mathrm{P}^\xi \in \mathbb{P}} \mathrm{E}^\xi \left[\int_0^{T^*} e^{-\beta t} \left(\alpha \frac{c_t^{1-\gamma}}{1-\gamma} + \frac{(1-\gamma)U_t}{2\theta} |\xi_t|^2 \right) dt + (1-\alpha) e^{-\beta T^*} \frac{c_{T^*}^{1-\gamma}}{1-\gamma} \right], \tag{2.34}$$

[9]

where E^ξ is the expectation under P^ξ, β is the subjective discount rate, $\gamma > 1$ is the relative risk aversion coefficient, $\alpha \in [0, 1]$ represents the relative importance of the intermediate and terminal utility, $\theta > 0$ is termed as the relative ambiguity aversion coefficient, and U_t is the utility process defined recursively as follows:

$$U_t = \mathrm{E}_t^\xi \left[\int_t^{T^*} e^{-\beta(s-t)} \left(\alpha \frac{c_s^{1-\gamma}}{1-\gamma} + \frac{(1-\gamma)U_s}{2\theta} |\xi_s|^2 \right) ds + (1-\alpha) e^{-\beta(T^*-t)} \frac{c_{T^*}^{1-\gamma}}{1-\gamma} \right]. \tag{2.35}$$

Assumption 2. *The investor maximizes the following homothetic robust utility under the real budget constraint (2.31).*

The investor's consumption-investment problem and the value function are defined by

$$V(\mathbf{X}_0) = \sup_{(c,\bar{\sigma}) \in \mathcal{B}(\mathbf{X}_0)} \inf_{\mathrm{P}^\xi \in \mathbb{P}} U_0. \tag{2.36}$$

When $\alpha = 0$, the above problem is termed a portfolio choice problem.

Let $\tilde{U}_t = e^{-\beta t} U_t$. Then, eq.(2.35) is rewritten as

$$\tilde{U}_t = \mathrm{E}_t^\xi \left[\int_t^{T^*} \left(\alpha\, e^{-\beta s} \frac{c_s^{1-\gamma}}{1-\gamma} + \frac{(1-\gamma)\tilde{U}_s}{2\theta} |\xi_s|^2 \right) ds + (1-\alpha) e^{-\beta T^*} \frac{c_{T^*}^{1-\gamma}}{1-\gamma} \right]. \tag{2.37}$$

2.4 Optimal Robust Control and PDE for Indirect Utility

As the standard Brownian motion under P^ξ is given by $B_t^\xi = B_t - \int_0^t \xi_s \, ds$, the SDE (2.1) for the state vector under P^ξ is rewritten as

$$d\mathbf{X}_t = \left(\begin{pmatrix} \bar{W}_t(\bar{r}_t + \bar{\sigma}_t' \bar{\lambda}_t) - c_t \\ -\mathcal{K} X_t \end{pmatrix} + \begin{pmatrix} \bar{W}_t \bar{\sigma}_t' \\ I_N \end{pmatrix} \xi_t \right) dt + \begin{pmatrix} \bar{W}_t \bar{\sigma}_t' \\ I_N \end{pmatrix} dB_t^\xi. \tag{2.38}$$

[9]This representation of homothetic robust utility utilizes the expression shown by Skiadas (2003).

Let J denote the discounted indirect utility function. Then, the Hamilton–Jacobi–Bellman (HJB) equation for problem (2.36) is expressed as

$$\sup_{(c,\bar{\sigma})\in\mathcal{B}(\mathbf{X}_0)} \inf_{\mathrm{P}^{\xi}\in\mathbb{P}} \Bigg\{ J_t + \begin{pmatrix} \bar{W}_t\left(\bar{r}_t + \bar{\sigma}_t'\bar{\lambda}_t\right) - c_t \\ -\mathcal{K}X_t \end{pmatrix}' \begin{pmatrix} J_W \\ J_X \end{pmatrix}$$
$$+ \frac{1}{2}\operatorname{tr}\left[\begin{pmatrix} \bar{W}_t\bar{\sigma}_t' \\ I_N \end{pmatrix} \begin{pmatrix} \bar{W}_t\bar{\sigma}_t' \\ I_N \end{pmatrix}' \begin{pmatrix} J_{WW} & J_{WX} \\ J_{XW} & J_{XX} \end{pmatrix} \right]$$
$$+ \alpha\, e^{-\beta t}\frac{c_t^{1-\gamma}}{1-\gamma} + \frac{(1-\gamma)J}{2\theta}|\xi_t|^2 + \xi_t' \begin{pmatrix} \bar{W}_t\bar{\sigma}_t' \\ I_N \end{pmatrix}' \begin{pmatrix} J_W \\ J_X \end{pmatrix} \Bigg\} = 0, \quad (2.39)$$
$$\text{s.t.} \quad J(T^*, \mathbf{X}_{T^*}) = (1-\alpha)e^{-\beta T^*}\frac{\bar{W}_{T^*}^{1-\gamma}}{1-\gamma}.$$

Let $\tau = T^* - t$, hereafter. We have the following lemma.

Lemma 2. *Under Assumptions 1 and 2, the indirect utility function, optimal consumption, and optimal investment for the problem (2.36) satisfy eqs.(2.40), (2.41), and (2.42), respectively. Here, G in eq.(2.40) is a solution of the PDE (2.43).*

$$J(t,\mathbf{X}_t) = e^{-\beta t}\frac{\bar{W}_t^{1-\gamma}}{1-\gamma}\left(G(\tau,X_t)\right)^{\gamma}, \quad (2.40)$$
$$c_t^* = \alpha^{\frac{1}{\gamma}}\frac{\bar{W}_t^*}{G(\tau,X_t)}, \quad (2.41)$$
$$\bar{\sigma}_t^* = \frac{1}{\gamma+\theta}\bar{\lambda}_t + \left(1 - \frac{1}{\gamma+\theta}\right)\frac{\gamma}{\gamma-1}\frac{G_X(\tau,X_t)}{G(\tau,X_t)}, \quad (2.42)$$

$$\frac{G_\tau}{G} = \frac{1}{2}\operatorname{tr}\left[\frac{G_{XX}}{G}\right] + \frac{\theta}{2(\gamma-1)(\gamma+\theta)}\left|\frac{G_X}{G}\right|^2 - \left(\mathcal{K}X_t + \frac{\gamma+\theta-1}{\gamma+\theta}\bar{\lambda}_t\right)'\frac{G_X}{G}$$
$$+ \frac{\alpha^{\frac{1}{\gamma}}}{G} - \frac{\gamma-1}{2\gamma(\gamma+\theta)}|\bar{\lambda}_t|^2 - \frac{\gamma-1}{\gamma}\bar{r}_t - \frac{\beta}{\gamma}, \quad G(0,X_{T^*}) = (1-\alpha)^{\frac{1}{\gamma}}. \quad (2.43)$$

Proof. See Appendix A.1. □

Remark 1. *The PDE (2.43) is rewritten as*

$$G_\tau = \frac{1}{2}\operatorname{tr}\left[G_{XX}\right] + \frac{\theta}{2(\gamma-1)(\gamma+\theta)}\frac{G_X'}{G}G_X - \left(\mathcal{K}X_t + \frac{\gamma+\theta-1}{\gamma+\theta}\bar{\lambda}_t\right)' G_X$$
$$- \left(\frac{\gamma-1}{2\gamma(\gamma+\theta)}|\bar{\lambda}_t|^2 + \frac{\gamma-1}{\gamma}\bar{r}_t + \frac{\beta}{\gamma}\right)G + \alpha^{\frac{1}{\gamma}}, \quad G(0,X_{T^*}) = (1-\alpha)^{\frac{1}{\gamma}}.$$
$$(2.44)$$

The PDE (2.44) shows that in the case of CRRA utility, i.e., $\theta = 0$, the nonlinear term vanishes and the PDE is linear. In the next section, we show how to linearly approximate this nonlinear term.

