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The Games of Matching Pennies and of Stone, Paper, Scissors : Critical Reassessment of Zero-Sum Games

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The Games of Matching Pennies and of Stone, Paper, Scissors : Critical Reassessment of Zero-Sum Games

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Abstract This paper continues to critically reassess the significance and limitations of zero-sum games, which was jointly established by mathematician John von Neumann and economist Oscar Morgenstern during the difficult times of Second World War. First of all, we will intensively discuss several Zero-Sum Two-Person Games, with special reference to the Games of Matching Pennies. New graphical illustrations will be attempted for clarification of the matter. Then, we will turn to novelist Edgar Alan Poe's best story *The Purloined Letter*, showing how the Poe story may challenge the validity of Neumann-Morgenstern approach. Next, we will shed new light on Conan Doyle's famous story *The Final Case* as a variant of the Games of Matching Pennies. Finally, we will pick up the Game of Stone, Paper, Scissors, which constitutes another important member of Zero-Sum Two Person Games.

Keywords zero-sum two-person games, graphical illustrations, the game of matching pennies, Poe's challenge , Doyle's story *The Final Case*, the game of stone, paper, scissors

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1. Historical Background of the Theory of Games

The Year of 1944 — this is certainly the most memorial year of the history of the theory of games. It is in this year that the collaboration between mathematician von Neumann and economist Oskar Morgenstern successfully produced the great book *Theory of Games and Economic Behavior* at the Princeton University Press.

In historical perspective, the devastating Second World War began in 1939 and finally ended in 1945. So, we could say that presumably, game theory was an academic by-product of the global war involving so many countries across the Atlantic, the Indian, and the Pacific Oceans. Yasuhiro Sakai, the author of the present book, was born in Japan one year after the Poland Invasion by Adolf Hitler and four year before the publication of *Game Theory* by Neumann and Morgenstern. Let us recall the political and economic situations surrounding Japan in 1944. It is true that at the beginning of the Pacific War with the United States, the Empire of Japan won a number of battles after it made a surprise attack on the Pearl Harbor in December 8, 1945 (Asia/West-Pacific time zones). As the saying goes, however, happy events are often accompanied by difficulties.

We can recall what Admiral Isoroku Yamamoto (1884-1943), a legendary Japanese tragic hero, told his fellow career soldiers just before the Pacific War began, " In the first six to twelve months of a war with the United States and Great Britain, Yamamoto will run wild and win victory upon victory. But then, if the war continues after that, I have no expectation of success". In retrospect, Yamamoto was quite right, and his prediction came true. After so many Japanese aircraft carriers were destroyed at the notorious Midway Battle in June 1942, the Japanese navy was doomed to total defeat in power struggles against the American counterpart, and even Admiral Yamamoto himself was killed in action in the South Pacific in 1943.

Under those difficult situations under the war period, all the scholars around the world had to keep trying their best in order to pull through the crisis. In Japan, which engaged in the war against the United States, there were a selected number of people who made every possible effort to continue their research activities. Among those devoted people was Professor Takata Yasuma (1883-1972) of Kyoto Imperial University, who as an influential economist and sociologist made a great contribution to the Power Theory of Social Science. His productive power as a serious scholar during the hectic war period was exceptionally great, for he published so many books on social power, race and economy, interest theory, controlled economy, and many other topics as well-illustrated by Takata (1937,1940a,1940b,1941,1942,1943,1944). In spite of the decline of a nation as a whole, Takata's personal spirits were still so high that he bravely declared as follows. " There might be a sudden collapse in the fortunes of the country. Takata is nevertheless ready to stand up alone against the stream." 1)

When the United States became a formidable enemy of Japan, it embarked on a new project, called the Manhattan Project, whose main purpose was to produce weapons of mass destruction including atomic bombs. On the one hand, von Neumann himself gave everything he acquired in order to invent a very powerful computer that was necessary to implement the project, thus putting less emphasis on the project itself. On the other hand, von Neumann' s commitment to Game Theory had to become less and less as the war went on. Nevertheless, his collaboration with economist Morgenstern somehow continued at Princeton until their bulky book *Theory of Games and Economic Behavior* was finally finished in 1944. In short, game theory was truly the joint product of mathematical talent of von Neumann and humanist intelligence of Morgenstern. 2)

As the reader has noticed, Chapters 1 and 2 are a sort of "twin chapters." Taken together, those chapters are intended to critically reassess the significance and limitations of Zero-Sum Games, which was jointly established by mathematician John von Neumann and economist Oskar Morgenstern during the difficult times of Second World War.

Let us write down the contents of this chapter. The next section will carefully examine how their joint book has been organized, thus unveiling the real intention of the two authors. Section 3 will intensively discuss several zero-sum two-person games, with special reference to the Games of Matching Pennies. New graphical illustrations will be attempted for clarification of the matter. In Section 4, we will turn to novelist Edgar Alan Poe's best story *The Purloined Letter,* discussing how the Poe story may challenge the validity of Neumann-Morgenstern approach to human behavior. We note that this would be probably the first time to connect the creative writer Poe and the strong Neumann-Morgenstern alliance. In Section 5 , we are in a position to safely return to our favorite story *The Final Problem* as a variant of the Game of Matching Pennies, which has continuously been our major topic in the last chapter, namely Chapter 1. In Section 6, we will pick up the Game of Stone, Paper and Scissors, which clearly constitutes another important member of the family of Zero-Sum Two-Person Games. And final remarks will be made in Section 7.

2. The Great Book *Theory of Games and Economic Behavior*: Its Contents Critically Examined

We are now ready to critically examine the whole contents of the great book *Theory of Games and Economic Behavior* (1944) from a set of fresh angles. Honestly speaking, it does not look like a "real thing," but rather like an "imaginary creature." It is really a bulky book containing 600 and more pages, full of strange notations and pretentious equations. In today's house building, one of the most popular styles is a "barrier-free style," in the sense that many walking obstacles in the house should be avoided. Contrary to such a modern style, there exist so many obstacles and barriers in the game-theory architecture made by von Neumann and Morgenstern. We should stress to say, however, that it is no less than a kind of "Bible of Game Theory," which is worthy of attempting to overcome mathematical obstacles/barriers.

In the following, let us more carefully shed new light on the original book of game theory. As the saying goes, seeing is believing. Take a very close look at Table 1. Then, we will see that broadly speaking, it consists of three parts, namely General Intro, Zero-Sum Games, and Non-Zero-Sum Games.

First of all, we are most likely to be shocked to see the apparently strange compactness of "Preface to First Edition," which was written in January 1943. Contrary to our expectation, this important preface was very compactly written, thus being unduly condensed into following two paragraphs:

> This book contains an exposition and various applications of a mathematical theory of games . The theory has been developed by one of us since 1928 and is now published for the first time in its entity. The applications are of two kinds: On the one hand to games in the proper sense, on the other hand to economic and sociological problems which, as we hope to show, are best approached from this direction.
>
> The applications which we shall make to games serve at least as much to corroborate the theory as to investigate these games. The nature of this reciprocal relationship will become clear as the investigation proceeds. Our main interest is, of course, in the economic and sociological direction. Here we can approach only the simplest questions. However, these questions are of a fundamental character. Furthermore, our aim is primarily to show that there is a rigorous approach to these subjects, involving, as they do, questions of parallel or opposite interest, perfect or imperfect information, free rational decision or chance influences..
>
> (von Neumann & Morgenstern 1944, Preface to first edition)

Table 1 *The Theory of Games and Economic Behavior*:
Zero-Sum Games as the Main Core of its Contents

Contents	Chapter	Pages	Title
GENERAL	1	1 - 45	Formation of the Economic Problem
INTRO	2	46-84	General Description of Games of Strategy
	3	85-168	Zero-Sum Two-Person Games: Theory
	4	169-219	Zero-Sum Two-Person Games: Examples
	5	220-237	Zero-Sum Three-Person Games
ZERO-	6	238-290	General Theory of Zero-Sum n-Person Games
SUM	7	291-329	Zero-Sum Four-Person Games
	8	330-338	Some Remarks on n ≧ 5 participants
GAMES	9	339-419	Composition and Decomposition of Games
	10	420-503	Simple Games
NON-ZERO-	11	504-586	General Non-Zero-Sum Games
SUM GAMES	12	587-616	Concepts of Domination and Solution

Interestingly enough, such unduly compactness of Preface was not confined to the First Edition. For instance, "Preface to Second Edition" (written in September 1046) and "Preface to Third Edition" (written in January 1953) were respectively one page long and two page long. Honestly speaking, the authors had little interest in making the prefaces well-appealing to the reader.

Secondly, in sharp contrast to the brief Preface, the Main Text was quite long and powerful. Metaphorically speaking, if we are allowed to compare the full contents of the book to the full course of Western-style cooking, then Preface might be regarded as a "light appetizer" , but the Main Text must be a "heavy main dish," and the Math Appendix a "colorful dessert."

As can be seen in Table 1, the contents of the book are divided into three parts. The first part deals with General Intro, consisting of the first two chapters, namely Chapters 1 and 2, The second part systematically discusses Zero-Sum Games, containing eight chapters from Chapter 3 through Chapter 10. The third part is concerned with the final two chapters, i.e., Chapters 11 and 12. Easy calculation can tell us that the shares of Part 1, Part 2 and Part 3 are respectively 13.64%, 68.02% and 18.34%. Therefore, Zero-Sum Games, and especially Zero-Sum Two-Person Games, have been the main target of the joint work of von Neumann and Morgenstern. This should clearly indicate the third characteristic of the 1944 book.

