CONVERGENCE THEOREMS USING ISHIKAWA ITERATION FOR FINDING COMMON FIXED POINTS OF DEMICLOSED AND 2-DEMICLOSED MAPPINGS IN HILBERT SPACES

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Abstract. This paper presents weak and strong convergence theorems for finding common fixed points of two nonlinear mappings, where one mapping is demiclosed, and the other is 2-demiclosed. For this purpose, we use Ishikawa type iteration and obtain weak convergence theorems. Nakajo and Takahashi's hybrid method and Takahashi, Takeuchi, and Kubota's shrinking projection method are also employed alongside Ishikawa iteration to derive strong convergence. Our proofs do not require the mappings to be commutative or continuous, and the results obtained in this paper extend many theorems in the literature.

1. Introduction

In this paper, we let H denote a real Hilbert space equipped with an inner product $\langle \cdot, \ \cdot \rangle$ and the associated norm $\|\cdot\|$ defined by $\|x\| = \sqrt{\langle x, \ x \rangle}$. Let $\mathbb{N}$ and $\mathbb{R}$ be the sets of natural and real numbers, respectively. Many researchers have studied approximation methods for finding fixed points of various types of nonlinear mappings. Let C be a nonempty subset of H. A mapping $S : C \to H$ is called

(i) *firmly nonexpansive* if $\|Sx - Sy\|^2 \leq \langle x - y, \ Sx - Sy \rangle$ for all $x, y \in C$,

(ii) *nonexpansive* if $\|Sx - Sy\| \leq \|x - y\|$ for all $x, y \in C$,

(iii) *nonspreading* [21, 22] if

$$2 \|Sx - Sy\|^2 \leq \|x - Sy\|^2 + \|Sx - y\|^2 \quad \text{for all } x, y \in C,$$

(iv) *hybrid* [43] if

$$3 \|Sx - Sy\|^2 \leq \|x - y\|^2 + \|x - Sy\|^2 + \|Sx - y\|^2 \quad \text{for all } x, y \in C,$$

(v) *generalized hybrid* [19] if there exist $\alpha, \beta \in \mathbb{R}$ such that

$$\alpha \|Sx - Sy\|^2 + (1 - \alpha) \|x - Sy\|^2 \leq \beta \|Sx - y\|^2 + (1 - \beta) \|x - y\|^2$$

for all $x, y \in C$,

(vi) *normally generalized hybrid* [48] if there exist $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ such that $\alpha + \beta + \gamma + \delta \geq 0$ and

$$\alpha \|Sx - Sy\|^2 + \beta \|x - Sy\|^2 + \gamma \|Sx - y\|^2 + \delta \|x - y\|^2 \leq 0$$

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for all $x, y \in C$, where $\alpha + \beta > 0$, or $\alpha + \gamma > 0$.

A firmly nonexpansive mapping is nonexpansive, nonspreading, and hybrid; for a proof, see Takahashi and Yao [49]. The class of generalized hybrid mappings includes all types of mappings (i)–(iv). Indeed, a generalized hybrid mapping with $\alpha = 1$ and $\beta = 0$ is a nonexpansive mapping. If $\alpha = 2$ and $\beta = 1$, then a generalized hybrid mapping is nonspreading. Nonspreading mappings arise out of necessity from optimization problems. A generalized hybrid mapping with $\alpha = 3/2$ and $\beta = 1/2$ is hybrid. A normally generalized hybrid mapping (vi) with $\alpha + \beta = 1$ and $\gamma + \delta = -1$ is generalized hybrid, and hence, the class of normally generalized hybrid mappings contains all types of mappings (i)–(v) as special cases.

A mapping $T : C \to C$ is called

(vii) 2-*generalized hybrid* [34] if there exist $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{R}$ such that

$$\begin{aligned} &\alpha_1 \left\| T^2 x - Ty \right\|^2 + \alpha_2 \left\| Tx - Ty \right\|^2 + (1 - \alpha_1 - \alpha_2) \left\| x - Ty \right\|^2 \\ &\le \beta_1 \left\| T^2 x - y \right\|^2 + \beta_2 \left\| Tx - y \right\|^2 + (1 - \beta_1 - \beta_2) \left\| x - y \right\|^2 \end{aligned} \tag{1.1}$$

for all $x, y \in C$,

(viii) *normally* 2-*generalized hybrid* [26] if there exist $\alpha_0, \beta_0, \alpha_1, \beta_1, \alpha_2, \beta_2 \in \mathbb{R}$ such that $\sum_{l=0}^{2} (\alpha_l + \beta_l) \ge 0$, $\alpha_2 + \alpha_1 + \alpha_0 > 0$, and

$$\begin{aligned} &\alpha_2 \left\| T^2 x - Ty \right\|^2 + \alpha_1 \left\| Tx - Ty \right\|^2 + \alpha_0 \left\| x - Ty \right\|^2 \\ &\qquad + \beta_2 \left\| T^2 x - y \right\|^2 + \beta_1 \left\| Tx - y \right\|^2 + \beta_0 \left\| x - y \right\|^2 \le 0 \end{aligned} \tag{1.2}$$

for all $x, y \in C$. The class of normally 2-generalized hybrid mappings contains all types of mappings (i)–(vii) introduced here. Indeed, if $\alpha_2 + \alpha_1 + \alpha_0 = 1$ and $\beta_2 + \beta_1 + \beta_0 = -1$, then a normally 2-generalized hybrid mapping is 2-generalized hybrid. Furthermore, if $\alpha_2 = \beta_2 = 0$, then it is normally generalized hybrid. It is known that mappings (i)–(viii) with fixed points are quasi-nonexpansive. The categories (iii)–(viii) contain mappings that are not continuous; see [13, 15, 23, 24, 25] and Section 3 of this paper. Convergence results concerning these types of mappings have been studied by many researchers; for recent results, see [1, 2, 3, 4, 8, 9, 27, 36, 37, 38]. Denote by $F(S)$ a set that collects all fixed points of a mapping S, that is,

$$F(S) = \{x \in C : Sx = x\}.$$

Let C be a nonempty, closed, and convex subset of H. According to Takahashi *et al.* [48], a mapping $S : C \to H$ of type (i)–(vi) is *demiclosed*, that is,

$$Sx_n - x_n \to 0 \text{ and } x_n \rightharpoonup v \Longrightarrow v \in F(S),$$

where $\{x_n\}$ is a sequence in C. For generalized hybrid mappings, see Kocourek *et al.* [19]. Kondo [23] called a self-mapping $T : C \to C$ 2-*demiclosed* if

$$Tx_n - x_n \to 0, \; T^2 x_n - x_n \to 0, \text{ and } x_n \rightharpoonup v \Longrightarrow v \in F(T).$$

According to Kondo and Takahashi [26], a mapping $T : C \to C$ of type (vii)–(viii) is 2-*demiclosed*; see also [34]. Because demiclosed mappings are 2-demiclosed, all types of mappings (i)–(viii) are 2-demiclosed.