Example 1. *Suppose an investor plans to invest in the 10-year TIPS* $Q_t(10)$ *and S&P 500* S_t *in addition to the money market account. Let* Φ_t *and* $\Sigma(X_t)$ *denote the optimal portfolio weights and volatility matrix, respectively.*

$$\Phi_t = \begin{pmatrix} \Phi_t^Q(10) \\ \Phi_t \end{pmatrix}, \quad \Sigma(X_t) = \begin{pmatrix} (\sigma_q(10) + \Sigma_q(10)X_t)' \\ (\sigma + \Sigma X_t)' \end{pmatrix}. \tag{2.45}$$

Let Φ_t^* *denote the optimal robust portfolio weights. Then, from eqs.(2.11), (2.32) and (2.42),* Φ_t^* *is given by*

$$\begin{aligned}\Phi_t^* = {} & \frac{1}{\gamma+\theta}\Sigma(X_t)'^{-1}\Big(\lambda + \Lambda X_t\Big) \\ & + \left(1 - \frac{1}{\gamma+\theta}\right)\frac{\gamma}{\gamma-1}\Sigma(X_t)'^{-1}\frac{G_X(\tau, X_t)}{G(\tau, X_t)} \\ & + \left(1 - \frac{1}{\gamma+\theta}\right)\Sigma(X_t)'^{-1}\Big(\sigma_p + \Sigma_p X_t\Big).\end{aligned} \tag{2.46}$$

Remark 2. *In eq.(2.46), the optimal robust portfolio is decomposed into the three demands. The first and second ones are myopic and intertemporal hedging demands, respectively. We call the third one inflation-deflation hedging demand. All the demands are nonlinear functions of the state vector because the inverse matrix of volatilities is its nonlinear function. This suggests that market timing effects cannot be incorporated into the portfolio without dynamically rebalancing the portfolio weights among risky securities in response to various phases created by the variation of the state vector process. The importance of such market timing effects is highlighted by Batbold et al. Batbold* et al. *(2022) in the optimal portfolio for CRRA utility.*

Remark 3. *Eq.(2.46) shows that the optimal robust portfolio is proportional to* $\Sigma(X_t)'^{-1}$*. Thus, as the determinant of the matrix* $\Sigma(X_t)'$ *approaches zero, the inverse matrix* $\Sigma(X_t)'^{-1}$ *diverges and the estimates of the optimal robust portfolio become unstable. In Section 4, we propose a method to estimate our quadratic security market model that achieves stability of the optimal robust portfolio estimates.*

3 Linear Approximate Robust Optimal Portfolio

In this section, we first introduce a time-dependent linear approximation method. Then, we derive the approximate optimal robust portfolio.

3.1 Time-dependent Linear Approximation

In the PDE (2.44), inserting eqs.(2.11) and (2.12) into $\bar{\lambda}_t$ and $\bar{r}_t$, respectively, leads to the following PDE:

$$G_\tau = \frac{1}{2}\operatorname{tr}\left[G_{XX}\right] + \frac{\theta}{2(\gamma-1)(\gamma+\theta)}\frac{G_X'}{G}G_X - \left(\mathcal{K}X_t + \frac{\gamma+\theta-1}{\gamma+\theta}\left(\bar{\lambda}+\bar{\Lambda}X_t\right)\right)' G_X$$
$$- \left\{\frac{\gamma-1}{2\gamma(\gamma+\theta)}|\bar{\lambda}+\bar{\Lambda}X_t|^2 + \frac{\gamma-1}{\gamma}\left(\bar{\rho}_0 + \bar{\rho}'X_t + \frac{1}{2}X_t'\bar{\mathcal{R}}X_t\right) + \frac{\beta}{\gamma}\right\} G + \alpha^{\frac{1}{\gamma}},$$
$$G(0, X_{T^*}) = (1-\alpha)^{\frac{1}{\gamma}}. \quad (3.1)$$

Let $\tilde{G}$ denote our time-dependent linear approximate solution of the PDE (3.1). We approximate $\dfrac{G_X}{G}$ in the nonlinear term of the PDE (3.1) by a linear function of X_t:

$$\frac{\tilde{G}_X}{\tilde{G}} \approx a(\tau) + A(\tau)X_t, \quad (3.2)$$

where $(a(\tau), A(\tau))$ is specified at the end of this subsection.

Then, we obtain the following approximate nonhomogeneous linear PDE:

$$\tilde{G}_\tau = \mathcal{L}\tilde{G} + \frac{\theta}{2(\gamma-1)(\gamma+\theta)}\left(a(\tau)+A(\tau)X_t\right)'\tilde{G}_X + \alpha^{\frac{1}{\gamma}}, \quad \tilde{G}(0,X) = (1-\alpha)^{\frac{1}{\gamma}}, \quad (3.3)$$

where $\mathcal{L}$ is the linear differential operator defined by

$$\mathcal{L}\tilde{G} = \frac{1}{2}\operatorname{tr}\left[\tilde{G}_{XX}\right] - \left(\mathcal{K}X_t + \frac{\gamma+\theta-1}{\gamma+\theta}\left(\bar{\lambda}+\bar{\Lambda}X_t\right)\right)'\tilde{G}_X$$
$$- \left\{\frac{\gamma-1}{2\gamma(\gamma+\theta)}|\bar{\lambda}+\bar{\Lambda}X_t|^2 + \frac{\gamma-1}{\gamma}\left(\bar{\rho}_0 + \bar{\rho}'X_t + \frac{1}{2}X_t'\bar{\mathcal{R}}X_t\right) + \frac{\beta}{\gamma}\right\}\tilde{G}. \quad (3.4)$$

To solve the nonhomogeneous linear PDE (3.3), we first consider the following homogeneous linear PDE:

$$\tilde{g}_\tau = \mathcal{L}\tilde{g} + \frac{\theta}{2(\gamma-1)(\gamma+\theta)}\left(a(\tau)+A(\tau)X_t\right)'\tilde{g}_X, \quad \tilde{g}(0,X) = 1. \quad (3.5)$$

An analytical solution of the PDE (3.5) is expressed as

$$\tilde{g}(\tau, X) = \exp\left(b_0(\tau) + b(\tau)'X + \frac{1}{2}X'B(\tau)X\right), \quad (3.6)$$

where $B(\tau)$ is a symmetric matrix. Then, a semi-analytical solution of the PDE (3.3) is expressed as

$$\tilde{G}(\tau, X_t) = \alpha^{\frac{1}{\gamma}}\int_0^\tau \tilde{g}(s, X_t)\, ds + (1-\alpha)^{\frac{1}{\gamma}}\tilde{g}(\tau, X_t). \quad (3.7)$$

Define $b^*(\tau, X_t)$ and $B^*(\tau, X_t)$ by

$$
\begin{aligned}
b^*(\tau, X_t) &= \frac{1}{\tilde{G}(\tau, X_t)} \left(\int_0^\tau \alpha^{\frac{1}{\gamma}} \tilde{g}(s, X_t) b(s)\, ds + (1-\alpha)^{\frac{1}{\gamma}} \tilde{g}(\tau, X_t) b(\tau) \right), \\
B^*(\tau, X_t) &= \frac{1}{\tilde{G}(\tau, X_t)} \left(\int_0^\tau \alpha^{\frac{1}{\gamma}} \tilde{g}(s, X_t) B(s)\, ds + (1-\alpha)^{\frac{1}{\gamma}} \tilde{g}(\tau, X_t) B(\tau) \right).
\end{aligned}
\tag{3.8}
$$

In eq.(3.2), we set $(a(\tau), A(\tau)) = (b^*(\tau, 0), B^*(\tau, 0))$, that is,

$$
\frac{\tilde{G}_X}{\tilde{G}} \approx b^*(\tau, 0) + B^*(\tau, 0) X_t. \tag{3.9}
$$