We wonder why the two giants of modern times had been so obsessive about zero-sum games. There must be some reasons for such obsession. The first reason we can think of seems to be historical. Let us recall the historical backgrounds of the two superstars. Both von Neumann and Oskar Morgenstern were born in Europe or the "Old World." While von Neumann spent his young days as the son of a rich Jewish family, he had to experience the suffering and tribulations of human life as the Nazis led by Fürer Adolf Hitler (1889-1945) began to assert itself in the region. Shortly after Morgenstern became a professor of economics at the prestigious University of Vienna, his chair was suddenly sacked because he was accused of being "politically unbearable." Although both von Neumann and Morgenstern later settled down at Institute of Advanced Study, Princeton University in the United States, i.e. the "New World," their suffering and hardship in the "Old World" never escaped their memories. Besides, when the Polish Invasion by Dictator Adolf Hitler triggered the Second World War, the peaceful private lives of von Neumann and Morgenstern were no longer isolated from the busy public duties. In short, together with the general public, the two great men were also drove into a corner, having the mental state that " To fight or not, that is the question." We strongly believe that behind the birth and development of game theory, there existed a number of fierce rivalries between states, between classes, and between ideologies.

The second reason for the rise of game theory is not historic, but rather more technical. According to von Neumann and Morgenstern, zero-sum games and non-zero-sum games should not be so hostile as they appear. Indeed, if we additionally assume the existence of a "third hypothetical player," then it is theoretically feasible to *expand* non-zero-sum two-person games to Zero-Sum Three-Person Games. In other words, in general, it is technically possible to *embed non-zero-sum n-person games into*

Zero-Sum n+1-Person Games. In this connection, von Neumann and Morgenstern remarked the following :

> We shall primarily construct a theory of the zero-sum games, but it will be found possible to dispose, with its help. of all games, without restriction. Precisely: We shall show that the general (hence in particular the variable sum) n-person game can be reduced to a zero-sum n + 1 - person game. Now the theory of the zero-sum n-person game will be based on the special case of the zero-sum two-person game. Hence our discussion will begin with a theory of these games. (Von Neumann & Morgenstern 1944, p. 47)

Remarkably, very similar remarks can be seen again much later in the book:

> Any given general games can be re-interpreted in a zero-sum game. This may seem paradoxical since the general games form a much more extensive family than the zero-sum games. However, our procedure will be to interpret an n-person general game as an n + 1- person zero-sum game. ... The procedure by which a given general n-person game is re-interpreted as an n + 1- person zero-sum game is a very simple and natural one. It consists of introducing a — fictitious —n + 1- st player who is assumed to lose the amount which the totality of the other n —real —players wins and *vice versa.*
>
> (von Neumann & Morgenstern 1944, pp. 505-506)

The operations of *expansion and embedding* , which should be very important, were favored and often used by von Neumann and Morgenstern. To our astonishment, however, these operations have been all but forgotten in the recent discussions on game theory. It is now high time to recover and further develop the skillful operations of expansion and embedding. As the saying goes, better late than never !

3. Zero-Sum Two-Person Games : Pure Strategies and Strictly Determined Games

In the Great Book *Theory of Games and Economic Behavior,* there is one important subject which plays the leading role throughput the book. That subject is nothing but the one named *Zero-Sum Two-Person Games.*

While we limit attention to those zero-sum two-person games, we have to keep in mind that there are two different kinds of games. The first kind is called *a strictly determined game or a closed game*, in which an equilibrium solution can be found in the specific set of *pure strategies.* The second kind is named a *non-strictly determined* *game or an open game,* where an equilibrium solution cannot be determined by the limited range of pure strategies, but should be found in the largely expanded set of *mixed strategies or the stochastic combinations of pure strategies.* [3)]

Frankly speaking, it is true that von Neumann and Morgenstern placed more emphasis on non-strictly determined games than strictly determined games. In order to climb up to the top of a high mountain peak, however, it would be a good idea to start with a sequence of lower hills and mountains. Likewise, we would like to begin with a group of simpler strictly determined games before challenging more general yet difficult (non-strictly determined) games.

In what follows, we would like to carefully discuss the basic structure of a zero-sum two-person game. To this end, let us suppose that the two players, Player A and Player B, engage in a certain game. As is seen in Fig. 1, whereas Player A has two pure strategies, α_1 and α_2, Player B has two strategies, β_1 and β_2. If A chooses α_i and B chooses β_j, then A's outcome is shown by a_{ij} $(i, j = 1, 2)$. Since this is a zero-sum game, the sum of A's outcome and B's outcome must be zero. This implies that under the present situation, B's outcome must be given by $(-a_{ij})$.

Von Neumann and Morgenstern assume that each player, Player A and Player B, is a very defensive person in the sense that he follows the rule of "safety first," We think that this would be a debatable assumption, perhaps provoking someone to raise objections. However, we will be content here to just obey the Neumann-Morgenstern assumption that each player always "play safe" in any possible situation.

When Player A chooses strategy α_1., which strategy , β_1 or β_2, A' s opponent B chooses cannot be known to A in advance. All we can say at this point is that A's payoff is a_{11} if B chooses strategy β_1, and a_{12} if B chooses strategy β_2. Hence, whichever strategy B chooses, *at least* the amount of $\min_j a_{1j}$ is guaranteed as a "security level" by A's choice of α_1. Therefore, we may call $\min_j a_{1j}$ the *security level for Player A of strategy* α_1 In a similar fashion, the *security level for Player A of strategy* α_2 is shown by $\min_j a_{2j}$. The purpose of Player A's choice behavior is to choose a strategy that makes his security level as much as possible. In other words, Player A' s behavior principle is summarized as the *max-min strategy*, and *the value of the game for Player A* is thus given by the following equation:

$$v_A = \text{Max}_i \text{ Min}_j \, a_{ij} \quad . \qquad (1)$$

PLAYER B

Pure Strategies

		β_1	β_2	row max (security level)	
PLAYER A	α_1	a_{11}	a_{12}	$\mathrm{Min}_j\ a_{1j}$ ⇒	$v_A = \mathbf{Max}_i\ \mathbf{Min}_j\ \mathbf{a}_{ij}$
Pure Strategies	α_2	a_{21}	a_{22}	$\mathrm{Min}_j\ a_{2j}$	
Column max (security level) ⇒		$\mathrm{Max}_i\ a_{i1}$	$\mathrm{Max}_i\ a_{i2}$		
		$v_B = \mathbf{Min}_j\ \mathbf{Max}_i\ \mathbf{a}_{ij}$			

Fig. 1 Zero-Sum Two-Person Game

If $v_A = v_B$, then the common value shows the value of a strictly determined game

Now, we recall that the behavior rules of the two players, Player A and Player B, should be just symmetrical. For Player B, an analogous situation must obtain. It is reasonable to call $\max_j\ a_{ij}$ the security level of strategy β_j , for any loss greater than level is effectively avoided.

. The purpose of Player B's choice behavior is to choose a strategy that makes his loss level as less as possible. In other words, Player A' s behavior principle is summarized as the *min-max strategy*, and *the value of the game for Player B* is thus given by the following equation:

$$v_B = \mathrm{Min}_j\ \mathrm{Max}_i\ a_{ij} \quad . \tag{2}$$

It is quite clear that in general, we have $v_A \geqq v_B$. In particular, if v_A and v_B happen to be just equal, then we say that *the solution of the game, v ,* exist, thus being written as follows:

$$v = v_A = v_B \quad . \tag{3}$$

Given any zero-sum two-person game, there is in general no guarantee that the solution of the game is provided by pure strategies only. The solution exists in some cases, but it does not exist in other cases. When we carefully read *Theory of Games and Economic Behavior* (1944), we find the interesting fact that there is only one solvable case which has been dealt with by von Neumann and Morgenstern. Such an "exceptional case" is shown in Fig. 2.

Fig. 2 consists of two charts, Chart (A) and Chart (B). First of all, by looking at Chart (A), we find " Max of Row Min = − 1 " on the right of the outcome matrix, and "Min of Column Max = − 1 " on the bottom of the matrix. This common value, circled by a dotted line, must represent the value of the game; namely, $v = v_A = v_B = -1$. Besides, the corresponding payoff (−1) in the payoff matrix is vividly circled by a solid line.

Let us figure out a good way to handle the present situation. Our unique device is based on drawing a three-dimensional diagram as shown in Chart (B). Take a careful look at two group of directions; one group, containing α_1- direction and α_2- direction, going from the near side to the far side, and another group, consisting of β_1- direction and β_2- direction, going from left side to the right side. Then, interestingly enough, we will find that Point a_{21} , indicated by black ball ● , is a *saddle point* in the sense that it is minimal in α - direction and maximal in β - direction. This clearly teaches us that any solution of the game, v , may be characterized as a saddle point.

We wonder if there might exist some suspicious readers who cast skeptical eyes upon the view of regarding a_{21} as a saddle point. In Fig. 2, since all the four points, a_{11}, a_{12}, a_{21}, and a_{22}, are *end points*, the unique convex-concave shape of a saddle point is not well-pictured in such a simple diagram. It is in this sense that a_{21} might not look like sort of "full-fledged saddle point" to some non-mathematical readers. In order to mend such imperfection, let us introduce our own device of newly introducing "supplementary strategies" into Chart (B).