Let $S : C \to C$ be a normally generalized hybrid mapping and let $T : C \to C$ be a normally 2-generalized hybrid mapping. Following Mann's work [32] in 1954, Kondo and Takahashi [28] used the following iteration scheme:

$$x_{n+1} = a_n x_n + b_n S x_n + c_n T x_n + d_n T^2 x_n \quad \text{for all } n \in \mathbb{N}, \tag{1.3}$$

where $x_1 \in C$ is given and $a_n, b_n, c_n, d_n \in [0, 1]$ are coefficients of a convex combination. They proved a weak convergence theorem to common fixed points of S and T. Very recently, Kondo [23] proved a strong convergence theorem for finding a common fixed point of a demiclosed mapping S and a 2-demiclosed mapping T using the Nakajo and Takahashi type hybrid method [35]:

$$\begin{aligned} x_1 &= x \in C \text{ given}, \\ y_n &= a_n x_n + b_n S x_n + c_n T x_n + d_n T^2 x_n \in C, \\ C_n &= \{z \in C : \|y_n - z\| \leq \|x_n - z\|\}, \\ Q_n &= \{z \in C : \langle x - x_n,\ x_n - z\rangle \geq 0\}, \text{ and} \\ x_{n+1} &= P_{C_n \cap Q_n} x \end{aligned} \tag{1.4}$$

for all $n \in \mathbb{N}$. He also used Takahashi, Takeuchi, and Kubota's type shrinking projection method [45]. The following recent contributions on the hybrid method and the shrinking projection method deserve mention: [11, 24, 31, 47]. To read further about approximation methods for finding common fixed points of nonlinear mappings, see [5, 12, 14, 18, 20, 28, 29, 30, 39, 40, 44].

On the other hand, in 1974, Ishikawa [16] introduced the following iteration: given $x_1 \in C$,

$$\begin{aligned} z_n &= \lambda_n x_n + (1 - \lambda_n) T x_n \text{ and} \\ x_{n+1} &= a_n x_n + (1 - a_n) T z_n \end{aligned} \tag{1.5}$$

for all $n \in \mathbb{N}$, where $a_n, \lambda_n \in [0, 1]$ are supposed to satisfy certain conditions. The iteration (1.5) coincides with Mann's iteration if $\lambda_n = 1$. Using Ishikawa iteration, Alizadeh and Moradlou [1, 2] demonstrated the following weak convergence theorem for a 2-generalized hybrid mapping:

Theorem 1.1 ([1, 2]). *Let C be a nonempty, closed, and convex subset of H. Let $T : C \to C$ be a 2-generalized hybrid mapping such that $F(T) \neq \emptyset$ and*

$$\left\|T^2 x - Tx\right\| \leq \|Tx - x\| \quad \textit{for all } x \in C. \tag{1.6}$$

Let $P_{F(T)}$ be the metric projection from H onto $F(T)$. Let $a, b \in \mathbb{R}$ such that $0 < a < b < 1$ and let $\{a_n\}$ be a sequence of real numbers in the interval $[a, b]$. Let $\{\lambda_n\}$ be a sequence in the interval $[0, 1]$ such that

$$\varliminf_{n \to \infty} \lambda_n (1 - \lambda_n) > 0.$$

Define a sequence $\{x_n\}$ in C as follows:

$$\begin{aligned} x_1 &\in C: \text{ given}, \\ z_n &= \lambda_n x_n + (1-\lambda_n) T x_n, \text{ and} \\ x_{n+1} &= a_n x_n + (1-a_n) T z_n \end{aligned}$$

for all $n \in \mathbb{N}$. Then, the sequence $\{x_n\}$ converges weakly to an element $\widehat{x}$ of $F(T)$, where $\widehat{x} \equiv \lim_{n\to\infty} P_{F(T)} x_n$.

When the condition (1.6) holds, the mapping T becomes demiclosed. Weak and strong convergence theorems using Ishikawa iteration have been studied by many researchers; for examples see a series of papers [6, 25, 33, 50, 51, 52] and Chapter 5 in Berinde [7].

In this paper, we present weak and strong convergence theorems for finding common fixed points of two nonlinear quasi-nonexpansive mappings. One of the mappings must be demiclosed, and the other 2-demiclosed. After introducing the required preliminaries in Section 2, we present examples of demiclosed and 2-demiclosed mappings in Section 3. In Section 4, we combine the iterations introduced by Kondo and Takahashi (1.3) and the Ishikawa type iteration (1.5) to prove weak convergence theorems that generalize the results in the author's previous paper [23]. A variant of Theorem 1.1 is also obtained (Corollary 4.1). In Section 5, Nakajo and Takahashi's method is developed, and, in Section 6, Takahashi, Takeuchi, and Kubota type results are derived. Our methods do not depend on the assumption that the mappings are commutative or continuous. Results obtained in this paper extend many theorems in the literature.

2. Preliminaries

Prerequisite information required for reading this paper is briefly summarized in this section. A more systematic explanation is provided by Takahashi in [41, 42]. For a sequence $\{x_n\}$ in a real Hilbert space H, we denote by $x_n \to x$ and $x_n \rightharpoonup x$ strong and weak convergence to x, respectively, where $x \in H$. It is known that $x_n \to x$ is characterized by the following condition: for any subsequence $\{x_{n_i}\}$ of $\{x_n\}$, there exists a subsequence $\{x_{n_j}\}$ of $\{x_{n_i}\}$ such that $x_{n_j} \to x$. Let C be a nonempty, closed, and convex subset of H. For any $x \in H$, there exists a unique point $p \in C$ such that $\|x-p\| = \inf_{w\in C} \|x-w\|$. This mapping is called a *metric projection* from H onto C and is denoted by P_C. A metric projection is firmly nonexpansive. In general, a firmly nonexpansive mapping is nonexpansive, nonspreading, and hybrid; see Takahashi and Yao [49]. When working with the metric projection P_C from H onto C, the following inequalities are useful:

$$\langle x - P_C x,\ P_C x - w\rangle \geq 0 \text{ and} \tag{2.1}$$

$$\|x-P_C x\|^2 + \|P_C x - w\|^2 \leq \|x-w\|^2 \tag{2.2}$$

for all $x \in H$ and $w \in C$.

The following lemma will be employed to prove the main theorems in this paper.

Lemma 2.1 ([46]). *Let F be a nonempty, closed, and convex subset of H, let P_F be the metric projection from H onto F, and let $\{x_n\}$ be a sequence in H. Suppose that*

$$\|x_{n+1} - q\| \leq \|x_n - q\| \tag{2.3}$$

for all $q \in F$ and $n \in \mathbb{N}$. Then, $\{P_F x_n\}$ is convergent in F.

The next lemma was proved by Maruyama *et al.* [34] to deal with 2-generalized hybrid mappings; see also Kondo and Takahashi [28] and Zegeye and Shahzad [53].