3.2 Linear Approximate Solution

Define functions m_2, m_1, and m_0 by

$$
\begin{aligned}
m_2(B) &= B^2 - \left(\mathcal{K} + \frac{\gamma+\theta-1}{\gamma+\theta}\bar{\Lambda}\right)' B - B\left(\mathcal{K} + \frac{\gamma+\theta-1}{\gamma+\theta}\bar{\Lambda}\right) - \frac{\gamma-1}{\gamma}\left(\frac{1}{\gamma+\theta}\bar{\Lambda}'\bar{\Lambda} + \bar{\mathcal{R}}\right), \\
m_1(B, b) &= \left(B - \left(\mathcal{K} + \frac{\gamma+\theta-1}{\gamma+\theta}\bar{\Lambda}\right)'\right) b, - \frac{\gamma+\theta-1}{\gamma+\theta} B\bar{\lambda} - \frac{\gamma-1}{\gamma}\left(\frac{1}{\gamma+\theta}\bar{\Lambda}'\bar{\lambda} + \bar{\rho}\right), \\
m_0(B, b) &= \frac{1}{2}\big(\mathrm{tr}[B] + |b|^2\big) - \frac{\gamma+\theta-1}{\gamma+\theta}\bar{\lambda}' b - \frac{1}{\gamma}\left(\frac{\gamma-1}{2(\gamma+\theta)}|\bar{\lambda}|^2 + (\gamma-1)\bar{\rho}_0 + \beta\right).
\end{aligned}
\tag{3.10}
$$

The solution of the approximate PDE (3.3) is called the linear approximate optimal control and is denoted by $(\tilde{c}^*, \tilde{\sigma}^*)$. We obtain the following proposition.

Proposition 1. *Under Assumptions 1 and 2, the linear approximate optimal consumption and investment for problem (2.36) satisfy eqs.(3.11) and (3.12), respectively.*

$$
\tilde{c}_t^* = \frac{\alpha^{\frac{1}{\gamma}} W_t^*}{\alpha^{\frac{1}{\gamma}} \displaystyle\int_0^\tau g(s, X_t)\, ds + (1-\alpha)^{\frac{1}{\gamma}} g(T^* - t, X_{T^* - t})}, \tag{3.11}
$$

where g is given by eq.(3.6), and

$$
\tilde{\sigma}_t^* = \frac{1}{\gamma+\theta}\big(\bar{\lambda} + \bar{\Lambda} X_t\big) + \left(1 - \frac{1}{\gamma+\theta}\right)\frac{\gamma}{\gamma-1}\big(b^*(\tau, X_t) + B^*(\tau, X_t) X_t\big), \tag{3.12}
$$

where (b^*, B^*) is given by eq.(3.8), and (B, b, b_0) is a solution of the system *of ODEs:*

$$\begin{aligned}
\frac{dB}{d\tau} &= m_2(B) + \frac{\theta}{(\gamma-1)(\gamma+\theta)} B^*(\tau,0)' B(\tau),\\
\frac{db}{d\tau} &= m_1(B,b) + \frac{\theta}{2(\gamma-1)(\gamma+\theta)} \big(B^*(\tau,0)' b(\tau) + B(\tau)' b^*(\tau,0)\big),\\
\frac{db_0}{d\tau} &= m_0(B,b) + \frac{\theta}{2(\gamma-1)(\gamma+\theta)} b^*(\tau,0)' b(\tau),
\end{aligned} \tag{3.13}$$

with $(B(0), b(0), b_0(0)) = (0,0,0)$*, where* (m_2, m_1, m_0) *is given by eq.(3.10).*

Proof. See Appendix A.2. □

Let us consider the investor in Example 1. Let $\tilde{\Phi}^*_t$ denote the approximate optimal portfolio weights. Then, from eqs.(2.11), (2.32), and (3.12), the loglinear approximate optimal portfolio weights are given by

$$\begin{aligned}
\tilde{\Phi}^*_t = {} & \frac{1}{\gamma+\theta}\Sigma(X_t)'^{-1}\Big(\lambda + \Lambda X_t\Big)\\
& + \left(1 - \frac{1}{\gamma+\theta}\right)\frac{\gamma}{\gamma-1}\Sigma(X_t)'^{-1}\Big(b^*(\tau, X_t) + B^*(\tau, X_t) X_t\Big)\\
& + \left(1 - \frac{1}{\gamma+\theta}\right)\Sigma(X_t)'^{-1}\Big(\sigma_p + \Sigma_p X_t\Big).
\end{aligned} \tag{3.14}$$

Note that in eq.(3.14), myopic and inflation-deflation hedging demands are rigorously evaluated, while intertemporal hedging demand is approximated.

4 Estimation of the Quadratic Security Market Model

Here, we represent the quadratic security model as a state-space model and estimate it using the quasi-maximum likelihood method based on a nonlinear Kalman filter. Batbold *et al.* (2022) estimate the quadratic security market model using the spot rates of bonds and the logarithm of a stock index based on the Kalman filter. However, the Kalman filter is not stationary because the logarithm of the stock index is nonstationary. Therefore, we estimate the model for the dividend rate of the stock index instead of the logarithm of the stock index.

4.1 State-space Model Representation of the Quadratic Security Market Model

We assume the investor in Example 1 and estimate the two-factor quadratic security market model. Given that we treat the latent process as the state process, estimating the security market model relies on the simultaneous estimation of model parameters and the state process. To do so, we first represent our model as a state-space model.

To provide the state-space model representation, we define the following:

$$
\begin{aligned}
&Y_t = \begin{pmatrix} s_t(0.5) \\ s_t(5) \\ s_t^Q(10) \\ D_t/S_t \end{pmatrix}, \qquad H_2(X_t) = \frac{1}{2}\begin{pmatrix} -0.5^{-1}X_t'\Sigma(0.5)X_t \\ -5^{-1}X_t'\Sigma(5)X_t \\ -10^{-1}X_t'\Sigma_q(10)X_t \\ X_t'\mathit{\Delta}X_t \end{pmatrix}, \\
&H_1 = \begin{pmatrix} -0.5^{-1}\sigma(0.5)' \\ -5^{-1}\sigma(5)' \\ -10^{-1}\sigma_q(10)' \\ \delta' \end{pmatrix}, \qquad H_0 = \begin{pmatrix} -0.5^{-1}\sigma_0(0.5) \\ -5^{-1}\sigma_0(5) \\ -10^{-1}\sigma_{0q}(10) \\ \delta_0 \end{pmatrix},
\end{aligned}
\tag{4.1}
$$

where $s_t(\tau)$ and $s_t^Q(\tau)$ are the treasury spot rates and the TIPS real spot rates with time τ to maturity at time t; $(\Sigma(\tau), \sigma(\tau), \sigma_0(\tau))$ and $(\Sigma_q(\tau), \sigma_q(\tau), \sigma_{0q})$ are solutions to eqs.(2.20)-(2.22) and (2.24)-(2.26); and $(\Delta, \delta, \delta_0)$ is given by eq.(2.9).

Eq. (2.1) can be transformed as follows.

$$
d\big(e^{t\mathcal{K}}X_t\big) = e^{t\mathcal{K}}\, dB_t. \tag{4.2}
$$

Let h be the observation time interval. Integrating the above equation over the interval $[nh, (n+1)h]$, we obtain

$$
e^{(n+1)h\mathcal{K}}X_{(n+1)h} - e^{nh\mathcal{K}}X_{nh} = \int_{nh}^{(n+1)h} e^{s\mathcal{K}}\, dB_s. \tag{4.3}
$$

Dividing both sides of the above equation by $e^{(n+1)h\mathcal{K}}$, we get

$$
X_{(n+1)h} = e^{-h\mathcal{K}}X_{nh} + \int_{nh}^{(n+1)h} e^{\{s-(n+1)h\}\mathcal{K}}\, dB_s, \tag{4.4}
$$

By definition of yield-to-maturity, the following holds:

$$
\begin{aligned}
s_t(\tau) &= -\frac{1}{\tau}\log P_t(\tau), \\
s_t^Q(\tau) &= -\frac{1}{\tau}\log Q_t(\tau).
\end{aligned}
\tag{4.5}
$$