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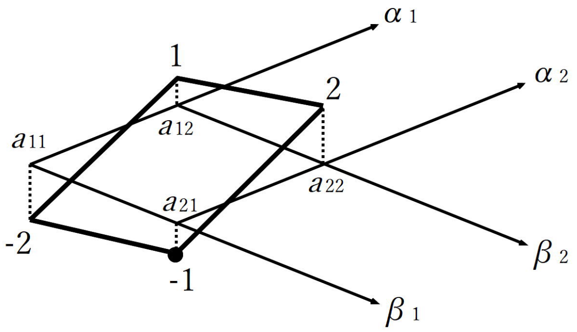
We wonder if there might exist some suspicious readers who cast critical eyes

PLAYER *B*

		β_1	β_2	Row Max	
PLAYER *A*	α_1	−2	1	−2	
	α_2	$\boxed{-1}$	2	$\boxed{-1}$	Max
Column Max		$\boxed{-1}$	2		
		Min			

(A) Pure strategies and payoff matrix



(B) Graphical representation of the original game

Fig. 2 A strictly determined game: the solution graphically illustrated upon the view of regarding a_{21} as a saddle point. In Fig. 2, since all the four point, a_{11}, a_{12}, a_{21}, and a_{22}, are *end points*, the unique convex-concave shape of a saddle point is not well-pictured in such a simple diagram. It is in this sense that a_{21} might not look like sort of "full-fledged saddle point" to some non-mathematical readers. In order to mend such imperfection, let us introduce our own device of newly introducing "supplementary strategies" into Chart (B).

More specifically, as is seen in Chart (A) of Fig. 3, we are ready to draw an "expanded outcome matrix" in which we introduce α_3 as a supplementary strategy of Player A, and β_0 as a supplementary strategy of Player B. Then, the reader will easily understand the great impact of supplementary strategies, for the new strategy α_3 is a strategy inferior to the old strategy α_1, and the new strategy β_0, inferior to the old strategy β_1. We should keep in mind that the "expanded game" added with supplementary strategies has essentially the same structure as the "original game." The graphical representation of such an expanded game is depicted in Chart (B) of Fig. 3. We are now happy to see that Point a_{21}, indicated by black ball ●, is really a charming *saddle point* to everybody's eyes. Thus, introduction of the two newly added strategies, α_3 and β_0, are quite appealing to our eyes. As the saying goes, seeing is believing indeed !

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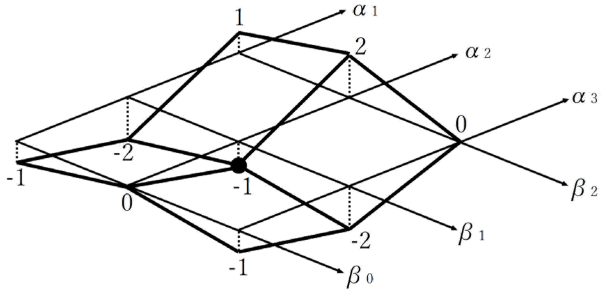
PLAYER B

		β_0 (newly added)	β_1	β_2	Row Min
	α_1	−1	−2	1	**−2**
PLAYER A	α_2	0	$\boxed{-1}$	2	$\boxed{-1}$ **Max**
	α_3 (newly added)	−1	−2	0	**−2**
	Column Max	0	$\boxed{-1}$ **Min**	0	

(A) New strategies α_3 and β_0 added to the matrix



(B) The point a_{21} is a saddle point in the expanded game

Fig. 3 The expanded game with newly added strategies, α_3 and β_0

4. The Game of Matching Pennies： Mixed Strategies and Non-Strictly Determined Games

According to von Neumann & Morgenstern (1944), *zero-sum two-person games* are clearly the most important games. Remarkably, those games per se can be further divided into two different kinds. The first kind of games are classified as *strictly determined games*, whose solutions are determined by *pure strategies only.* Although those kinds of games have been exclusively discussed so far, however, they are relatively a few. In the real world, however, the second kind of games, or exactly *non-strictly determined games,* whose solutions should be written as *the mixtures of pure-strategies,* or briefly *mixed strategies* , are more important.

Like many other scientists, von Neumann and Morgenstern got wild with excitement over many indoor games. Among those games, they had much interest in the following two games. They are the Game of Matching Pennies and the Game of Stone, Paper, Scissors. In fact, von Neumann and Morgenstern wrote the following impressive remark:

> In order to overcome the difficulties in the non-strictly determined case, it is best to reconsider the simplest examples of this phenomenon. These are games of Matching Pennies and of Stone, Paper, Scissors, Since an empirical, common-sense attitude with respect to the "problems" of these games exists, we may hope to get a clue for the solution of non-strictly determined (zero-sum two-person) games by observing and analyzing these attitudes.
>
> (von Neumann & Morgenstern 1944, p. 143)

Honestly speaking, the great book *Theory of Games and Economic Behavior* is not an easy book to read. First, it is a bulky book containing 600 and more pages, written in apparently German-like complicated English. Second, it looks like a Book of Applied Mathematics, with heavy mathematical equipment. There are many strange notations, overwhelming figures and involved computations. Third, it is not so well-organized as it should be. Unnecessary repetition of similar, and even the same, topics can be seen almost everywhere throughout the book. For instance, the two authors' pet subject "The Game of Matching Pennies" has been repeatedly referred to so many times; in page 94, page 111, pages 143-144, page 164, page 166, page 169, pages 175-178, page 185, and so on. As the saying goes, perseverance may accomplish many things. We do not quite understand, however, why very similar subjects are scattered in so many pages.

Von Neumann & Morgenstern (1944) is generous enough to discuss four different cases for 'The Game of Matching Pennies." While the first three cases will be dealt with in this section, the last fourth case, which is nothing but the story *The Final Problem* in Conan Doyle's *Memoirs of Sherlock Holmes* (1892), will very carefully be reinvestigated in the next section.

Let us take a close look at Fig. 4. It clearly shows *The First Game of Matching Pennies,* or *Matching Pennies in Ordinary Form.* In Chart (A), there are two players in the game, namely Player A and Player B. We assume that each player has two strategies — " H " (head) and " T " (tail). On the one hand, if A ' s strategy and B ' s strategy are *just matching* like " *H versus H,* " or "*T versus T ,"* then A gains the amount (+1) as the winner 's prize (but B suffers the amount (−1) as the loser's loss). On the other hand, if the two persons' strategies are *non-matching* like " *H versus T," or " T versus H," then A* suffers the amount (−1) as the loser's loss (but B gains the amount (+1) as the winner 's prize). As Fig. 4 tells us, the relevant outcome matrix becomes a symmetric matrix with all the diagonal elements (+1) and all the off-diagonal elements (−1). [4)]

Let us calculate the values of the game for Players A and B. Then, we will easily find the following:

$$v_A = \mathrm{Max}_i \ \mathrm{Min}_j \ a_{ij} = -1 \ , \tag{4}$$

$$v_B = \mathrm{Min}_j \ \mathrm{Max}_i \ a_{ij} = 1 \ . \tag{5}$$

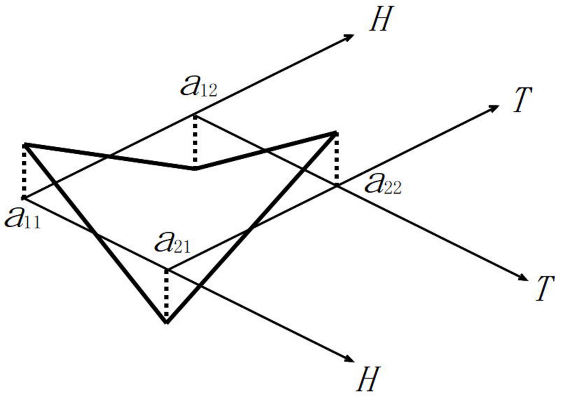
Apparently, the two values, v_A and v_B , are not equal. This implies that in "The Standard Form," which limits our analysis to pure strategies, does not yield any solution at all. In other words, as Chart (B) tells us, there are no saddle points in the chart. In fact, while the two points a_{11} and a_{22} represent maximal peaks, the two points a_{12} and a_{21} indicate minimal bottoms. None of those four points can be attractive saddle points !

Now, we are caught in a sort of dilemma. Probably, there would be two different ways to get out of it. First, we would accept non-existence of the solution as such, thus deeply introspecting its philosophical meaning. Second, we would not feel self-satisfied with such non-existence, but attempting to break the deadlock by expanding the limits of solutions.

PLAYER *B*

		H	T	Row Min		*DIFFERENCE* (*optimum ratio)*
PLAYER *A*	H	1	−1	$\boxed{-1}$	Max	*2* (*p = 1/2*)
	T	−1	1	$\boxed{-1}$	Max	*2* (*1−p = 1/2)*
Column Max		$\boxed{1}$	$\boxed{1}$			
		Min	Min			
DIFFERENCE		*2*	*2*			
(optimum ratio1/2)		*(q= 1/2)*	*(1/q = 1/2)*			

(A) Payoff matrix : limiting pure strategies does not yield solutions



(B) A graphical representation : saddle points do not exist

Fig. 4 The game of matching pennies (Case Ⅰ) : the standard case

It is von Neumann and Morgenstern who have taken the second way out. More specifically, they dare to assume that each player can play a *mixed strategy* in the sense that he takes a *stochastic mixture* of the two strategies " H " (Head) and " T " (Tail). For instance, if A chooses H with probability p, and T with probability $(1-p)$, A 's mixed strategy may be written as 【 H , T : p , $1-p$ 】 . Likewise, if B chooses H with probability q, and T with probability $(1-q)$, B's mixed strategy is formulated as 【 H, T: q, $1-q$ 】 . If we introduce mixed strategies into the game, then the key problems to ask would be what the *optimum strategy* is all about, and how to find it.