Lemma 2.2 ([34]). *Let $x, y, z, w, v \in H$ and let $a, b, c, d, e \in \mathbb{R}$. Then, the following hold:*

(1) *If $a + b + c = 1$, then*

$$\begin{aligned}&\|ax + by + cz\|^2\\ &= a\|x\|^2 + b\|y\|^2 + c\|z\|^2 - ab\|x - y\|^2 - bc\|y - z\|^2 - ca\|z - x\|^2.\end{aligned}$$

(2) *If $a + b + c + d = 1$, then*

$$\begin{aligned}&\|ax + by + cz + dw\|^2\\ = \;& a\|x\|^2 + b\|y\|^2 + c\|z\|^2 + d\|w\|^2\\ & -ab\|x - y\|^2 - ac\|x - z\|^2 - ad\|x - w\|^2\\ & -bc\|y - z\|^2 - bd\|y - w\|^2 - cd\|z - w\|^2.\end{aligned}$$

(3) *If $a + b + c + d + e = 1$, then*

$$\begin{aligned}&\|ax + by + cz + dw + ev\|^2\\ = \;& a\|x\|^2 + b\|y\|^2 + c\|z\|^2 + d\|w\|^2 + e\|v\|^2\\ & -ab\|x - y\|^2 - ac\|x - z\|^2 - ad\|x - w\|^2 - ae\|x - v\|^2\\ & -bc\|y - z\|^2 - bd\|y - w\|^2 - be\|y - v\|^2\\ & -cd\|z - w\|^2 - ce\|z - v\|^2 - de\|w - v\|^2.\end{aligned}$$

A mapping $T : C \to H$ with $F(T) \neq \emptyset$ is called *quasi-nonexpansive* if $\|Tx - q\| \leq \|x - q\|$ for all $x \in C$ and $q \in F(T)$. According to Itoh and Takahashi [17], a set of all fixed points of a quasi-nonexpansive mapping is closed and convex. Kondo and Takahashi [26] proved that a normally 2-generalized hybrid mapping that has a fixed point is quasi-nonexpansive. For completeness, we reproduce the proof.

Lemma 2.3 ([26]). *Let $T : C \to C$ be a normally 2-generalized hybrid mapping such that $F(T) \neq \emptyset$, where C is a nonempty subset of H. Then, T is quasi-nonexpansive.*

Proof. Let $x \in C$ and $q \in F(T)$. We show that $\|Tx - q\| \leq \|x - q\|$. As T is normally 2-generalized hybrid, there exist real numbers $\alpha_0, \alpha_1, \alpha_2, \beta_0, \beta_1, \beta_2$ such that $\sum_{l=0}^{2} (\alpha_l + \beta_l) \geq 0$, $\alpha_2 + \alpha_1 + \alpha_0 > 0$, and

$$\alpha_2 \left\|T^2 q - Tx\right\|^2 + \alpha_1 \|Tq - Tx\|^2 + \alpha_0 \|q - Tx\|^2$$
$$+\beta_2 \left\|T^2 q - x\right\|^2 + \beta_1 \|Tq - x\|^2 + \beta_0 \|q - x\|^2 \leq 0.$$

As $q = Tq = T^2 q$,

$$(\alpha_2 + \alpha_1 + \alpha_0) \|q - Tx\|^2 + (\beta_2 + \beta_1 + \beta_0) \|q - x\|^2 \leq 0.$$

From $\sum_{l=0}^{2} (\alpha_l + \beta_l) \geq 0$, it follows that

$$\begin{aligned}(\alpha_2 + \alpha_1 + \alpha_0) \|q - Tx\|^2 &\leq -(\beta_2 + \beta_1 + \beta_0) \|q - x\|^2 \\ &\leq (\alpha_2 + \alpha_1 + \alpha_0) \|q - x\|^2 .\end{aligned}$$

Dividing by $\alpha_2 + \alpha_1 + \alpha_0 \, (> 0)$, we obtain $\|q - Tx\|^2 \leq \|q - x\|^2$, which completes the proof. □

This lemma shows that all the types of mappings (i)–(viii) which were described in the introduction are quasi-nonexpansive if they have a fixed points. A mapping $S : C \to H$ is called *demiclosed* if

$$Sx_n - x_n \to 0 \text{ and } x_n \rightharpoonup v \Longrightarrow v \in F(S), \tag{2.4}$$

where $\{x_n\}$ is a sequence in C. The next lemma asserts that a normally generalized hybrid mapping is demiclosed. A similar result related to generalized hybrid mappings is demonstrated by Kocourek *et al.* [19].

Lemma 2.4 ([48]). *Let $S : C \to C$ be a normally generalized hybrid mapping, where C is a nonempty, closed, and convex subset of H. Then, S is demiclosed.*

Kondo [23] calls a mapping $T : C \to C$ 2-*demiclosed* if it satisfies

$$Tx_n - x_n \to 0, \quad T^2 x_n - x_n \to 0, \quad \text{and} \quad x_n \rightharpoonup v \Longrightarrow v \in F(T). \tag{2.5}$$

Clearly, a demiclosed mapping is 2-demiclosed. Although the following lemma was proved by Kondo and Takahashi [26], we provide an alternative proof for completeness.

Lemma 2.5 ([26]). *Let $T : C \to C$ be a normally 2-generalized hybrid mapping, where C is a nonempty, closed, and convex subset of H. Then, T is 2-demiclosed.*

Proof. Let $\{x_n\}$ be a sequence in C that satisfies $Tx_n - x_n \to 0$, $T^2 x_n - x_n \to 0$, and $x_n \rightharpoonup v$. As $\{x_n\}$ is a sequence in C, $x_n \rightharpoonup v$, and C is weakly closed, v is an element of C. Therefore, $Tv \, (\in C)$ exists. Our goal is to prove that $Tv = v$. As T is normally 2-generalized hybrid, there exist real numbers $\alpha_0, \alpha_1, \alpha_2, \beta_0, \beta_1, \beta_2$ such that $\sum_{l=0}^{2} (\alpha_l + \beta_l) \geq 0$, $\alpha_2+\alpha_1+\alpha_0 > 0$, and

$$\begin{aligned}\alpha_2 \left\|T^2 x_n - Tv\right\|^2 + \alpha_1 \left\|Tx_n - Tv\right\|^2 + \alpha_0 \left\|x_n - Tv\right\|^2 \\ +\beta_2 \left\|T^2 x_n - v\right\|^2 + \beta_1 \left\|Tx_n - v\right\|^2 + \beta_0 \left\|x_n - v\right\|^2 \leq 0\end{aligned}$$

for all $n \in \mathbb{N}$. It follows that

$$\begin{aligned}\alpha_2 \left(\left\|T^2 x_n - x_n\right\|^2 + 2\left\langle T^2 x_n - x_n,\ x_n - Tv\right\rangle + \left\|x_n - Tv\right\|^2\right) \\ +\alpha_1 \left(\left\|Tx_n - x_n\right\|^2 + 2\left\langle Tx_n - x_n,\ x_n - Tv\right\rangle + \left\|x_n - Tv\right\|^2\right) + \alpha_0 \left\|x_n - Tv\right\|^2 \\ +\beta_2 \left(\left\|T^2 x_n - x_n\right\|^2 + 2\left\langle T^2 x_n - x_n,\ x_n - v\right\rangle + \left\|x_n - v\right\|^2\right) \\ +\beta_1 \left(\left\|Tx_n - x_n\right\|^2 + 2\left\langle Tx_n - x_n,\ x_n - v\right\rangle + \left\|x_n - v\right\|^2\right) \\ +\beta_0 \left\|x_n - v\right\|^2 \leq 0.\end{aligned}$$