Thus, from eqs.(2.19), (2.23), (4.5), (2.9), and (2.27), we obtain

$$
Y_{nh} = H_2(X_{nh}) + H_1X_{nh} + H_0. \tag{4.6}
$$

In eq.(4.4), denoting

$$
x_n = X_{nh}, \quad F = e^{-h\mathcal{K}}, \quad w_n = \int_0^h e^{(s-h)\mathcal{K}}\, dB_s, \tag{4.7}
$$

we have the following state-transition equation.

$$x_{n+1} = F\,x_n + w_n, \tag{4.8}$$

where $w_n \sim N(0, \Omega_w)$ and

$$\Omega_w = \int_0^h e^{(s-h)\mathcal{K}}\big(e^{(s-h)\mathcal{K}}\big)' ds = (\mathcal{K} + \mathcal{K}')^{-1}\big(I_2 - e^{-h(\mathcal{K}+\mathcal{K}')}\big). \tag{4.9}$$

In eq.(4.6), denoting $y_n = Y_{nh}$ and adding the observation error term ε_n to the right-hand side, we obtain the following observation equation.

$$y_n = H_2(x_n) + H_1 x_n + H_0 + \varepsilon_n, \tag{4.10}$$

where $\varepsilon_n \overset{i.i.d.}{\sim} N(0, \Omega_\varepsilon)$ is independent of w_n and Ω_ε is given by

$$\Omega_\varepsilon = \begin{pmatrix} \omega_{11} & 0 & 0 & 0 \\ 0 & \omega_{22} & 0 & 0 \\ 0 & 0 & \omega_{33} & 0 \\ 0 & 0 & 0 & \omega_{44} \end{pmatrix}. \tag{4.11}$$

In this manner, our quadratic security market model can be interpreted as a state-space model consisting of state-transition eq.(4.8) and observation eq.(4.10). In this state-space model, the observation equation is nonlinear; thus, it cannot be estimated by using an ordinary Kalman filter. We estimate the state-space model using the quasi-maximum likelihood method based on unscented Kalman filter proposed by Julier, Uhlmann, and Durrant-Whyte (2000). The unscented Kalman filter is a nonlinear Kalman filter and it approximates a probability distribution using the unscented transformation unlike the extended Kalman filter, which approximates a nonlinear function linearly.[10]

Remark 4. *The TIPS-related component of the volatility matrix is calculated from estimates of the parameters* $(\Sigma_q(\tau), \sigma_q(\tau))$ *in the observation eq.(4.10). The stock-related component of the volatility matrix is calculated by substituting the estimates of the parameters* (Δ, δ) *in the observation eq.(4.10) into eqs.(2.28) and (2.29).*

4.2 Regularization

As already pointed out in Example 1, eq.(3.14) shows that as the determinant of matrix $\Sigma(X_t)'$ approaches zero, the inverse matrix $\Sigma(X_t)'^{-1}$ diverges and the estimates of the optimal robust portfolio become unstable.

[10]Ristic, Arulampalam, and Gordon (2004) evaluate that this approximation is more accurate than the linear approximation, and the performance of the unscented Kalman filter is better than the extended Kalman filter.

We, therefore, add a regularization term to the negative quasi-loglikelihood to stabilize the portfolio. Since the matrix $\Sigma(X_t)'^{-1}$ depends on the state process, we first consider a range of possible values of the state process.

SDE (4.2) for the state process is solved as

$$X_t = e^{-t\mathcal{K}} X_0 + \int_0^t e^{(s-t)\mathcal{K}} \, dB_s. \tag{4.12}$$

Thus, the stationary distribution of X_t is given by $N(0, (\mathcal{K}+\mathcal{K}')^{-1})$. Assume $X_t \sim N(0, (\mathcal{K}+\mathcal{K}')^{-1})$. Let $\mathcal{C}\mathcal{C}' = (\mathcal{K}+\mathcal{K}')^{-1}$ be the Cholesky decomposition and define $Z = \mathcal{C}^{-1} X_t$. Then, $Z \sim N(0, I_N)$. Therefore, it is reasonable to assume that the state vector takes the following values:

$$Z = \begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix} = j \begin{pmatrix} \cos 0.25\pi k \\ \sin 0.25\pi k \end{pmatrix}, \tag{4.13}$$

where $j = -2.00, -1.75, -1.50, \cdots, 2.00$ and $k = 0, 1, 2, 3.$.

A candidate for a regularization term could be the square of the Frobenius norm of $\Sigma(X_t)'^{-1}$ or the absolute value of the determinant of $\Sigma(X_t)'^{-1}$. After various trials and errors, we adopt the following as a regularization term that can reconcile a large loglikelihood and a stable portfolio.

$$\nu \Big(|\mathrm{det}\Sigma(0)'^{-1}| + \sum_{j=0}^{3} |\mathrm{det}\Sigma(\mathcal{C}^{-1} Z^{(k)})'^{-1}| \Big), \tag{4.14}$$

where $\nu = 10$ and

$$Z^{(k)} = \begin{pmatrix} \cos 0.5\pi k \\ \sin 0.5\pi k \end{pmatrix}. \tag{4.15}$$

Note that the parameter $\nu = 10$ was adopted based on our subjective evaluation; thus, the following estimation results are not absolute.

We estimate the two-factor quadratic security market model by minimizing the negative quasi-likelihood with the regularization term based on the unscented Kalman filter on 262 month-end data from January 1999 to October 2020, observed in the U.S. security markets. The time-series data used for estimation are 6-month, 5-year, and 10-year treasury spot rates[11], 5-year and 10-year TIPS real spot rates[12], and the dividends of S&P 500[13].

5 Numerical Analysis

In this section, we examine market timing effects in optimal robust portfolios and the relationship between relative uncertainty tolerance and optimal consumption-investment using our approximate optimal portfolio.

[11]These spot rates data are available on the FRB website. They are computed based on the estimation method by Gürkaynak, Sack, and Wright (2007).

[12]These TIPS real spot rate data are available on the FRB website. They are computed based on the estimation method by Gürkaynak, Sack, and Wright (2010).

[13]These data are available on the website of Robert Shiller.

We consider a long-term investor who has an initial asset W_0 and plans to invest in the 10-year TIPS and S&P 500 in addition to the money market account over the 35 years. We set $T^{*}= 35$ and $\alpha = 0.5, \beta = 0.04$. We also assume $\gamma = 2.5$ and $\theta = 1.5$. To analyze the variation in the optimal robust portfolio allocations owing to the change in the state vector, we assume, based on the results of the above analysis, that the state vector $X_t = \mathcal{C}Z$ changes, as shown in eq.(4.13).

5.1 Market Timing Effects

5.1.1 Market Timing Effects in Optimal Allocations

The estimated optimal allocations to equity, inflation-indexed bonds, and the money market account plotted against the state vector are shown in Figs.1-3.

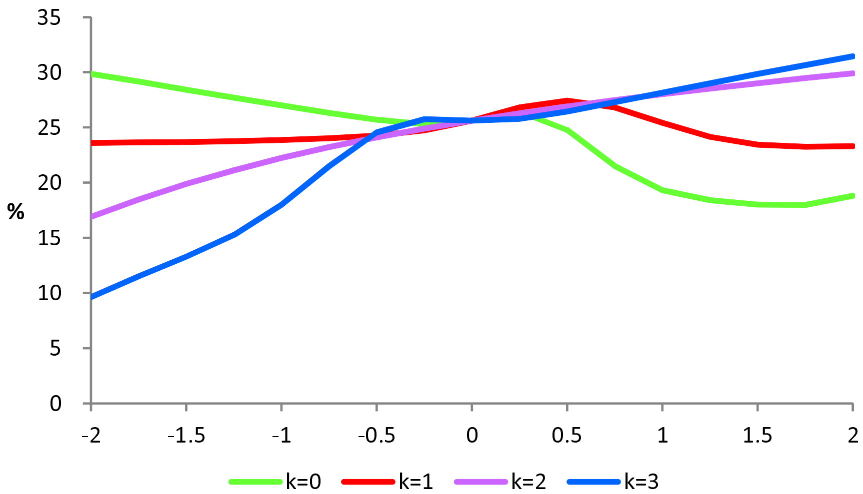

Figure 1: Optimal allocation (%) to equity plotted against the state vector.