Fortunately, a very convenient method to find the optimum strategy is available. Let us take a careful look at a pair of numbers written on the right of the dotted line, and at another pair on the outside of the bottom line. On the one hand, when Player A chooses strategy H , its payoff is 1 if B also chooses H , and it is (−1) if B chooses T. Therefore, we find that *the difference of the two payoffs* must be 2 in terms of the *absolute value*. On the other hand, when A chooses T, its payoff is (−1) if B chooses H, and it is 1 if B also chooses T, whence *the difference of the two outcomes* must be 2 in terms of the *absolute value.* If we compare those two values, then we immediately find that the corresponding *optimal ratios of A' strategies* should be $p = 1/2$ and $1- p = 1/2$. In an analogous fashion, we can find that the corresponding optimal ratios of B should be $q = 1/2$ and $1-q = 1/2$. 5)

For convenience, let us suppose that Player A ' s mixed strategies are limited to three: namely, $p = 0, 1/2, 1$. Similarly, Player B's strategies are also limited to three: namely, $q = 0, 1/2, 1$. Then, the corresponding outcome matrix (b_{ij}) is given by Chart (A) of Fig. 2.5. Making use of the last chart, namely Chart (A) of Fig. 4, we can compute the values of each b_{ij} in the following way.

$$b_{ij} = (1)\, p\, q + (-1)\, p\, (1-q) + (-1)(1-p)\, q + (1)(1-p)(1-q)\, . \qquad (6)$$

For instance, if $p = 0$ and $q = 0$, then we can compute b_{11} as follows:

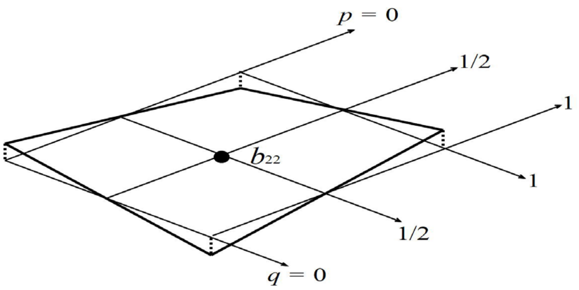
$$b_{11} = (1)(0)(0) + (-1)(0)(1) + (-1)(1)(0) + (1)(1)(1) = 1.$$

The remaining values of the outcome matrix in Chart (A) can be computed in an analogous way.

PLAYER B

		q = 0	1/2	1	Row Min	
	p = 0	1	0	−1	−1	
PLAYER A	1/2	0	$\boxed{0}$	0	$\boxed{0}$	Max
	1	−1	0	1	−1	
Column Max		1	$\boxed{0}$	1		
			Min			

(A) Stochastic mixed strategies and the payoff matrix



(B) A graphical representation : b_{22} is a saddle point.

Fig. 5 A mixed-strategy complication of Case 1 : a solution surely exists

Now, regarding the new outcome matrix (b_{ij}) , we can find the *values of the game* for Players A and B as follows:

$$v_A \;=\; \mathrm{Max}_i\ \mathrm{Min}_j\ b_{ij} \;=\; 0 \quad , \tag{7}$$

$$v_B \;=\; \mathrm{Min}_j\ \mathrm{Max}_i\ b_{ij} \;=\; 0 \quad . \tag{8}$$

As is quite clear, the two values, v_A and v_B, are just equal , implying that the *value of the game* is simply confirmed as follows:

$$v \;=\; v_A \;=\; v_B \;=\; 0\ . \tag{9}$$

Chart (B) tells us that the middle point b_{22} stands for a charming "saddle point" or the "equilibrium point" of the game under question. In fact, given any zero-sum two-person game, if we deals with it in an *expanded game* by introducing mixed strategies, we can always find a solution of the game in a theoretical framework, namely a saddle point in a graphical framework. This is no less than the primary purpose of writing the great book *Theory of Games and Economic Behavior* by the collaboration of mathematician Von Neumann and economist Oskar Morgenstern. 6).

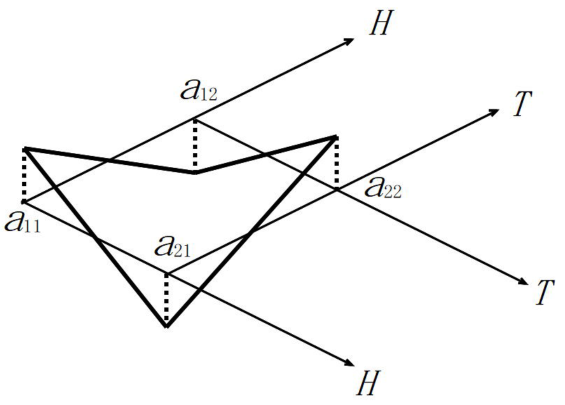
Now, let us turn to Case 2. As is seen in Fig. 6, this is a "biased case" in that matching on heads gives a *double premium*. Hence the outcome matrix of Fig. 6 differs from the one of Fig. 4 by the doubling of its element a_{11} . Apparently, as both Chart (A) and Chart (B) can tell us, limiting our analysis on pure strategies yields neither a solution nor a saddle point. Although this is exactly the same result as Case 1, we can nevertheless see a difference between Cases 1 and 2. Since the element a_{11} of Case 2 is now doubled, the corresponding point in Chart (B) should tower twice higher, so that the *optimal ratios of each player's mixed strategies* should also be changed accordingly.

More specifically, following the convenient method mentioned above, we can easily compute such optimal ratios of mixed strategies. As Chart (A) of Fig. 6 tells us, we are able to derive $p = 2/5$ and $q = 2/5$. For illustrative convenience, let us limit the mixed strategies available to both Players A and B be the following six :

$$p\,,\,q \;=\; 0,\;\; 1/5,\;\; 2/5,\, 3/5,\, 4/5,\, 1\ . \tag{10}$$

		PLAYER *B*				
		H	T			*DIFFERENCE*
				Row Min		(*optimum ratio*)
PLAYER *A*	H	2	−1	$\boxed{-1}$	Max	*3* (*p = 2/5*)
	T	−1	1	$\boxed{-1}$	Max	*2* (*1−p = 3/5*)
Column Max		$\boxed{2}$	$\boxed{1}$			
			Min			
DIFFERENCE		*3*	*2*			
(*optimum ratio*)		(*q= 2/5*)	(*1/q = 3/5*)			

(A) Payoff matrix : limiting pure strategies does not yield solutions



(B) A graphical representation : saddle points do not exist

Fig. 6 The special game of matching pennies (Case 2) : The *H*-*H* matching doubles the prize

Then, the relevant outcome matrix (b_{ij}) will be given by the 6 by 6 matrix in Fig. 7, in which each outcome is multiplied by five here. For example, if $p = 2/5$ and $q = 2/5$, then Player A ' s outcome is given as follows:

$$b_{33} = (2)(2/5)(2/5) + (-1)(2/5)(3/5) + (-1)(3/5)(2/5) + (1)(3/5)(3/5) = 1/5 .$$

Hence, multiplying the outcome by five, we may obtain $5\,b_{22} = 1$ as we wish. Besides, we can find the *values of the game* for Players A and B as follows:

$$5\,v_A = 5\,v_B = 1 .$$

Therefore, a solution surely exists in the *expanded game with mixed strategies*, and is shown by $v = 1/5$. Graphically speaking, it is also a charming saddle point. Let us make a comparison between Example 1 and Example 2. This way, we will be able to well-analyze the impact of giving a double premium for matching heads.

Now, we are ready to turn our attention to Example 3. As is clearly seen in Fig. 2.8, this is a "further twisted game" in which matching on heads gives a *double premium,* but failing to match on a choice (by Player A) of heads gives a *triple penalty.* Interestingly enough, this game is " *a game twisted in both positive and negative ways, with a penalty overpowering a premium.* "

Let us take a look at Fig. 8. Then, like in previous two cases, limiting our analysis on pure strategies guarantee neither the existence of a solution nor a charming saddle point. Following the convenient method aforementioned, we can obtain the optimal ratios as follows:

$$p = 4/7 , \quad 1-p = 3/7 ; \quad q = 4/7 , \quad 1-q = 3/7 .$$

For simplicity, let us limit the possible values of p and q to the following eight:

$$p , q = 0, 1/7, 2/7, 3/7, 4/7, 6/7, 1 .$$

Then, the outcome matrix of mixed strategies will be shown in Fig. 8, in which each outcome is multiplied by seven. For instance, if $p = 2/7$ and $q = 4/7$ then Player A ' s outcome is provided as follows:

Player *B*

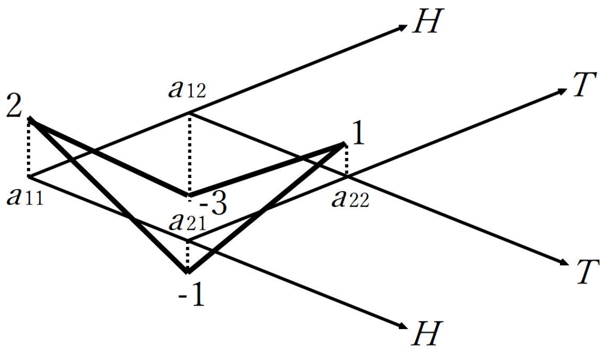
	$q = 0$	1/5	2/5	3/5	4/5	1	Row Min	
p = 0	5	3	1	−I	−3	−5	−5	
1/5	3	2	1	0	−1	2	−2	
2/5	1	1	$\boxed{1}$	1	1	1	$\boxed{1}$	Max
3/5	−1	0	1	2	3	4	−1	
4/5	−3	−1	1	3	5	7	−3	
1	−5	−2	1	4	7	10	−1	
Column Max	5	3	$\boxed{1}$	4	7	10		
			Min					

Fig. 7 Allowing mixed strategies to find a solution of the game of Case 2

Remark. For convenience, all the payoffs are quintupled.