This yields

$$\begin{aligned}\alpha_2 \left(\left\|T^2 x_n - x_n\right\|^2 + 2\left\langle T^2 x_n - x_n,\ x_n - Tv\right\rangle\right) \\ +\alpha_1 \left(\left\|Tx_n - x_n\right\|^2 + 2\left\langle Tx_n - x_n,\ x_n - Tv\right\rangle\right) \\ +\left(\alpha_2 + \alpha_1 + \alpha_0\right) \left\|x_n - Tv\right\|^2 \\ +\beta_2 \left(\left\|T^2 x_n - x_n\right\|^2 + 2\left\langle T^2 x_n - x_n,\ x_n - v\right\rangle\right) \\ +\beta_1 \left(\left\|Tx_n - x_n\right\|^2 + 2\left\langle Tx_n - x_n,\ x_n - v\right\rangle\right) \\ +\left(\beta_2 + \beta_1 + \beta_0\right) \left\|x_n - v\right\|^2 \leq 0.\end{aligned}$$

Furthermore, it is true that

$$\begin{aligned}\alpha_2 \left(\left\|T^2 x_n - x_n\right\|^2 + 2\left\langle T^2 x_n - x_n,\ x_n - Tv\right\rangle\right) \\ +\alpha_1 \left(\left\|Tx_n - x_n\right\|^2 + 2\left\langle Tx_n - x_n,\ x_n - Tv\right\rangle\right) \\ +\left(\alpha_2 + \alpha_1 + \alpha_0\right) \left(\left\|x_n - v\right\|^2 + 2\left\langle x_n - v,\ v - Tv\right\rangle + \left\|v - Tv\right\|^2\right) \\ +\beta_2 \left(\left\|T^2 x_n - x_n\right\|^2 + 2\left\langle T^2 x_n - x_n,\ x_n - v\right\rangle\right) \\ +\beta_1 \left(\left\|Tx_n - x_n\right\|^2 + 2\left\langle Tx_n - x_n,\ x_n - v\right\rangle\right) \\ +\left(\beta_2 + \beta_1 + \beta_0\right) \left\|x_n - v\right\|^2 \leq 0,\end{aligned}$$

and hence,

$$\begin{aligned}&\alpha_2\left(\left\|T^2x_n-x_n\right\|^2+2\left\langle T^2x_n-x_n,\ x_n-Tv\right\rangle\right)\\&+\alpha_1\left(\|Tx_n-x_n\|^2+2\langle Tx_n-x_n,\ x_n-Tv\rangle\right)\\&+(\alpha_2+\alpha_1+\alpha_0)\left(2\langle x_n-v,\ v-Tv\rangle+\|v-Tv\|^2\right)\\&+\beta_2\left(\left\|T^2x_n-x_n\right\|^2+2\left\langle T^2x_n-x_n,\ x_n-v\right\rangle\right)\\&+\beta_1\left(\|Tx_n-x_n\|^2+2\langle Tx_n-x_n,\ x_n-v\rangle\right)\\&+\left(\sum_{l=0}^{2}(\alpha_l+\beta_l)\right)\|x_n-v\|^2\leq 0.\end{aligned}$$

Subtracting $\left(\sum_{l=0}^{2}(\alpha_l+\beta_l)\right)\|x_n-v\|^2\,(\geq 0)$ from the left-hand side, we obtain

$$\begin{aligned}&\alpha_2\left(\left\|T^2x_n-x_n\right\|^2+2\left\langle T^2x_n-x_n,\ x_n-Tv\right\rangle\right)\\&+\alpha_1\left(\|Tx_n-x_n\|^2+2\langle Tx_n-x_n,\ x_n-Tv\rangle\right)\\&+(\alpha_2+\alpha_1+\alpha_0)\left(2\langle x_n-v,\ v-Tv\rangle+\|v-Tv\|^2\right)\\&+\beta_2\left(\left\|T^2x_n-x_n\right\|^2+2\left\langle T^2x_n-x_n,\ x_n-v\right\rangle\right)\\&+\beta_1\left(\|Tx_n-x_n\|^2+2\langle Tx_n-x_n,\ x_n-v\rangle\right)\leq 0.\end{aligned}$$

Because $\{x_n\}$ is weakly convergent, it is bounded. Using that $Tx_n-x_n\to 0$, $T^2x_n-x_n\to 0$, and $x_n\rightharpoonup v$, it holds in the limit as $n\to\infty$ that

$$(\alpha_2+\alpha_1+\alpha_0)\|v-Tv\|^2\leq 0.$$

Dividing by $\alpha_2+\alpha_1+\alpha_0\,(>0)$, we obtain

$$\|v-Tv\|^2\leq 0,$$

which means that $v=Tv$. Hence the proof is completed. □

In the theorems presented in this paper, we require the nonlinear mappings to have common fixed points. A set of sufficient conditions that guarantee the existence of common fixed points of normally 2-generalized hybrid mappings is given by the next theorem.

Theorem 2.1 ([10])**.** *Let $S,T:C\to C$ be commutative normally 2-generalized hybrid mappings, where C is a nonempty, closed, and convex subset of H. Suppose that there exists an element $x\in C$ such that $\{S^kT^lx:k,l\in\mathbb{N}\cup\{0\}\}$ is bounded. Then, $F(S)\cap F(T)$ is nonempty.*

3. Examples of mappings

In this section, we provide examples of demiclosed and 2-demiclosed mappings that are not continuous. The examples are slightly modified versions of those given by Berinde [7], Igarashi *et al.* [15], Hojo *et al.* [13], and Kondo [23, 24, 25]. The first two examples are nonspreading. Remember that a mapping $S : C \to H$ is called nonspreading if

$$2\|Sx - Sy\|^2 \leq \|x - Sy\|^2 + \|Sx - y\|^2 \tag{3.1}$$

for all $x, y \in C$. Because nonspreading mappings are a special case of normally generalized hybrid mappings, the mappings are demiclosed by Lemma 2.4.