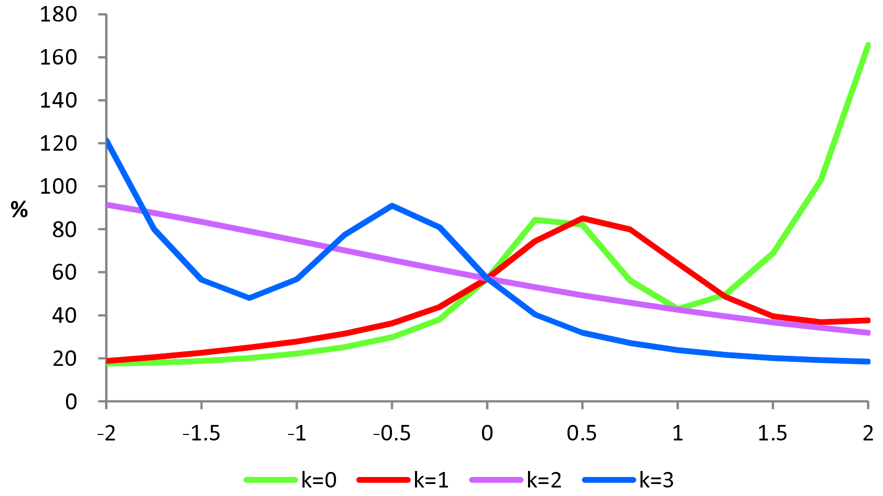

Figure 2: Optimal allocation (%) to inflation-indexed bonds plotted against the state vector.

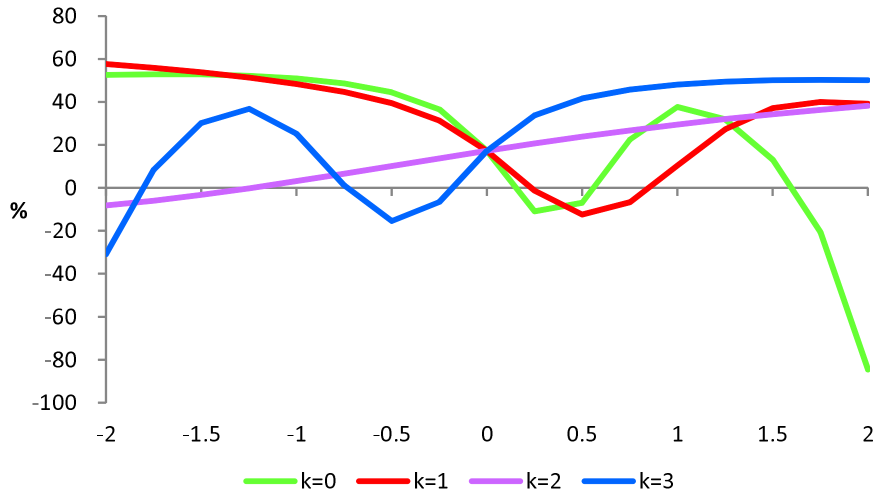

Figure 3: Optimal allocation (%) to the money market account plotted against the state vector.

The optimal allocations to both equity and inflation-indexed bonds respond significantly and nonlinearly to changes in state vectors, suggesting that market timing effects are significant and nonlinear. In particular, the market timing effects in optimal allocation to inflation-indexed bonds are substantially large and highly nonlinear. These results strongly suggest the effectiveness of constantly estimating the state vector with high accuracy, calculating the optimal asset allocation based on the estimated state vector, and recomposing the asset allocation into the calculated optimal asset allocation; that is, the effectiveness of leveraging market timing effects in asset allocation.

In this study, the investor is assumed to possess no human capital and to invest only in domestic securities. If the investor possesses human capital, which is generally considered safe, and can invest in foreign equities which have different risk profiles from domestic ones, then the optimal allocations to domestic and foreign equities would be considerably larger than the optimal allocation to equity shown in Fig.1. In the end, market timing effects in optimal allocations to domestic and foreign equities would then also be considerably larger than those in the optimal allocation to equity shown in Fig.1.

5.1.2 Market Timing Effects on Equity

Figs.4-5 show the factor decomposition of optimal allocation to equity plotted against the state vector.

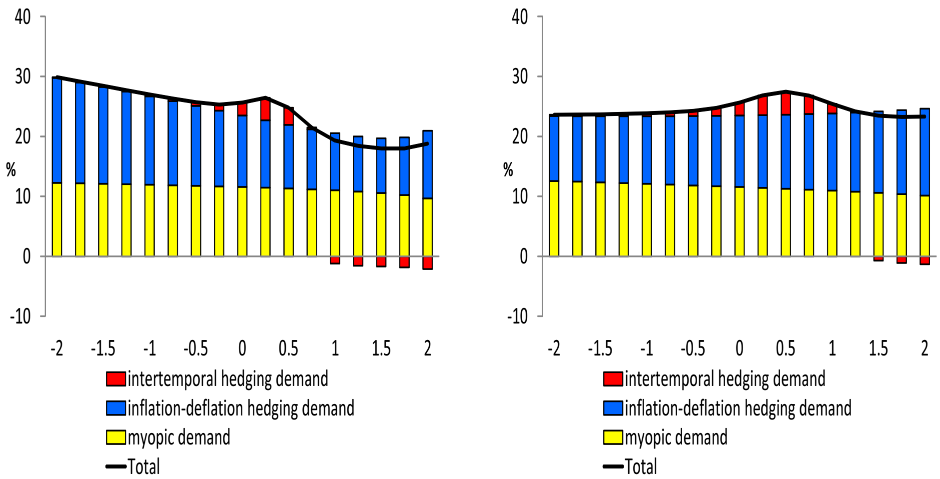

Figure 4: Factor decomposition of optimal allocation (%) to equity plotted against the state vector ($k = 0, 1$).

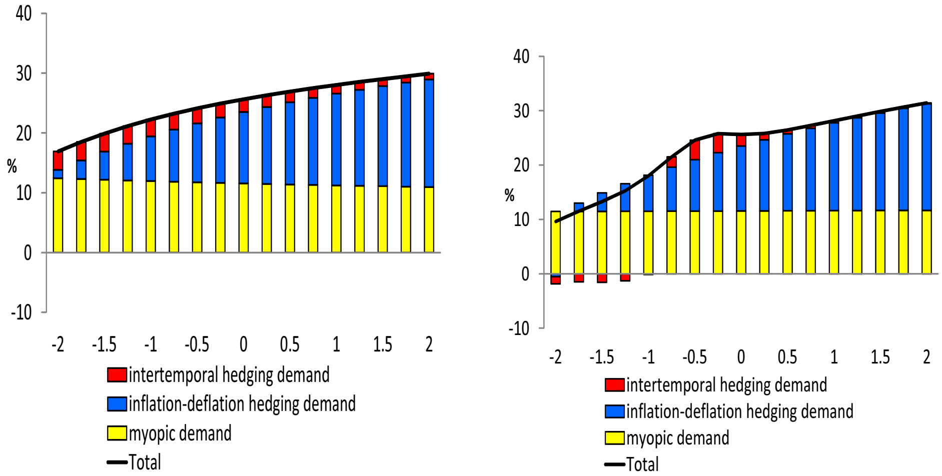

Figure 5: Factor decomposition of optimal allocation (%) to equity plotted against the state vector ($k = 2, 3$).

As the state vector changes, inflation-deflation hedging demand changes significantly and nonlinearly, whereas myopic and intertemporal hedging demands do not change significantly, suggesting that most of the market timing effects in optimal allocation to equity are driven by changes in inflation-deflation hedging demand.

In the present estimation results, the level of myopic demand is large, but its contribution to the market timing effects is not large. However, as mentioned above, the present estimation results are based on the parameter representing the weight of the regularization term adopted by our subjective evaluation. Therefore, under the estimation results obtained under different parameters, the contribution of myopic demand to the market timing effects could be larger.

5.1.3 Market Timing Effects on Inflation-indexed Bond

Figs.6-7 illustrate the factor decomposition of optimal allocation to inflation-indexed bonds plotted against the state vector.