			PLAYER *B*			
		H	T			*DIFFERENCE*
				Row Min		*(optimum ratio)*
PLAYER *A*	H	2	−3	$\boxed{-3}$	Max	*5* (*p = 2/7)*
	T	−1	1	−1		*2* (1−*p*=5/7)
	Column Max	2	$\boxed{1}$			
			Min			
	DIFFERENCE	*3*	*4*			
	(optimum ratio)	*(q= 4/7)*	*(1/q = 3/7)*			

(A)　Payoff matrix：　limiting pure strategies does not yield solutions



(B)　A graphical representation：　saddle points do not exist

Fig. 8　Another special game of matching pennies (CASE 3)：

While *H*-*H* matching doubles the prize, *H*-*L* matching triples the penalty.

$$b_{35} \;=\; (2)(2/7)(4/7) + (-3)(2/7)(3/7) + (-1)(5/7)(4/7) + (1)(5/7)(3/7) \;=\; -1/7\,.$$

Hence, we obtain $7b_{35} = -1$ as we wish. Other elements of the outcome matrix $(b_{ij.})$ will be found in a similar way. Moreover, we can find the *values of the game* for Players A and B as follows:

$$7\,v_A \;=\; 7\,v_B \;=\; -\,1\,.$$

Therefore, a solution surely exists in the *expanded game with mixed strategies*, and is shown by $v = -\,1/7$. Graphically speaking, it is also a charming saddle point. A comparison between Examples 2 and 3 will lead to a series of interesting results. To save the space, however, we omit them altogether here.

PL *B*

			q = 0	1/7	2/7	3/7	4/7	5/7	6/7	**Row Min** 1	
	p =0	7	5	3	1	-1	-3	-5	-7	0 -7	
	1/7	3	2	1	0	-1	-2	-3	-4	-4	
	2/7	-1	-1	-1	-1	$\boxed{-1}$	-1	-1	-1	$\boxed{-1}$	**MAX**
PL	3/7	-5	-4	-3	-2	-1	0	1	2	-5	
A	4/7	-9	-7	-5	-3	-1	1	3	5	-9	
	5/7	-13	-10	-7	-4	-1	2	5	9	-13	
	6/7	-17	-11	-9	-5	-1	3	7	11	-17	
	1	-21	-19	-11	-9	-1	4	9	14	-21	
Column		7	5	3	1	$\boxed{-1}$	4	9	14		
Max						**MIN**					

Fig. 9 Allowing mixed strategies to find a solution of the game of CASE 5

Remark. For convenience, the payoffs are all multiplied by seven.

PL *A* and PL *B* mean Player *A* and Player *B*, respectively.

5. Novelist Edgar Alan Poe on "The Game of Even and Odd Numbers"

As is now well-known, the birth and development of game theory has been closely related to detective or mystery stories. In his 1928 paper, Morgenstern (1928), economist Oskar Morgenstern paid attention to the famous novel *The Final Problem* (1893) by writer Conan Doyle, discussing "the duel between detective Sherlock Holmes and Professor Moriarty " from a game theoretical point of view. Facing the fundamental difficulty to find a reasonable solution to the Holmes-Moriarty problem in the traditional framework of general equilibrium theory, he strongly felt the need to establish a brand new theory of games. Besides, a detailed discussion on this problem was done again in the great book *The Theory of Games and Economic Behavior* (1944) by Morgenstern's collaboration with mathematician von Neumann.

We would strongly believe that *The Final Problem* should be a problem worthy of the name, so that it should be dealt with as the very final problem for two-person zero-sum games. So, before proceeding to that one, we would like to discuss here the second from the last problem, which is the " Game of Even and Odd Numbers" invented by American novelist Edgar Alan Poe (1809-1849). Poe is around 50 years older than Doyle, and regarded as one of mystery story writers of world fame. [6)]

"The Game of Even and Odd Numbers" is a very interesting game, which was first introduced by Poe (1844) in his mystery story *The Purloined Letter*. This game, characterizing Poe as a sharp-eyed observer of human behavior, is concerned with the game-theoretic situation in which each of the two persons is expected to outguess the strategy of his/her opponent. In his long yet nicely-written paragraph, Poe remarked:

> The Prefect errs by being too deep or too shallows, for the matter in hand; and many a schoolboy is a better reasoner than he. I [Detective Dupan] knew one about eight years of age whose success at guessing in the game of 'even and odd' attracted universal admiration. This game is a simple, and is played with marbles. One player holds in his hands a number of these toys, and demands of another whether that number is even or odd. If the guess is right, the guesser wins one; if wrong, he loses one. The boy to whom I allude won all the marbles of the school. Of course he had some principle of guessing; and this lay in mere observation and admeasurements of the astuteness of his opponents. For example, an arrant simpleton is his opponent, and, holding up his closed hand, asks, ' are they even or odd? ' Our schoolboy replies, ' odd,' and loses; but upon the second trial he wins, for he then says to himself, the simpleton had them even upon the first trial, and his amount of cunning is just sufficient to make him have them odd upon the second: I will therefore guess odd'; − he

> guesses odd, and wins. Now. with a simpleton a degree above the first, he would have reasoned thus: This fellow finds that in the first instance I guessed odd, and, in the second he will propose to himself upon the first impulse, a simple variation from even to odd, as did the first simpleton, but then a second thought will suggest that this is too simple a variation, and finally he will decide upon putting it even as before. I will therefore guess even guesses even, and wins. Now this mode of reasoning in the schoolboy, whom his fellows termed ' lucky, ' -- what, in its last analysis, is it ?" "It is merely, " I said, " an identification of reasoner's intellect with that of his opponent." (Poe 1845)

Remarkably, this story by Poe describes a game-theoretic situation where a " wonder boy " and his opponent, either " an arrant simpleton" or " a bit clever boy," play the simple " Game of Even and Odd Numbers." Each player has only two strategies, " Even (Number) " and " Odd (Number.). " As is shown in Fig. 10, there are typically two trials of games. At the first trial, the opponent Simpleton holds in his hand a number of marbles, asks Wonder Boy whether the number is "Even" or "Odd." Suppose that Wonder Boy replies "Odd" and loses, with his outcome being (−1). Then, the critical second trial will begin. Presumably, there are two different kind of opponents, an Arrant Simpleton and a Clever Kid. On the one hand, if the opponent is an Arrant Simpleton, he is expected to simply change his strategy from " Even " to " Odd, " and Wonder Boy rightly guesses " Odd " and wins. On the other hand, if the opponent is a Clever Kid, his next strategy becomes more complicated than an Arrant Simpleton. The Clever Kid will have the *second thought* that a simple variation from "Even" to "Odd," as did the first Simpleton, is too simple a variation to believe, and finally he will change again to choose "Even" again. As a result, the Wonder Boy" s guessing is " Old " and eventually wins. Consequently, whether the opponent is Simpleton or Clever Kid, the Wonder Boy eventually wins the game.

Undoubtedly, Poe's Game of Even and Odd Numbers is no less than a Game of Matching Pennies, its outcome matrix being identical to that of Fig. 4 above. It is in this sense that Poe may be regarded as one of pioneers of Game Theory. In Poe's Game, as was repeatedly discussed above, limiting our analysis on pure strategies guarantees neither a solution nor a saddle point. If we expand the scope of strategies by allowing mixed strategies, however, as was shown by Fig. 5, a solution (or a saddle point) surely exist when each player chooses a specific mixed strategy of " Even " and " Odd " with a 50-50 chance. Then, the solution of the game would be just zero. Hence, if this kind of game is played repeatedly, the game would be fair in the sense that each player

【*I . The First Trial* 】

		THE OPPONENT	
		$\boxed{\text{EVEN}}$	ODD
THE WONDER BOY	EVEN	1	−1
	$\boxed{\text{ODD}}$	$\boxed{-1}$	1

The Wonder Boy loses and the opponent wins.