Example 3.1. Let $H = C = \mathbb{R}$. For $a > 0$, define $S : \mathbb{R} \to \mathbb{R}$ as follows;

$$Sx = \begin{cases} 0 & \text{if } x \leq a, \\ \frac{a}{\sqrt{2}} & \text{if } x > a \end{cases}$$

such that S is a nonspreading mapping on $\mathbb{R}$. Indeed, if $x, y \leq a$ or $x, y > a$, then the LHS of (3.1) is 0, and therefore, the inequality holds. Assume without loss of generality that $x \leq a < y$. Then, $Sx = 0$ and $Sy = a/\sqrt{2}$. Consequently, the $LHS = a^2$. On the other hand,

$$RHS = |x - Sy|^2 + |y|^2 \geq y^2 > a^2.$$

This shows that the condition (3.1) is fulfilled. Therefore, S is nonspreading, and hence demiclosed. □

Example 3.2. Let H be a Hilbert space, let $C = H$, and let P_U be the metric projection from H onto U, where $U = \{x \in H : \|x\| \leq 1\}$ is the unit sphere in H. Define $S : H \to H$ as follows:

$$Sx = \begin{cases} P_U x & \text{if } \sqrt{2} < \|x\|, \\ 0 & \text{if } \|x\| \leq \sqrt{2}. \end{cases}$$

Now let us show that S is nonspreading: if $\|x\|, \|y\| \leq \sqrt{2}$, then $Sx = Sy = 0$, and the condition (3.1) follows. If $\|x\|, \|y\| > \sqrt{2}$, then the condition (3.1) is

$$2\|P_U x - P_U y\|^2 \leq \|x - P_U y\|^2 + \|P_U x - y\|^2.$$

This inequality is true, because the metric projection is firmly nonexpansive, and thus, it is nonspreading; see Takahashi and Yao [49]. Finally, if we assume that $\|x\| \leq \sqrt{2} < \|y\|$, then $Sx = 0$ and $Sy = P_U y = y/\|y\|$. In this case, the LHS of (3.1) is $2\|Sx - Sy\|^2 = 2$. The RHS is

$$RHS = \|x - Sy\|^2 + \|y\|^2 \geq \|y\|^2 > 2.$$

This means that the condition (3.1) is satisfied for all $x, y \in C$, and therefore, S is a demiclosed mapping. □

Next, we present examples of normally 2-generalized hybrid mappings that are 2-demiclosed. Letting $\alpha_2 = \alpha$, $\beta_2 = -\beta$ with $0 < \beta < \alpha < 2\beta$, and all the other coefficients be 0 in (1.2), we have

$$\alpha \left\| T^2 x - Ty \right\|^2 \leq \beta \left\| T^2 x - y \right\|^2 . \tag{3.2}$$

Example 3.3. Let $H = C = \mathbb{R}$, and define $T : \mathbb{R} \to \mathbb{R}$ as follows:

$$Tx = \begin{cases} 0 & \text{if } x \leq \sqrt{\alpha}, \\ \sqrt{\beta} & \text{if } x > \sqrt{\alpha}. \end{cases}$$

Notice that T does not satisfy condition (3.1) because $\alpha < 2\beta$. However, the mapping T satisfies (3.2). Indeed, it holds that $T^2 x = 0$ for all $x \in \mathbb{R}$ because of the hypothesis $\beta < \alpha$. Thus,

$$\begin{aligned} (3.2) \iff & \alpha (Ty)^2 \leq \beta y^2 \\ \iff & |Ty| \leq \sqrt{\frac{\beta}{\alpha}} |y| . \end{aligned} \tag{3.3}$$

If $y \leq \sqrt{\alpha}$, then $|Ty| = 0$. Consequently, (3.3) holds. If $y > \sqrt{\alpha}$, then the LHS of (3.3) is $|Ty| = \sqrt{\beta}$ whereas the RHS is

$$RHS = \sqrt{\frac{\beta}{\alpha}} |y| > \sqrt{\frac{\beta}{\alpha}} \sqrt{\alpha} = \sqrt{\beta}.$$

Thus, (3.3) holds. Because T is normally 2-generalized hybrid, it is 2-demiclosed. □

Example 3.4. Let H be a Hilbert space, let $C = H$, and let P_U be the metric projection from H onto the unit sphere U. Define $T : H \to H$ as follows:

$$Tx = \begin{cases} P_U x & \text{if } \sqrt{\frac{\alpha}{\beta}} < \|x\| , \\ 0 & \text{if } \|x\| \leq \sqrt{\frac{\alpha}{\beta}}, \end{cases}$$

where $0 < \beta < \alpha < 2\beta$. Then, the mapping T satisfies (3.2). Indeed, it holds that $T^2 x = 0$ for all $x \in H$. Thus,

$$\begin{aligned} (3.2) \iff & \alpha \|Ty\|^2 \leq \beta \|y\|^2 \\ \iff & \|Ty\| \leq \sqrt{\frac{\beta}{\alpha}} \|y\| . \end{aligned} \tag{3.4}$$

If $\|y\| \leq \sqrt{\alpha/\beta}$, then $\|Ty\| = 0$. Consequently, (3.4) holds. If $\|y\| > \sqrt{\alpha/\beta}$, then the LHS of (3.4) is $LHS = \|Ty\| = \|P_U y\| = 1$. The RHS is

$$RHS = \sqrt{\frac{\beta}{\alpha}} \|y\| > \sqrt{\frac{\beta}{\alpha}} \sqrt{\frac{\alpha}{\beta}} = 1.$$

Thus, (3.4) follows, and T is 2-demiclosed. □

Obviously, the mappings presented in this section are not continuous.

4. Weak convergence

This section presents weak convergence theorems for finding common fixed points of two quasi-nonexpansive mappings. One of the mappings is assumed to be demiclosed (2.4), and the other to be 2-demiclosed (2.5). From Lemma 2.3, we know a normally generalized hybrid mapping and a normally 2-generalized hybrid mapping are quasi-nonexpansive if they have fixed points. Lemma 2.4 shows that a normally generalized hybrid mapping is demiclosed, and Lemma 2.5 shows that a normally 2-generalized hybrid mapping is 2-demiclosed. The basic elements of the following proof were established and polished by many researchers; in particular, see [1, 2, 11, 19, 28, 29, 34, 48, 52].

Theorem 4.1. *Let C be a nonempty, closed, and convex subset of H. Let $S : C \to C$ be a quasi-nonexpansive and demiclosed mapping and let $T : C \to C$ be a quasi-nonexpansive and 2-demiclosed mapping. Suppose that $F(S) \cap F(T)$ is nonempty. Let $P_{F(S)\cap F(T)}$ be the metric projection from H onto $F(S) \cap F(T)$. Let $\{\lambda_n\}$, $\{\mu_n\}$, $\{\nu_n\}$, and $\{\xi_n\}$ be sequences of real numbers in the interval $[0,1]$ that satisfy*

$$\begin{gathered}\lambda_n + \mu_n + \nu_n + \xi_n = 1 \quad \textit{for all } n \in \mathbb{N}, \\ \varliminf_{n\to\infty} \lambda_n\mu_n > 0, \quad \varliminf_{n\to\infty} \lambda_n\nu_n > 0, \quad \textit{and} \quad \varliminf_{n\to\infty} \lambda_n\xi_n > 0. \end{gathered} \tag{4.1}$$

Let $a \in (0,1]$ and let $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, and $\{d_n\}$ be sequences of real numbers in the interval $[0,1]$ such that $a_n + b_n + c_n + d_n = 1$ and $a \leq b_n + c_n + d_n$ for all $n \in \mathbb{N}$. Define a sequence $\{x_n\}$ in C as follows:

$$\begin{aligned} &x_1 \in C: \ \textit{given}, \\ &z_n = \lambda_n x_n + \mu_n S x_n + \nu_n T x_n + \xi_n T^2 x_n, \ \textit{and} \\ x_{n+1} = a_n x_n &+ b_n S^L z_n + c_n T^M z_n + d_n \frac{1}{n}\sum_{l=0}^{n-1} T^l z_n \end{aligned} \tag{4.2}$$

for all $n \in \mathbb{N}$, where $L, M \in \mathbb{N} \cup \{0\}$. Then, $\{x_n\}$ converges weakly to an element $\widehat{x}$ of $F(S) \cap F(T)$, where $\widehat{x} \equiv \lim_{n\to\infty} P_{F(S)\cap F(T)} x_n$.