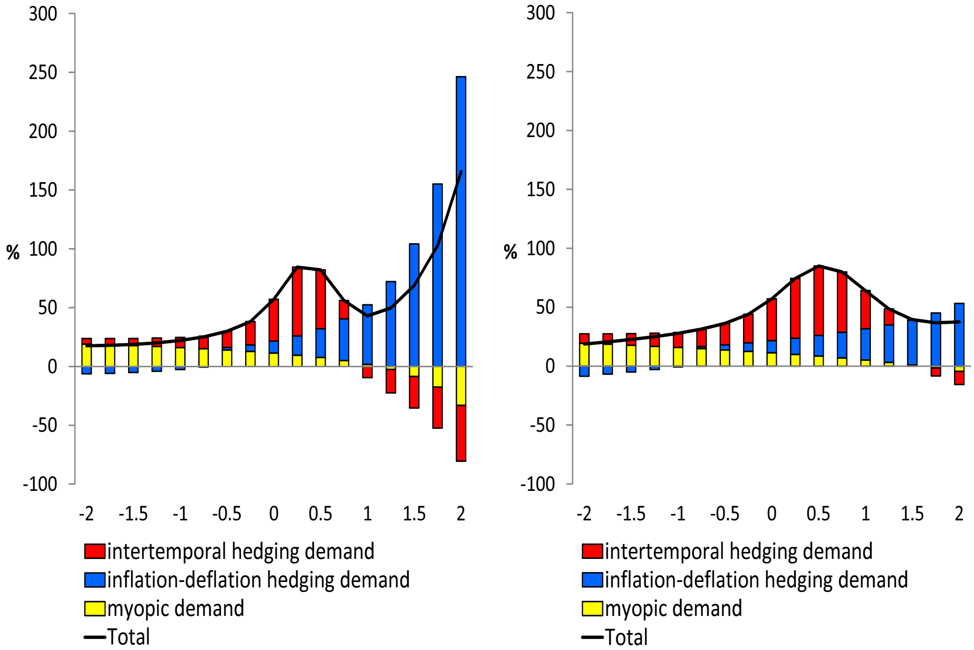

Figure 6: Factor decomposition of optimal allocation (%) to inflation-indexed bonds plotted against the state vector ($k = 0, 1$).

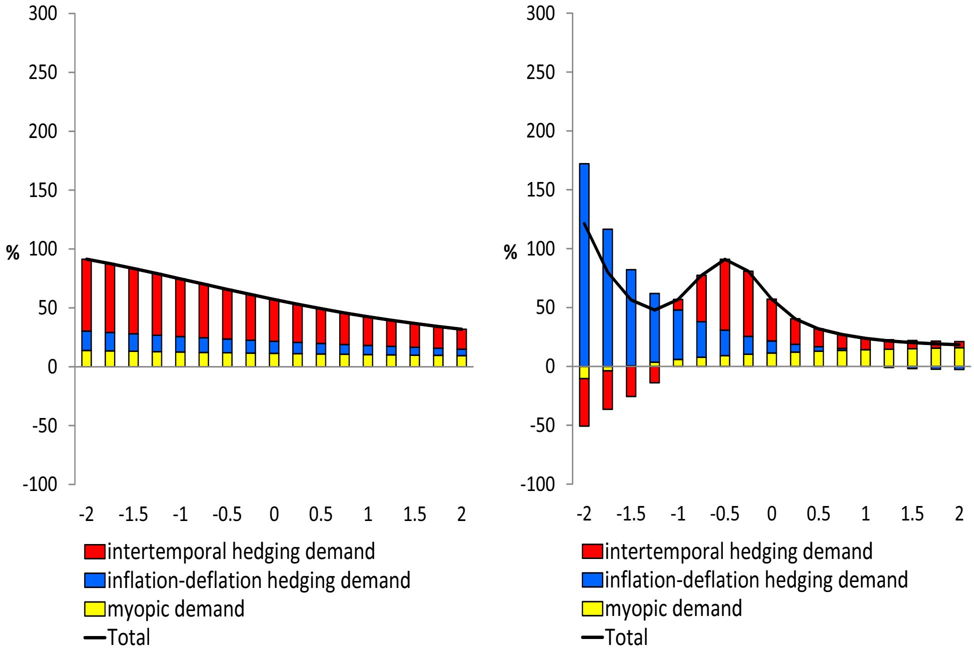

Figure 7: Factor decomposition of optimal allocation (%) to inflation-indexed bonds plotted against the state vector ($k = 2, 3$).

All the demands contribute to the level of optimal allocation to inflation-indexed bonds. Furthermore, all demands change significantly and nonlinearly concerning changes in the state vector and contribute to the market timing effects in optimal allocation to inflation-indexed bonds, with inflation hedging demands contributing the most.

5.2 Uncertainty Tolerance and Optimal Consumption-Investment

To analyze the impact of changes in relative ambiguity aversion on optimal consumption and investment, we fix $\gamma = 2.5$. Then, the relationship between relative ambiguity aversion and uncertainty tolerance is shown in Table 1.

Table 1: Relationship between relative ambiguity aversion and uncertainty tolerance at relative risk aversion of 2.5.

θ	0	$\cdots$	2.5	$\cdots$	7.5	$\cdots$	∞
$(\gamma+\theta)^{-1}$	0.4	$\cdots$	0.2	$\cdots$	0.1	$\cdots$	0

To examine the relationship between relative uncertainty tolerance and the optimal consumption-investment, we set $X_0 = 0$.

5.2.1 Relative Uncertainty Tolerance and Optimal Investment

The relationship between the level of relative uncertainty tolerance and the optimal allocation to equity at relative risk aversion of 2.5 is shown in Fig.8

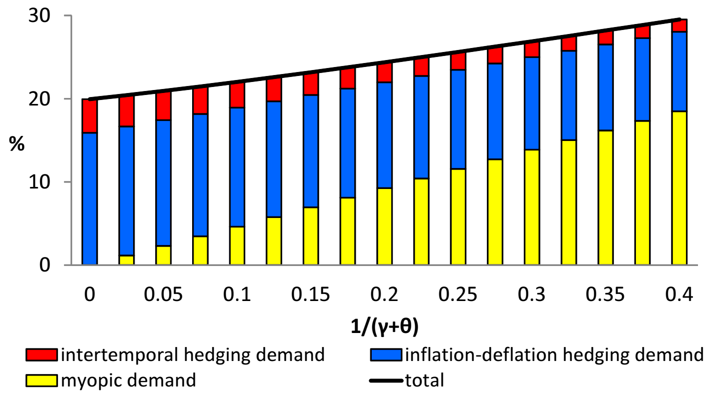

Figure 8: Optimal allocation (%) to equity plotted against relative uncertainty tolerance at relative risk aversion of 2.5.

The increase in myopic demand associated with increased uncertainty tolerance outweighs the decrease in intertemporal hedging and inflation-deflation hedging demands. As a result, the optimal allocation to equity increases.

The relationship between the level of relative uncertainty tolerance and the optimal allocation to inflation-indexed bonds is shown in Fig.9

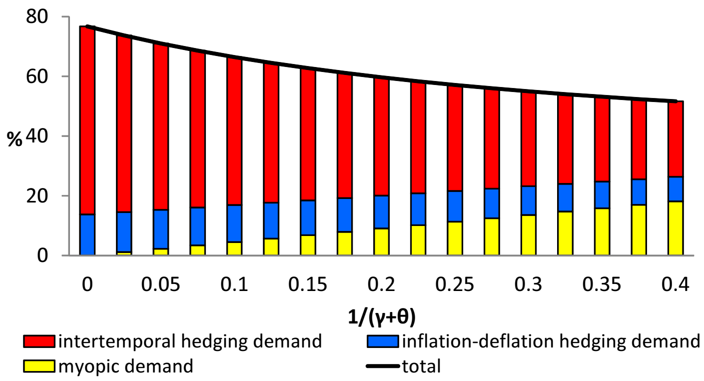

Figure 9: Optimal allocation (%) to inflation-indexed bonds plotted against relative uncertainty tolerance at relative risk aversion of 2.5.