【*II (A) . The Second Trial against a mere simpleton* 】

If THE OPPONENT is *an arrant simpleton* , then THE WONDER BOY simply changes his strategy: **EVEN ⇒ ODD**

		THE OPPONENT	
		EVEN ⇒	$\boxed{\text{ODD}}$
THE WONDER BOY	EVEN	1	−1
	$\boxed{\text{ODD}}$	−1	$\boxed{1}$

The Wonder Boy wins

【 *II (B) . The Second Trial against for a little cleverer kid* 】

If the opponent is *a little cleverer kid*, then the wonder boy has to take a more complicated strategy. Although the boy might be tempted to change his strategy from **EVEN to ODD,** he must have a second thought to change his strategy again from **ODD** to **EVEN , namely EVEN ⇒ ODD ⇒ EVEN .**

		THE OPPONENT	
		EVEN ⇒	ODD
		$\boxed{\text{EVEN}}$ ⇐	ODD as a second thought
THE WONDER BOY	$\boxed{\text{EVEN}}$	$\boxed{1}$	−1
	ODD	−1	1

The Wonder Boy also wins

Fig. 10 Edgar Alan Poe on the Game of Even and Odd Numbers

would be a winner or a loser with a 50-50 chance. Hence, there should be neither a one-sided winner nor a one-sided loser.

Here, we find that Poe's sharp observation is in sharp contrast to the Neumann-Morgenstern way of thought. According to Poe, if the Game of Even and Odd Numbers is played repeatedly, then the winning ratio (or the losing ratio) of each player will never converge to a half, but tends either to expand to unity or to contract to zero. As a result, as is seen in Fig. 10, the "wonder boy" will dominate the game and monopolize all the prizes, whereas the "less intelligent kid" will be exploited by the "wonder boy." Such an unfair distribution of the prizes, according to Poe as a critical observer, originates in unequal distribution of intelligence and anticipation between the two players.

In short, Edgar Alan Poe as a critical observer of the real world, has constantly emphasized the cruel fact that in any two-person game, the coexistence of a one-sided winner and a one-sided loser appears. This radically differs the situation game theorists have ardently advocated. Therefore, we might see a dilemma between "the practical solution" and "the theoretical solution." We wonder which solution the reader would be inclined to support. In our opinion, Poe's objection against an easy theoretical solution is really worth paying attention to. In general, theory is no more than theory, and should not be almighty. If there occurs an irreconcilable discrepancy between the theory and the reality, we must rely on the reality, not on the theory.

2.6 Conan Doyle's Story *The Final Problem* as a Variant of Matching Pennies

We are now in a position to reexamine Conan Doyle's story *The Final Problem* as a variant of matching pennies. Although we already discussed it in Chapter 1 of this book, we wish to shed new light on it from the viewpoint of the Game of Matching Pennies. Exactly speaking, in von Neumann & Morgenstern (1944), *The Final Case* is regarded as Example 4, just following the previous Examples 1, 2, and 3.

In *The Final Problem*, there are the two players, Professor Moriarty and Sherlock Holmes. While their levels of intelligence are almost equal, their strategies should be different, thus requiring cool manipulation and shrewd anticipation. As is seen in Fig. 11, each person has two strategies, "Proceeding to Dover" and "Quitting at Canterbury. The fields (D,D) and (C,C) correspond to Moriarty catching Holmes, which it is reasonable to describe by a certain positive value. Whereas the field (C,D) shows that Holmes successfully escaped to Dover (and presumably to the Continent), Moriarty had to stop at the intermediate station Canterbury. Clearly, this means Moriarty's defeat, thus being describable by a certain negative value . Finally, the field (D,C) means that Holms escapes Moriarty at Canterbury, but fails to reach the Continent. This can be thought of as a draw.

The most important problem to ask here is specifically what numbers should be filled in the four fields, namely (D,D), (CC), (D,C) and (C,D). Conceivably, there would be a variety of ways of giving numbers. The *first interpretation* by von Neumann and Morgenstern (1944) is given by Fig. 11.

As is plainly clear, the outcome matrix in Chart (A) of Fig. 11 shows that overall, Professor Moriarty has an advantage over Sherlock Holmes in their confrontation. In fact, whereas the positive value of (+2) appears twice, namely in the fields (D,D) and (C,C), the negative value of (−2) does exist at all, and the negative value (−1) appears only once in the field (C,D), with the zero value in the field (D,C). [7]

As can easily be seen, limiting our analysis on pure strategies does not guarantee a solution of the game. This is because the following inequality has to hold here:

$$v_M = \mathrm{Min}_i \mathrm{Max}_j a_{ij} = 2 > 0 = \mathrm{Max}_j \mathrm{Min}_i a_{ij} = v_H \quad . \qquad (11)$$

The value of the game for Moriarty is greater than that for Holmes, implying that the game per se is not fair at all in the sense that it actually more advantageous to Moriarty than Holmes.

Now, in line with the idea of von Neumann and Oskar Morgenstern, let us expand the scope of strategies of each player to introduce the "mixture of pure strategies," or simply "mixed strategies" into the game. Specifically, let us consider the mixed strategy of mixing the pure strategies D and C, with the ratio p and $(1-p)$, which can be denoted by $[D, C; p, 1-p]$. For the sake of convenience, let us limit the values of p and q on the five numbers: 0. 1/5, 2/5, 3/5, 4/5, and 1. Then, we obtain the outcome matrix of mixed strategies (b_{ij}), as is seen in Fig. 11.

In this expanded game, the pair of mixed strategies (p , q) = (3/5. 2/5) gives a solution, thus being a saddle point. For convenience, each value of Panel (B) is multiplied by 1/5 . Then, it is an easy job to derive the following equation:

$$v_M = \mathrm{Min}_i \mathrm{Max}_j b_{ij}/5 = 4/5 = \mathrm{Max}_j \mathrm{Min}_i b_{ij}/5 = v_H \quad . \qquad (12)$$

Since the game of the expanded game is a positive value, we are confirmed again that the game per se is a unfair game, guaranteeing a one-sidedly victory for Moriarty and a one-sided defeat for Holmes. Moriarty's best strategy is the strategy of

HOLMES

		Dover	**Canterbury**	Row Min	
MORIARTY	**Dover**	2	0	$\boxed{0}$	**Max**
	Canterbury	−1	2	−1	
Column Max		$\boxed{2}$	$\boxed{2}$		
		Min	**Min**		

(A) Payoff matrix : limiting pure strategies does not yield solutions

HOLMES

		$q = 0$	1/5	2/5	3/5	4/5	1	Row Min	
	p = 0	10	7	4	1	−2	−5	−5	
	1/5	8	6	4	2	0	−2	−2	
	2/5	6	5	4	3	2	1	1	
MORI-	3/5	4	4	$\boxed{4}$	4	4	4	$\boxed{4}$	**Max**
ARTY	4/5	2	3	4	5	6	7	2	
	1	0	2	4	6	8	10	0	
Column Max		10	7	$\boxed{4}$	6	8	10		
				Min					

(B) Allowing mixed strategies to find a solution of the game (A)

Remark. For convenience, all the payoffs are multiplied by five.

Fig. 11 *The Final Problem* as a Game of Matching Pennies (FIRST Interpretation): Showdown between Moriarty and Holmes proceeding to Dover with probability 60 % , whereas Holmes's best strategy is the same strategy with probability 40 %. In plain English, while Moriarty has to make a special lottery containing " three Dover balls and two Canterbury balls, " Holms must make another symmetric lottery containing " two Dover balls and three Canterbury balls." Of course, those lotteries should be free and fair, being independent of any possible interferences. Although such lot drawing is perhaps conceivable in a fiction, it never exists in the real world in which flesh-and-blood persons like ourselves live. It is in this sense that the validity of introducing mixed strategies into the game must be discussed very seriously.

Now, as the *second possible interpretation*, let us consider Fig. 12 in which, contrary to the *first interpretation,* Sherlock Holmes is in a more advantageous position than Moriarty. If we decease *every outcome* in Panel (A) of the last Fig. 11 by the equal amount of one unit, then we immediately obtain the outcome matrix in Panel (A) of the present Fig. 12. Then, the positive value (+2) is now replaced by a smaller value (+1) in the fields (D,D) and (C,C), and the negative value (− 1) is lessened to (−2). Not only that, the positive value (+2) is also reduced to (+1). Overall, Holmes' position is considerably improved in comparison with the last situation.

As is quite clear in Panel (A) of Fig.12, limiting on pure strategies doest not guarantee a solution of the game. If we introduce mix strategies into the matrix, it is clear in Panel (B) that a solution surely exist, with $(p , q) = (3/5, 2/5)$ becoming a saddle point as before. The solution, (−1/5) , of the expanded game, certainly confirms Holmes's advantage against Moriarty.

		HOLMES			
		Dover	Canterbury	Row Min	
MORIARTY	Dover	1	−1	$\boxed{-1}$	Max
	Canterbury	−2	1	−2	
Column Max		$\boxed{1}$	$\boxed{1}$		
		Min	Min		

(A) Limiting pure strategies does not yield solutions

For convenience, all the payoffs are multiplied by (1/50) here.

		HOLMES							
		$q = 0$	1/5	2/5	3/5	4/5	1	Row Min	
	p = 0	5	2	−1	−4	−7	−10	−10	
	1/5	3	1	−1	−3	−5	−7	−7	
	2/5	1	0	−1	−2	−3	−4	−4	
MORI-	3/5	−1	−1	$\boxed{-1}$	−1	−1	−1	$\boxed{-1}$	Max
ARTY	4/5	−3	−2	−1	0	1	2	−3	
	1	−5	−3	−1	1	3	5	−5	
Column Max		5	2	$\boxed{-1}$	1	3	5		
				Min					

(B) Allowing mixed strategies to find a solution of the game (A)

Remark. For convenience, all the payoffs are multiplied by five .