Proof. First, note that because the mappings S and T are quasi-nonexpansive, $F(S) \cap F(T)$ is closed and convex. As $F(S) \cap F(T) \neq \emptyset$ is assumed, the metric projection $P_{F(S)\cap F(T)}$ is defined. We verify that

$$\left\| \frac{1}{n}\sum_{l=0}^{n-1} T^l z_n - q \right\| \leq \|z_n - q\| \tag{4.3}$$

for all $q \in F(T)$ and $n \in \mathbb{N}$. Indeed, as T is quasi-nonexpansive, it follows that

$$\begin{aligned}\left\|\frac{1}{n}\sum_{l=0}^{n-1} T^l z_n - q\right\| &\leq \frac{1}{n}\left\|\sum_{l=0}^{n-1} T^l z_n - nq\right\| \\ &= \frac{1}{n}\left\|\sum_{l=0}^{n-1}\left(T^l z_n - q\right)\right\| \leq \frac{1}{n}\sum_{l=0}^{n-1}\left\|T^l z_n - q\right\| \\ &\leq \frac{1}{n}\sum_{l=0}^{n-1}\|z_n - q\| = \|z_n - q\|.\end{aligned}$$

This shows that (4.3) holds. Furthermore, the following is also true:

$$\|z_n - q\| \leq \|x_n - q\| \tag{4.4}$$

for all $q \in F(S) \cap F(T)$ and $n \in \mathbb{N}$. Indeed, as S and T are quasi-nonexpansive,

$$\begin{aligned}&\|z_n - q\| \\ &= \left\|\lambda_n x_n + \mu_n S x_n + \nu_n T x_n + \xi_n T^2 x_n - q\right\| \\ &= \left\|\lambda_n (x_n - q) + \mu_n (S x_n - q) + \nu_n (T x_n - q) + \xi_n \left(T^2 x_n - q\right)\right\| \\ &\leq \lambda_n \|x_n - q\| + \mu_n \|S x_n - q\| + \nu_n \|T x_n - q\| + \xi_n \left\|T^2 x_n - q\right\| \\ &\leq \lambda_n \|x_n - q\| + \mu_n \|x_n - q\| + \nu_n \|x_n - q\| + \xi_n \|x_n - q\| \\ &= \|x_n - q\|.\end{aligned}$$

This demonstrates that (4.4) is correct. Next, we use these two inequalities to prove that

$$\|x_{n+1} - q\| \leq \|x_n - q\| \tag{4.5}$$

for all $q \in F(S) \cap F(T)$ and $n \in \mathbb{N}$. As S and T are quasi-nonexpansive, it follows from (4.3) and (4.4) that

$$\begin{aligned}&\|x_{n+1} - q\| \qquad (4.6)\\ &= \left\|a_n x_n + b_n S^L z_n + c_n T^M z_n + d_n \frac{1}{n}\sum_{l=0}^{n-1} T^l z_n - q\right\| \\ &= \left\|a_n (x_n - q) + b_n \left(S^L z_n - q\right) + c_n \left(T^M z_n - q\right) + d_n \left(\frac{1}{n}\sum_{l=0}^{n-1} T^l z_n - q\right)\right\| \\ &= a_n \|x_n - q\| + b_n \left\|S^L z_n - q\right\| + c_n \left\|T^M z_n - q\right\| + d_n \left\|\frac{1}{n}\sum_{l=0}^{n-1} T^l z_n - q\right\| \\ &\leq a_n \|x_n - q\| + b_n \|z_n - q\| + c_n \|z_n - q\| + d_n \|z_n - q\| \\ &\leq a_n \|x_n - q\| + b_n \|x_n - q\| + c_n \|x_n - q\| + d_n \|x_n - q\| \\ &= \|x_n - q\|.\end{aligned}$$

Thus, (4.5) is verified. From (4.5), we know that $\{\|x_n - q\|\}$ is convergent in $\mathbb{R}$ for all $q \in F(S) \cap F(T)$. The sequence $\{x_n\}$ is bounded. Lemma 2.1 with (4.5) implies that $\{P_{F(S)\cap F(T)} x_n\}$ is convergent in $F(S) \cap F(T)$, and thus, $\widehat{x} \equiv \lim_{n\to\infty} P_{F(S)\cap F(T)} x_n$ exists.

Our next aim is to demonstrate that

$$\begin{aligned} \|z_n - q\|^2 \leq\ & \|x_n - q\|^2 - \lambda_n\mu_n \|x_n - Sx_n\|^2 \\ & -\lambda_n\nu_n \|x_n - Tx_n\|^2 - \lambda_n\xi_n \left\|x_n - T^2x_n\right\|^2 \end{aligned} \tag{4.7}$$

for all $q \in F(S) \cap F(T)$ and $n \in \mathbb{N}$. Indeed, as S and T are quasi-nonexpansive and $q \in F(S) \cap F(T)$, we can use (2) in Lemma 2.2 to obtain

$$\begin{aligned} & \|z_n - q\|^2 \\ = & \left\|\lambda_n x_n + \mu_n Sx_n + \nu_n Tx_n + \xi_n T^2 x_n - q\right\|^2 \\ = & \left\|\lambda_n (x_n - q) + \mu_n (Sx_n - q) + \nu_n (Tx_n - q) + \xi_n \left(T^2x_n - q\right)\right\|^2 \\ = & \lambda_n \|x_n - q\|^2 + \mu_n \|Sx_n - q\|^2 + \nu_n \|Tx_n - q\|^2 + \xi_n \left\|T^2x_n - q\right\|^2 \\ & -\lambda_n\mu_n \|x_n - Sx_n\|^2 - \lambda_n\nu_n \|x_n - Tx_n\|^2 - \lambda_n\xi_n \left\|x_n - T^2x_n\right\|^2 \\ & -\mu_n\nu_n \|Sx_n - Tx_n\|^2 - \mu_n\xi_n \left\|Sx_n - T^2x_n\right\|^2 - \nu_n\xi_n \left\|Tx_n - T^2x_n\right\|^2 \end{aligned} \tag{4.8}$$

$$\begin{aligned} \leq & \lambda_n \|x_n - q\|^2 + \mu_n \|x_n - q\|^2 + \nu_n \|x_n - q\|^2 + \xi_n \|x_n - q\|^2 \\ & -\lambda_n\mu_n \|x_n - Sx_n\|^2 - \lambda_n\nu_n \|x_n - Tx_n\|^2 - \lambda_n\xi_n \left\|x_n - T^2x_n\right\|^2 \\ & -\mu_n\nu_n \|Sx_n - Tx_n\|^2 - \mu_n\xi_n \left\|Sx_n - T^2x_n\right\|^2 - \nu_n\xi_n \left\|Tx_n - T^2x_n\right\|^2 \\ = & \|x_n - q\|^2 \\ & -\lambda_n\mu_n \|x_n - Sx_n\|^2 - \lambda_n\nu_n \|x_n - Tx_n\|^2 - \lambda_n\xi_n \left\|x_n - T^2x_n\right\|^2 \\ & -\mu_n\nu_n \|Sx_n - Tx_n\|^2 - \mu_n\xi_n \left\|Sx_n - T^2x_n\right\|^2 - \nu_n\xi_n \left\|Tx_n - T^2x_n\right\|^2. \end{aligned}$$