Contrary to the optimal allocation to equity, the decrease in intertemporal hedging and inflation-deflation hedging demands associated with increased relative uncertainty tolerance outweighs the increase in myopic demand, and the optimal allocation to inflation-indexed bonds decreases.

5.2.2 Relative Uncertainty Tolerance and Optimal Consumption

Finally, the relationship between the level of relative uncertainty tolerance and the consumption-wealth ratio at relative risk aversion of 2.5 is shown in Fig.10.

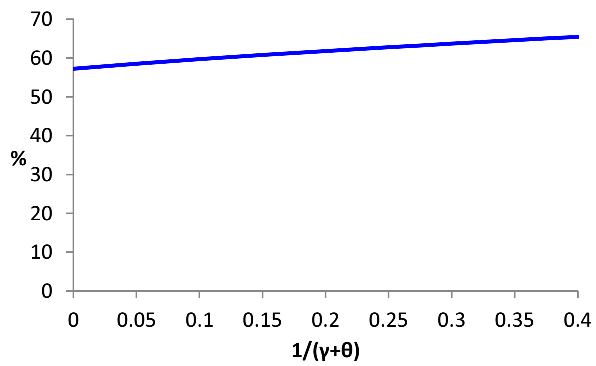

Figure 10: Optimal consumption-wealth ratio plotted against relative uncertainty tolerance at relative risk aversion of 2.5.

The optimal consumption-wealth ratio increases as relative uncertainty tolerance increases.

6 Conclusion and Future Works

We considered a finite-time consumption-investment problem for homothetic robust utility under a quadratic security market model with stochastic volatilities and inflation rates. Since the PDE for indirect utility is nonlinear and nonhomogeneous, we proposed a time-dependent linear approximation method to derive an approximate solution.

Next, we proposed an estimation method for the quadratic security market model, noting that the optimal robust portfolio is proportional to the inverse of the volatility matrix, and that the optimal robust portfolio diverges when the determinant of the volatility matrix approaches zero. In other words, we proposed an estimation method in which the absolute value of the determinant of the inverse of the volatility matrix is added to the negative quasi-loglikelihood as a regularization term. Based on the quadratic security market model estimated by the proposed estimation method, we performed a numerical analysis of the approximate optimal robust portfolio.

The market timing effects in optimal allocation to equity were significant and nonlinear, and primarily owing to inflation-deflation hedging demand. Additionally, the market timing effects in optimal allocation to inflation-indexed bonds were substantially large and highly nonlinear. All demands contribute to the market timing effect, with inflation-deflation hedging demand contributing the most. In the market timing effects claimed by strategic asset allocation, we focused on intertemporal hedging demand and ignored inflation-deflation hedging demand; however, this numerical analysis suggests that inflation-deflation hedging demand does matter in strategic asset allocation.

We interpret the reasons for such large market timing effects in inflation-deflation hedging demand as follows. To begin with, inflation-deflation hedging demand is closely tied to monetary policy. When the economy heats up and inflation fears rise, interest rates are raised, resulting in downward pressure on stock prices and a slowdown in GDP. When inflation subsides and the economy cools, interest rates are lowered, which in turn puts upward pressure on stock prices and accelerates GDP. Thus, the market timing effects in inflation-deflation hedging demand can be interpreted as being amplified by monetary policy. We interpret the amplification effect of monetary policy as having been greatly enhanced by the quantitative easing policy implemented by the Fed against the backdrop of deflationary fears in the immediate aftermath of the global financial crisis.

In this study, the optimal allocation to equity is not large because the investor does not possess human capital, which is generally considered safe, and is not allowed to invest in foreign equities, which have different risk profiles from domestic equities. If the investor possesses human capital and can invest in foreign equity, the optimal allocations to domestic and foreign equities are expected to be larger, resulting in larger market timing effects in the optimal allocation to domestic and foreign equities. Future work is required to examine optimal robust portfolios when investors possess human capital and can invest in foreign equity.

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Competing Interests

The authors have no competing interests to declare that are relevant to the content of this article.

A Proofs

A.1 Proof of Lemma 2

The worst-case probability P^{ξ^*} satisfies

$$\xi_t^* = -\frac{\theta}{(1-\gamma)J}\begin{pmatrix}\bar{W}_t\bar{\sigma}_t' \\ I_N\end{pmatrix}'\begin{pmatrix}J_W \\ J_X\end{pmatrix}. \tag{A.1}$$

Substituting P^* into the HJB equation (2.39) yields

$$\sup_{(c,\bar{\sigma})\in\mathcal{B}(X_0)}\left[J_t+\begin{pmatrix}\bar{W}_t\left(\bar{r}_t+\bar{\sigma}_t'\bar{\lambda}_t\right)-c_t \\ -\mathcal{K}X_t\end{pmatrix}'\begin{pmatrix}J_W \\ J_X\end{pmatrix}+\frac{1}{2}\operatorname{tr}\left[\begin{pmatrix}\bar{W}_t\bar{\sigma}_t' \\ I_N\end{pmatrix}\begin{pmatrix}\bar{W}_t\bar{\sigma}_t' \\ I_N\end{pmatrix}'\begin{pmatrix}J_{WW} & J_{WX} \\ J_{XW} & J_{XX}\end{pmatrix}\right]\right.$$
$$\left.+\alpha e^{-\beta t}\frac{c^{1-\gamma}}{1-\gamma}-\frac{\theta}{2(1-\gamma)J}\left|\begin{pmatrix}\bar{W}_t\bar{\sigma}_t' \\ I_N\end{pmatrix}'\begin{pmatrix}J_W \\ J_X\end{pmatrix}\right|^2\right]=0. \tag{A.2}$$

It is apparent that optimal control $u_t^* = (c_t^*, \bar{\sigma}_t^*)$ in thTe HJB equation (A.2) satisfies

$$c_t^* = \alpha^{\frac{1}{\gamma}}e^{-\frac{\beta}{\gamma}t}J_W^{-\frac{1}{\gamma}}, \tag{A.3}$$
$$\bar{\sigma}_t^* = \mathcal{T}_t\left(\bar{\lambda}_t+\frac{J_{XW}}{J_W}+\frac{\theta}{\gamma-1}\frac{J_X}{J}\right), \tag{A.4}$$

where T_t is given by

$$\mathcal{T}_t = \left(-\frac{\bar{W}_t^*J_{WW}}{J_W}+\theta\frac{\bar{W}_t^*J_W}{(1-\gamma)J}\right)^{-1}. \tag{A.5}$$

The consumption-related terms in the HJB equation (A.2) are computed as

$$-c_t^*J_W+\alpha\, e^{-\beta t}\frac{c_t^{*1-\gamma}}{1-\gamma}=\frac{\gamma}{1-\gamma}c_t^*J_W. \tag{A.6}$$

The investment-related terms in the HJB equation (A.2) are computed as

$$\bar{W}_t^* J_W \bar{\Lambda}_t' \bar{\sigma}_t^* + \frac{1}{2}\,\mathrm{tr}\left[\begin{pmatrix}\bar{W}_t^*(\bar{\sigma}_t^*)' \\ I_N\end{pmatrix}\begin{pmatrix}\bar{W}_t^*(\bar{\sigma}_t^*)' \\ I_N\end{pmatrix}'\begin{pmatrix}J_{WW} & J_{WX} \\ J_{XW} & J_{XX}\end{pmatrix}\right]$$

$$- \frac{\theta}{2(1-\gamma)J}\left|\begin{pmatrix}\bar{W}_t^*(\bar{\sigma}_t^*)' \\ I_N\end{pmatrix}'\begin{pmatrix}J_W \\ J_X\end{pmatrix}\right|^2$$

$$= \frac{1}{2}\,\mathrm{tr}\left[J_{XX}\right] - \frac{\theta}{2(1-\gamma)J}|J_X|^2 - \frac{|\psi_t|^2}{2\bar{W}_t^{*2}\left(J_{WW} - \dfrac{\theta J_W^2}{(1-\gamma)J}\right)}, \quad \text{(A.7)}$$

where

$$\psi_t = -\bar{W}_t^* J_W\left(\bar{\lambda}_t + \frac{J_{XW}}{J_W} + \frac{\theta}{\gamma-1}\frac{J_X}{J}\right). \quad \text{(A.8)}$$

By substituting optimal control (A.3) and (A.4) into the HJB equation (A.2) and using eqs. (A.6) and (A.7), the following PDE for J is obtained:

$$J_t + \frac{1}{2}\mathrm{tr}\left[J_{XX}\right] - \frac{\theta}{2(1-\gamma)J}|J_X|^2 - \frac{|\psi_t|^2}{2\bar{W}_t^{*2}\left(J_{WW} - \dfrac{\theta J_W^2}{(1-\gamma)J}\right)}$$

$$+ \bar{W}_t^* \bar{r}_t J_W - (\mathcal{K}X_t)' J_X + \frac{\gamma}{1-\gamma} c_t^* J_W = 0. \quad \text{(A.9)}$$

From the above PDE, we deduce that the indirect utility function is represented by (2.40).