Fig. 12 *The Final Case* as a Game of Matching Pennies (SECOND Interpretation):

Finally, let us attempt to give the *third interpretation*, by which the Game of *The Final Problem* is completely fair for either player. Concerning the two players, one player has neither an advantageous nor a disadvantage over the other.

Let us take a look at Panel (A) of Fig. 13, Then, we can see that there are the two positive values, (+2) and (+1), and the two negative values, (−2) and (−1), showing the fair distribution of outcomes in the matrix. In this game, limiting our analysis on pure strategies does not guarantee a solution at all. When we are allowed to take care of mixed strategies in the expanded game, however, a solution surely exist, and is indicated by a saddle point (p , q) = (2/3, 2/3). Note that the value of the game is $v = 0$. This clearly demonstrates that this game is a completely fair game for both players, namely Moriarty and Holmes.

In the above, we have given three different interpretations of *The Final Problem* as a variant of "The Game of Matching Pennies." They are the First, Second and Third interpretations. Unfortunately, von Neumann and Morgenstern (1944) discussed only the First interpretation, thus completely neglecting the Second and Third interpretations. We do think, however, that such unbalanced treatment is fairly groundless and should be amended. In our opinion, the Second and Third interpretations should be as important as the First interpretation. .

We are entitled to choose any one interpretation. Whichever interpretation we may choose, there nevertheless exists a hardly surmountable barrier before us. This barrier is related to the practical difficulty to determine the best mixed strategy from the relevant outcome matrix. On the one hand, knowing that Holmes is coming soon, Moriarty has to make a quick decision by making a strange lottery box including "three Dover balls and two Canterbury balls." We do not think that this is practically feasible for Moriarty as a "gentleman of high pride ." Presumably, his pride would be seriously wounded and hardly recoverable. Similarly for Sherlock Holmes as a archrival of Moriarty. They are engaging in an only once game that cannot be repeated. In such a life-or-death situation, relying on lotteries would hardly be conceivable. On the other hand, the situation in which the two persons of equal intelligence, like Moriarty and Holmes, are playing a life-or-death game would be exceptionally rare. In almost all situations, as is seen in the confrontation between highly intelligent Sherlock Holmes and arrant simpleton Cornel Moran, the IQs of the two opponents are likely to be quite different. Then, the result of the game would be what Poe's Game of Even and Odd Player would eloquently tell us: that is, one player will be a dominant winner and the other player a miserable underdog.

		HOLMES			
		Dover	**Canterbury**	Row Min	
MORIARTY	**Dover**	1	−1	$\boxed{-1}$	**Max**
	Canterbury	−2	2	−2	
	Column Max	$\boxed{1}$	2		
		Min	**Min**		

(A) **Limiting pure strategies does not yield solutions**

		HOLMES								
		$q = 0$	1/6	2/6	3/6	4/6	5/6	1	Row Min	
	$q = 0$	12	8	4	0	−4	−8	−12	−12	
	1/6	9	6	3	0	−3	−6	−9	−9	
MORIARTY	2/6	6	4	2	0	−2	−4	−6	−6	
	3/6	3	2	1	0	−1	−2	−3	−3	
	4/6	0	0	0	$\boxed{0}$	0	0	0	$\boxed{0}$	Max
	5/6	−3	−2	−1	0	1	2	3	−3	
	1	−6	−4	−2	0	2	4	6	−6	
	Column Max	12	8	4	$\boxed{0}$	2	4	6		
					Min					

(B) Allowing mixed strategies to find a solution of the game (A)

Remark. For convenience, all the payoffs are multiplied by six.

Fig. 13 *The Final Problem* as a Game of Matching Pennies (THIRD Interpretation)

To sum up, when we straightforwardly apply game theories to real situations, we should distinguish between the two different kinds of games. One kind of games, like outdoor games including baseball and tennis, are repeatable and regarded as "light games." The other games are more serious and non-repeatable, thus being thought of as " heavy games." It is in those "heavy games" that the validity of adopting mixed strategies would be seriously challenged. Further investigations on this important problem would be required.

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7. Reassessment of the Game of Stone, Paper, Scissors

We are now in a position to discuss another zero-sum two-person game which has been favored by von Neumann and Morgenstern. It is nostalgically called "The Game of Stone, Paper, Scissors." It is conventional for us to think that "Stone defeats Scissors," "Scissors defeat "Paper," and "Paper defeats Stone." Needless to say, there should be no ethical judgments on "Stone," "Paper," and "Scissors" *per se.*

It is quite easy to understand that "Matching Pennies" is regarded by von Neumann and Morgenstern as a very important illustration of zero-sum games, since it has been very popular in everyday life in any Western society. "Stone, Paper, Scissors," however, seems to be not so popular as "Matching Pennies. There might be some people asking why those two games require for equal treatment. In this respect, von Neumann & Morgenstern (1944) once remarked:

> These two examples show the difficulties which we encounter in a not-strictly determined games, in a particularly clear form; just because of their extreme simplicity the difficulty is perfectly isolated here, *in vitro.* The point is that in "Matching Pennies" and in "Stone, Paper, Scissors," any way of playing is just as good as any other: There is no intrinsic advantage or disadvantage in "heads" or "tails" *per se*, nor in "stone," "paper" or "scissors" *per se.* The only thing which matters is to guess correctly what the adversary is going to do; but how are we going to describe that without further assumptions about the players' "intellect" ? There are, of course, more complicated games which are not strictly determined and which are important from various more subtle, technical points. But as far as the main difficulty is concerned, the simple games of "Matching Pennies" and of "Stone, Paper, Scissors" are perfectly characteristic.
>
> (von Neumann & Morgenstern 1944, p. 111)

In historical perspective, at Princeton University in the 1940s, Dr. Shizuo Kakutani, then a Japanese rising star, worked hard as a research assistant to Professor John von

Neumann. This historical fact gives the conjecture that Kakutani personally told von Neumann and Morgenstern "the Exciting Game of Stone, Paper, Scissors." This conjecture might be correct or not correct, requiring still further investigation. [8)]

The outcome matrix of pure strategies for "The Game of Stone, Paper, Scissors" can be given in Fig. 14. Each of the two players, Players A and B, has the three strategies; "Stone," "Paper," and "Scissors." If we denote "win" by " +1", "loss" by "−1", and "draw" by "0" , the relevant outcome matrix is shown by Fig. 14. As is easily seen in Upper Chart (A), the following inequality holds:

$$v_A = \mathrm{Min}_i \mathrm{Max}_j a_{ij} = 1 > -1 = \mathrm{Max}_j \mathrm{Min}_i a_{ij} = v_B . \qquad (13)$$

Hence, limiting our analysis on pure strategies guarantees neither a solution nor a saddle point. Lower Chart (B) graphically shows the non-existence of a saddle point..

Now, let us expand the scope of strategies to allow the mixture of pure strategies, namely *mixed strategies*. Suppose that Player A chooses the mixed strategy " Stone with probability p_1, Paper with probability p_2, Scissors with probability p_3," where $p_1 + p_2 + p_3 = 1$, and that Player B chooses the mixed strategy " Stone with probability q_1, Paper with probability q_2, Scissors with probability q_3," where $q_1 + q_2 + q_3 = 1$. Then, the outcome of the mixed strategy of Player A is written as follows; [9)]

$$b_{ij} = \Sigma_i \Sigma_j \; p_i \; a_{ij} \; q_j \; . \qquad (14)$$

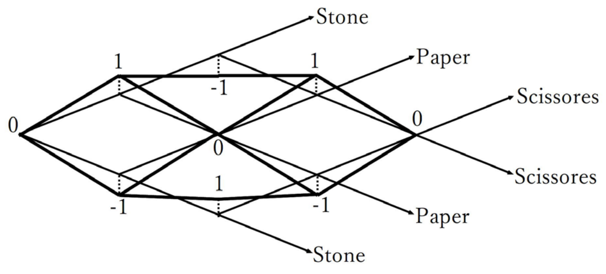
Given a_{ij}'s , the value of b_{ij} depends on Player A's strategy vector $p = (p_1, p_2, p_3)$. and Player B's strategy vector $q = (q_1, q_2, q_3)$. As can naturally be expected, the optimal mixed strategies of the two persons are provided as follows:

$$p = (1/3, 1/3, 1/3) \text{ and } q = (1/3, 1/3, 1/3) . \qquad (15)$$

In other word, Player A's optimal strategy must be the *fair strategy* in the sense that he chooses "Stone," "Paper," and "Scissors" with *equal probability*, that is 1/3 . In order to make this point clearer, let us simply write (1/3,1/3,1/3) as (1/3), (1,0,0) as (100), (0,1,0) as (010), and (0,0,1) as (001). Then, if we dare to ignore all other mixed strategies except those four, then the matrix of mixed strategies for "The Game of Stone, Paper, Scissors" will be given in Fig. 15. It is a easy job to see in Panel (A) that a solution of the mixed game exist and given by $v = v_A = v_B = 0$. Besides, we can show that the solution mentioned above is an attractive saddle point. To save the space, however, its graphical representation in the three-dimensional Euclidean space

		PLAYER B				
		Stone	Paper	Scissors	Row Min	
PLAYER A	Stone	0	−1	1	−1	Max
	Paper	1	0	−1	−1	Max
	Scissors	−1	1	0	− 1	Max
	Column Max	1	1	1		
		Min	Min	Min		

(A) Limiting pure strategies does not yield solutions



(B) A graphical representation: saddle points do not exist

Fig. 2.14 The "Final Problem" as a Game of Matching Pennies (THIRD Interpretation) :

Limiting pure strategies does not yield "saddle point solutions."

will be omitted here. To be fair, Theory is no more than Theory, and nothing else. Sometimes (even too often), however, Apparent Reality may differ from True Reality. Theoretically speaking, the Game of Stone, Paper, Scissors should be a *fair game* in the sense that it is neither advantageous nor disadvantageous to either player. If we play that game repeatedly, the winning probability (and also the losing probability) will tend to a half.