Because $\mu_n\nu_n \|Sx_n - Tx_n\|^2 + \mu_n\xi_n \left\|Sx_n - T^2x_n\right\|^2 + \nu_n\xi_n \left\|Tx_n - T^2x_n\right\|^2 \geq 0$, we have (4.7) as claimed.

Observe that

$$\begin{aligned} & a(\lambda_n\mu_n \|x_n - Sx_n\|^2 + \lambda_n\nu_n \|x_n - Tx_n\|^2 + \lambda_n\xi_n \left\|x_n - T^2x_n\right\|^2) \\ & \leq \|x_n - q\|^2 - \|x_{n+1} - q\|^2 \end{aligned} \tag{4.9}$$

for all $q \in F(S) \cap F(T)$ and $n \in \mathbb{N}$. As S and T are quasi-nonexpansive, using (4.3) and (4.7) yields

$$\begin{aligned}
&\|x_{n+1} - q\|^2 \qquad (4.10)\\
&= \left\| a_n (x_n - q) + b_n (S^L z_n - q) + c_n (T^M z_n - q) + d_n \left(\frac{1}{n} \sum_{l=0}^{n-1} T^l z_n - q \right) \right\|^2 \\
&\le a_n \|x_n - q\|^2 + b_n \|S^L z_n - q\|^2 + c_n \|T^M z_n - q\|^2 + d_n \left\| \frac{1}{n} \sum_{l=0}^{n-1} T^l z_n - q \right\|^2 \\
&\le a_n \|x_n - q\|^2 + (b_n + c_n + d_n) \|z_n - q\|^2 \\
&\le a_n \|x_n - q\|^2 + (b_n + c_n + d_n) (\|x_n - q\|^2 - \lambda_n \mu_n \|x_n - Sx_n\|^2 \\
&- \lambda_n \nu_n \|x_n - Tx_n\|^2 - \lambda_n \xi_n \|x_n - T^2 x_n\|^2).
\end{aligned}$$

Consequently,

$$\begin{aligned}
&(b_n + c_n + d_n) (\lambda_n \mu_n \|x_n - Sx_n\|^2 + \lambda_n \nu_n \|x_n - Tx_n\|^2 + \lambda_n \xi_n \|x_n - T^2 x_n\|^2) \\
&\le \|x_n - q\|^2 - \|x_{n+1} - q\|^2
\end{aligned}$$

Using the hypothesis that $a \le b_n + c_n + d_n$, we obtain (4.9). As $\{\|x_n - q\|\}$ is convergent and $a > 0$, it follows from (4.1) and (4.9) that

$$x_n - Sx_n \to 0, \quad x_n - Tx_n \to 0, \quad \text{and} \quad T^2 x_n - x_n \to 0 \qquad (4.11)$$

as $n \to \infty$.

Finally, to complete the proof we must demonstrate that

$$x_n \rightharpoonup \widehat{x} \left(\equiv \lim_{n \to \infty} P_{F(S) \cap F(T)} x_n \right).$$

It suffices to show that for any subsequence $\{x_{n_i}\}$ of $\{x_n\}$, there exists a subsequence $\{x_{n_j}\}$ of $\{x_{n_i}\}$ such that $x_{n_j} \rightharpoonup \widehat{x}$. Let $\{x_{n_i}\}$ be a subsequence of $\{x_n\}$. As $\{x_{n_i}\}$ is bounded, there exists a subsequence $\{x_{n_j}\}$ of $\{x_{n_i}\}$ such that $x_{n_j} \rightharpoonup v$ for some $v \in H$. As S is demiclosed (2.4) and T is 2-demiclosed (2.5), it follows from (4.11) that $v \in F(S) \cap F(T)$. Hence, from (2.1), we have

$$\left\langle x_{n_j} - P_{F(S) \cap F(T)} x_{n_j},\ P_{F(S) \cap F(T)} x_{n_j} - v \right\rangle \ge 0$$

for all $j \in \mathbb{N}$. As $x_{n_j} \rightharpoonup v$ and $P_{F(S) \cap F(T)} x_n \to \widehat{x}$, it holds in the limit as $j \to \infty$ that $\langle v - \widehat{x},\ \widehat{x} - v \rangle \ge 0$. This means that $v = \widehat{x}$, and thus, $x_n \rightharpoonup \widehat{x}$. This completes the proof. □

Remark 4.1. *The construction of* x_{n+1} *in* (4.2) *can be replaced by more general types; for instance,*

$$x_{n+1} = a_n x_n + b_n S^L z_n + c_n T^M z_n + d_n \frac{1}{n} \sum_{l=0}^{n-1} S^l z_n + e_n \frac{1}{n} \sum_{l=0}^{n-1} T^l z_n, \text{ or}$$

$$(4.12)$$

$$x_{n+1} = a_n x_n + b_n S^L z_n + c_n T^M z_n + d_n \frac{1}{n} \sum_{l=0}^{n-1} S^l z_n + e_n \frac{1}{n} \sum_{l=m}^{m+n-1} T^l z_n,$$

where $m \in \mathbb{N} \cup \{0\}$. *In* (4.12), $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{d_n\}$, *and* $\{e_n\}$ *are sequences of real numbers in* $[0,1]$ *such that* $a_n + b_n + c_n + d_n + e_n = 1$ *and* $a \leq b_n + c_n + d_n + e_n$. *To verify this point, carefully review* (4.6) *and* (4.10) *and the proof of* Theorem 4.1.