First, optimal consumption control (2.41) is obtained as follows:

$$c_t^* = \alpha^{\frac{1}{\gamma}} e^{-\frac{\beta}{\gamma}t} J_W^{-\frac{1}{\gamma}} = \alpha^{\frac{1}{\gamma}} e^{-\frac{\beta}{\gamma}t}\left(e^{-\beta t}\left(\frac{G}{\bar{W}_t^*}\right)^{\gamma}\right)^{-\frac{1}{\gamma}} = \alpha^{\frac{1}{\gamma}}\frac{\bar{W}_t^*}{G}. \quad \text{(A.10)}$$

Derivatives of J are given by

$$J_t = -J\left(\beta + \gamma\frac{G_\tau}{G}\right), \quad \bar{W} J_W = (1-\gamma)J, \quad J_X = \gamma\, J\frac{G_X}{G},$$

$$\bar{W}^2 J_{WW} = -\gamma(1-\gamma)J, \quad \bar{W} J_{XW} = \gamma(1-\gamma)J\frac{G_X}{G}, \quad J_{XX} = \gamma\, J\left((\gamma-1)\frac{G_X}{G}\frac{G_X'}{G} + \frac{G_{XX}}{G}\right).$$

Next, $\mathcal{T}_t$ in eq.(A.5) is expressed as $\mathcal{T}_t = (\gamma+\theta)^{-1}$. Therefore, by inserting $\mathcal{T}_t = (\gamma+\theta)^{-1}$ and derivatives of J into eq.(A.4), we obtain optimal investment control (2.42). ψ_t in eq.(A.8) is rewritten as

$$\psi_t = J\left((\gamma-1)\bar{\lambda}_t + \gamma(\gamma+\theta-1)\frac{G_X}{G}\right). \quad \text{(A.11)}$$

The second to fourth terms in the PDE (A.9) are calculated from eq.(A.11) as follows:

$$
\begin{aligned}
&\frac{1}{2}\operatorname{tr}[J_{XX}] - \frac{\theta}{2(1-\gamma)J}|J_X|^2 - \frac{|\psi_t|^2}{2\bar{W}_t^{*2}\left(J_{WW} - \dfrac{\theta J_W^2}{(1-\gamma)J}\right)} \\
=&J\left\{\frac{\gamma}{2}\operatorname{tr}\left[(\gamma-1)\frac{G_X}{G}\frac{G_X'}{G} + \frac{G_{XX}}{G}\right]\right. \\
&\left. + \frac{\gamma^2\theta}{2(\gamma-1)}\left|\frac{G_X}{G}\right|^2 - \frac{1}{2(\gamma-1)(\gamma+\theta)}\left|(\gamma-1)\bar{\lambda}_t + \gamma(\gamma+\theta-1)\frac{G_X}{G}\right|^2\right\} \\
=&J\left\{\frac{\gamma}{2}\operatorname{tr}\left[\frac{G_{XX}}{G}\right] - \frac{\gamma-1}{2(\gamma+\theta)}|\bar{\lambda}_t|^2\right. \\
&\left. - \frac{\gamma(\gamma+\theta-1)}{\gamma+\theta}\bar{\lambda}_t'\frac{G_X}{G} + \frac{\gamma}{2}\left(\gamma - 1 + \frac{\gamma\theta}{\gamma-1} - \frac{\gamma(\gamma+\theta-1)^2}{(\gamma-1)(\gamma+\theta)}\right)\left|\frac{G_X}{G}\right|^2\right\} \\
=&\gamma J\left\{\frac{1}{2}\operatorname{tr}\left[\frac{G_{XX}}{G}\right] - \frac{\gamma-1}{2\gamma(\gamma+\theta)}|\bar{\lambda}_t|^2 - \frac{\gamma+\theta-1}{\gamma+\theta}\bar{\lambda}_t'\frac{G_X}{G} + \frac{\theta}{2(\gamma-1)(\gamma+\theta)}\left|\frac{G_X}{G}\right|^2\right\}.
\end{aligned}
\tag{A.12}
$$

The seventh term in the PDE (A.9) is calculated from eq.(A.10) as follows:

$$
\frac{\gamma}{1-\gamma}c_t^* J_W = \frac{\gamma}{1-\gamma}\alpha^{\frac{1}{\gamma}}\frac{\bar{W}_t^*}{G}(1-\gamma)\frac{J}{\bar{W}_t^*} = \alpha^{\frac{1}{\gamma}}\gamma\frac{J}{G}. \tag{A.13}
$$

Substituting eqs.(A.12) and (A.13) into eq.(A.9) and dividing by γJ yields the PDE (2.43).

A.2 Proof of Proposition 1

Substituting eqs.(3.7) and $G_X = \big(b^*(\tau,0)+B^*(\tau,0)X_t\big)G$ into eqs.(2.41) and (2.42) yield the approximate optimal consumption (3.11) and investment (3.12). Substituting g and its derivatives into the PDE (3.5), we obtain

$$
\frac{db_0}{d\tau}+X'\frac{db}{d\tau}+\frac{1}{2}X'\frac{dB}{d\tau}X = m(X_t)+\frac{\theta}{2(\gamma-1)(\gamma+\theta)}\big(b^*(\tau,0)+B^*(\tau,0)X_t\big)'\big(b(\tau)+B(\tau)X_t\big), \tag{A.14}
$$

where $m(X_t)$ is given by

$$
\begin{aligned}
m(X_t) = {} & \frac{1}{2}\mathrm{tr}[B] + \frac{1}{2}\left(|b|^2 + 2X_t'Bb + X_t'B^2X_t\right) \\
& - \left\{\frac{\gamma+\theta-1}{\gamma+\theta}\bar{\lambda} + \left(\mathcal{K} + \frac{\gamma+\theta-1}{\gamma+\theta}\bar{\Lambda}\right)X_t\right\}' b - \frac{\gamma+\theta-1}{\gamma+\theta}\bar{\lambda}'BX_t \\
& - \frac{1}{2}X_t'\left(\mathcal{K} + \frac{\gamma+\theta-1}{\gamma+\theta}\bar{\Lambda}\right)' BX_t - \frac{1}{2}X_t'B\left(\mathcal{K} + \frac{\gamma+\theta-1}{\gamma+\theta}\bar{\Lambda}\right)X_t \\
& - \frac{\gamma-1}{2\gamma(\gamma+\theta)}\left(|\bar{\lambda}|^2 + 2\bar{\lambda}'\bar{\Lambda}X_t + X_t'\bar{\Lambda}'\bar{\Lambda}X_t\right) - \frac{\gamma-1}{\gamma}\left(\bar{\rho}_0 + \bar{\rho}'X_t + \frac{1}{2}X_t'\bar{\mathcal{R}}X_t\right) - \frac{\beta}{\gamma}.
\end{aligned}
\tag{A.15}
$$

As eq.(A.14) is identical on X, we get the system of ODEs (3.13).

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