			Player *B*				
		(100)	**(1/3)**	**(010)**	**(001)**	**Row Min**	
	(100)	0	0	−1	1	− 1	
Player *A*	**(1/3)**	0	$\boxed{0}$	0	0	$\boxed{0}$	**Max**
	(010)	1	0	0	−1	−1	
	(001)	−1	0	1	0	−1	
Column Max		1	$\boxed{0}$	1	1		
			Min				

Fig. 2.15 Allowing mixed strategies in the Game of Stone, Paper, Scissors surely yields a solution

Fortunately or unfortunately, however, such fairness is not likely to be supportable in most of daily games. In the real world, we often eyewitness the eventual dominance of a "wonder boy," who will tend to defeat many other "simpletons" after several rounds. The question of much interest is then why in the real world, a one-sided winning streak for the wonder boy and a one-sided losing streak for a simpleton are likely to happen.

This may sound strange to some people. We do think, however, that a simpleton who is overanxious to win is doomed to defeat, whereas a wonder boy who is highly intelligent and very good at outwitting is a perfect winner. It would be a ironic consequence to see that a robot-like simpleton with no ambition for victory would do a better job in the simple Game of Stone, Paper, Scissors. Believe or not, this is likely to agree with common sense.

8. Common Sense and Mathematical Rigor : Good Advice from von Neumann and Morgenstern to the General Reader

In the long history of Japanese Art and Sciences, the following maxim was very famous:

> Never forget the ideals with which you started out.
>
> (Ze-a-mi 1440)

This maxim was first told by Ze-a-mi (1363 - 1443), a famous Noh drama writer in pre-modern Japan. Today, it seems that the specter of Game Theory is haunting the economics profession, and haunting indeed with no relation to the original ideals of its founders, i.e. John von Neumann and Oskar Morgenstern. It is high time for us to get back to the original ideas which we started out. Otherwise, we would aimlessly be going up in higher and higher in the wind like a kite without its string. 10)

As was mentioned above, there are two memorable years for Game Theory. The first important year was 1928 as the Year of Birth, in which Neumann's paper on social game theory and Morgenstern's paper on economic forecasting were independently published. While the so-called Min-Max Theorem for zero-sum games was first established and proved by Neumann, the Holmes-Moriarty duel in Conan Doyle's Sherlock Holmes stories was first noticed as a "troublesome game" by Morgenstern. The second important year for Game Theory was 1944 as the Year of Maturity, in which the great book *Theory of Games and Economic Behavior* was finally published. This book was a bulky book with more than 600 pages, has been a sort of "Bible of Game Theory." To be honest, we have been "faithful readers of the Game Theory Classic." We now realize that there exist, and are still widening, the gaps between the "Game Classic 1944" and the apparently brilliant developments in more recent times. Thus, it is high time to get back to the original ideas which we started out. For related topics, see Morgenstern (1958, 1972, 1976), Nakayama (1997), Nakayama & Muto & Funaki (2000) , Nasar (1998), and Soros (2008). .

In the following, we will summarize what we have learned in Chapters 1 and 2.

⑴ The very core of the Classical Game Theory is made up of "Zero-Sum Game," but not by "Non-Zero-Sum Game." As a matter of fact, it is an easy job for us to introduce an "extra person" so that we can expand a given non-zero-sum n - person game into a zero-sum ($n + 1$) - person game.

⑵ Concerning the family of zero-sum games, the zero-sum Two-Person Games constitute the most important member.

⑶ Von Neumann and Morgenstern believe that the simple Games of Matching Pennies, and of Stone, Paper, Scissors are perfectly characteristic of all the two-person zero-sum games..

⑷ "The Final Case," featuring the Moriarty-Holms confrontation, can be regarded as a variant of the game of matching pennies.

⑸ The solution of the original game with *pure* strategies and that of the expanded game with *mixed* strategies should not be discussed on the same level of abstraction. A special caution for the interpretation of the mixed strategies must be taken.

⑹ In order for the solution of the expanded game to be meaningful and applicable to the economic reality, the game in question must be repeatable at many times. Moreover, the two players are required to have the equal degree of intelligence and outguessing ability. In reality, those conditions are usually not satisfied.

⑺ Overall, the gulf between Theory and Reality is too large to fill in, and gets larger and larger in such a complex world as we live in.

Finally, we must keep in mind that von Neumann and Morgenstern have left us the following warning:

> This is another instance of the characteristic relationship between common sense and mathematical rigor. ... if the mathematical proof fails to establish the commonsense result, then there is a strong case for rejecting the theory altogether. Thus the primate of mathematical procedure extends only to establish checks on the theories —in a way which would not be open to common sense alone. (von Neumann & Morgenstern 1944, p. 361)

As our everyday experience can tell us, we often see the conflict between common sense and mathematical rigor. We are quite confident that we should rely on common sense rather than mathematical rigor. As the saying goes, simple is beautiful !

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Footnotes

1) Yasuma Takata (1883-1972) , called "Marshall of Japan" by Martin Bronfenbrenner, has made an outstanding contribution to the "Theory of Social Power," For the establishment of a New Theory of Social Science, we have to combine power theory a la Takata and game theory a la Neumann & Morgenstern. As the saying goes, where there is a will, there is a way ! Also see Takata (1955) and, Morishima (1994, 1994).

2) It is said that human beings have three different mental temperaments, namely "intelligence", "passion" and "determination." While Morgenstern is dominantly "the man of strong passion," von Neumann is "the man of cool intelligence." The collaboration between the two persons seems to have yielded "solid determination." It is a sad historical fact that von Neumann, cool-headedly calculating the cost and balance of any project, once recommended the American Government to drop an atomic bomb to Kyoto, the ancient capital of Japan with so many temples and memorials. Unfortunately, there seemed to exist no upper bounds for cool calculation. For details, see Poundstone (1992)..

3) In the real world, we must notice that there is a *third kind of game* in the sense that it can be described neither by a closed game nor by an open game. Von Neumann & Morgenstern (1944) already paid attention to "some games of peculiar character" as follows: "Some games like Roulette are of an even more peculiar character. In Roulette the mathematical expectation of the players is clearly negative. Thus the motives for participating in that game cannot be understood if one identifies the monetary return with utility." (Neumann & Morgenstern 1944, p. 87, footnote 3) We remember that in the Pacific War, an apparently crazy "Kamikaze pilot" made a dash at an American battleship. The motive of the Kamikaze pilot for that action cannot be understood if his return is merely identified with money. Also see Morgenstern (1928, 1935, 1958, 1972, 1976), Morishima (1994), Luce & Raiffa (1957), Aumann $ Hart (1992), Dasgupta & Gale & Hart & Maskin (1992), Dixit & Nalebuff (1991), Okada (1996, 2nd ed. 2011), Poundstone (1992), Sakai (1982, 1996, 2000, 2001, 2005, 2010, 2019), Schotter (1976), Shubik (1967, 1982), Suzuki (1959, 1994, 1999), Taleb (2007), and Varoufakis (2001).

4) Among those four cases of Matching Pennies, the first case was especially referred to as "Matching Pennies in its ordinary form." The remaining three cases was discussed as "Matching Pennies in other forms." For details, see von Neumann and Morgenstern (1944), pp. 175 - 178..

5) For details, see Suzuki (1959) and Shimazu (1997).

6) Edgar Alan Poe was known as a bit strange person, but his passion for literature was so great

that he wrote many masterpieces such as *Black Cat*, *The Fall of the House of Usher*, *The Murders in the Rue Morgue*, and *The Raven*. We would like to point it out that his influence on Japanese world of letters was very conspicuous. Even a Japanese noted novelist changed his pen name to " Edogawa Rampo," somewhat sounding like " Edgar Alan Poe." Poe's outstanding contribution to the relation between literature and mathematics has attracted a number of mathematicians including Yoshioka (1966).

7) In von Neumann & Morgenstern (1944) , the big numbers of " plus 100 " and " minus 50 " were conspicuously employed. For analytical convenience, however, those numbers are multiplied by the fraction (1/50) here. .

8) Later, Kakutani became a famous mathematician who succeeded in establishing " a generalization of Brouwer's fixed point theorem. This generalization theorem was effective used to prove the existence of general equilibrium. For this point, see Neumann and Morgenstern (1944), page 154, long footnote 1. Personally speaking, a brief meeting with Kakutani at Rochester in the 1960s has been in my fond memory until today.

9) Since this is a zero-sum game, it follows that Player B's outcome is given as follows:

$$b_{ij} = \Sigma_i \Sigma_j \, p_i \, (1 - a_{ij}) \, q_j \ .$$

10) For this point, see Morishima (1994).