As a corollary of Theorem 4.1, we obtain the following:

Corollary 4.1. *Let* C *be a nonempty, closed, and convex subset of* H. *Let* $T : C \to C$ *be a 2-generalized hybrid mapping such that* $F(T) \neq \emptyset$. *Let* $P_{F(T)}$ *be the metric projection from* H *onto* $F(T)$. *Let* $\{\lambda_n\}$, $\{\nu_n\}$, *and* $\{\xi_n\}$ *be sequences of real numbers in the interval* $[0,1]$ *that satisfy*

$$\lambda_n + \nu_n + \xi_n = 1 \quad \text{for all } n \in \mathbb{N},$$
$$\varliminf_{n\to\infty} \lambda_n \nu_n > 0, \quad \text{and} \quad \varliminf_{n\to\infty} \lambda_n \xi_n > 0.$$

Let $a, b \in \mathbb{R}$ *such that* $0 < a < b < 1$ *and let* $\{a_n\}$ *be a sequence of real numbers in the interval* $[a,b]$. *Define a sequence* $\{x_n\}$ *in* C *as follows:*

$$\begin{aligned} x_1 &\in C : \text{ given,} \\ z_n &= \lambda_n x_n + \nu_n T x_n + \xi_n T^2 x_n, \text{ and} \\ x_{n+1} &= a_n x_n + (1 - a_n) T z_n \end{aligned} \tag{4.13}$$

for all $n \in \mathbb{N}$. *Then,* $\{x_n\}$ *converges weakly to an element* $\widehat{x}$ *of* $F(T)$, *where* $\widehat{x} \equiv \lim_{n\to\infty} P_{F(T)} x_n$.

Proof. A 2-generalized hybrid mapping is normally 2-generalized hybrid. Therefore, from Lemmas 2.3 and 2.5, T is quasi-nonexpansive and 2-demiclosed. Set $S = I$ and $\mu_n = 0$ for all $n \in \mathbb{N}$ in Theorem 4.1, where I is the identity mapping defined on C. Then, $F(S) \cap F(T) = F(T)$, and (4.9) in the proof of that theorem becomes

$$\alpha \left(\lambda_n \nu_n \|x_n - T x_n\|^2 + \lambda_n \xi_n \left\|x_n - T^2 x_n\right\|^2 \right) \leq \|x_n - q\|^2 - \|x_{n+1} - q\|^2$$

for some $\alpha \in (0,1)$, where $q \in F(T)$ and $n \in \mathbb{N}$. This yields $x_n - T x_n \to 0$ and $T^2 x_n - x_n \to 0$ in (4.11). Setting $M = 1$ and $b_n = d_n = 0$ for all $n \in \mathbb{N}$ in Theorem 4.1, we obtain the desired result. □

Corollary 4.1 is a variant of Theorem 1.1. We add the term $\xi_n T^2 x_n$ in (4.13) whereas the condition (1.6) in Theorem 1.1 is dispensable in our result. Similarly, the following result is derived from Theorem 4.1.

Corollary 4.2 ([52]). *Let C be a nonempty, closed, and convex subset of H. Let $S : C \to C$ be a generalized hybrid mapping such that $F(S) \neq \emptyset$. Let $P_{F(S)}$ be the metric projection from H onto $F(S)$. Let $\{\lambda_n\}$ be a sequence of real numbers in the interval $[0,1]$ that satisfies*

$$\varliminf_{n\to\infty} \lambda_n (1-\lambda_n) > 0.$$

Let $a, b \in \mathbb{R}$ such that $0 < a < b < 1$ and let $\{a_n\}$ be a sequence of real numbers in the interval $[a,b]$. Define a sequence $\{x_n\}$ in C as follows:

$$\begin{aligned} x_1 &\in C : \text{ given}, \\ z_n &= \lambda_n x_n + (1-\lambda_n) S x_n, \text{ and} \\ x_{n+1} &= a_n x_n + (1-a_n) S z_n \end{aligned}$$

for all $n \in \mathbb{N}$. Then, $\{x_n\}$ converges weakly to an element $\widehat{x}$ of $F(S)$, where $\widehat{x} \equiv \lim_{n\to\infty} P_{F(S)} x_n$.

Proof. Note that a generalized hybrid mapping is normally generalized hybrid. Therefore, from Lemma 2.3 and 2.4, S is quasi-nonexpansive and demiclosed. Letting $\nu_n = \xi_n = c_n = d_n = 0$ and $L = 1$ in Theorem 4.1, we obtain the desired result. □

Theorem 4.1 directly yields the following corollary:

Corollary 4.3 ([28]). *Let C be a nonempty, closed, and convex subset of H, let $S : C \to C$ be a normally generalized hybrid mapping, and let $T : C \to C$ be a normally 2-generalized hybrid mapping. Suppose that $F(S) \cap F(T)$ is nonempty. Let $P_{F(S)\cap F(T)}$ be the metric projection from H onto $F(S) \cap F(T)$. Let $a, b \in \mathbb{R}$ such that $0 < a < b < 1$ and let $\{\lambda_n\}$, $\{\mu_n\}$, $\{\nu_n\}$, and $\{\xi_n\}$ be sequences of real numbers in the interval $[a,b]$ such that $\lambda_n + \mu_n + \nu_n + \xi_n = 1$ for all $n \in \mathbb{N}$. Define a sequence $\{x_n\}$ in C as follows:*

$$\begin{aligned} x_1 &\in C : \text{ given}, \\ x_{n+1} &= \lambda_n x_n + \mu_n S x_n + \nu_n T x_n + \xi_n T^2 x_n \end{aligned}$$

for all $n \in \mathbb{N}$. Then, the sequence $\{x_n\}$ converges weakly to a common fixed point $\widehat{x} \in F(S) \cap F(T)$, where $\widehat{x} \equiv \lim_{n\to\infty} P_{F(S)\cap F(T)} x_n$.

Proof. From Lemma 2.3 and 2.4, a normally generalized hybrid mapping with a fixed point is quasi-nonexpansive and demiclosed. Similarly, from Lemma 2.3 and 2.5, a normally 2-generalized hybrid mapping with a fixed point is quasi-nonexpansive and 2-demiclosed. Letting $a_n = c_n = d_n = 0$, $L = 0$ in Theorem 4.1, we obtain the desired result. □

We can prove the following theorem concerning two demiclosed mappings in a similar way as we proved Theorem 4.1. In the proof of Theorem 4.2, the equality (1) in Lemma 2.2 is used. For this point, check (4.8) in the proof of Theorem 4.1 carefully.

Theorem 4.2. *Let C be a nonempty, closed, and convex subset of H. Let $S, T : C \to C$ be quasi-nonexpansive and demiclosed mappings with $F(S) \cap F(T) \neq \emptyset$. Let $P_{F(S)\cap F(T)}$ be the metric projection from H onto $F(S) \cap F(T)$. Let $\{\lambda_n\}$, $\{\mu_n\}$, and $\{\nu_n\}$ be sequences of real numbers in the interval $[0,1]$ that satisfy*

$$\lambda_n + \mu_n + \nu_n = 1 \quad \textit{for all } n \in \mathbb{N},$$
$$\varliminf_{n\to\infty} \lambda_n \mu_n > 0, \quad \textit{and} \quad \varliminf_{n\to\infty} \lambda_n \nu_n > 0.$$

Let $a \in (0,1]$ and let $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, and $\{d_n\}$ be sequences of real numbers in the interval $[0,1]$ such that $a_n + b_n + c_n + d_n = 1$ and $a \leq b_n + c_n + d_n$ for all $n \in \mathbb{N}$. Define a sequence $\{x_n\}$ in C as follows:

$$x_1 \in C: \textit{ given},$$
$$z_n = \lambda_n x_n + \mu_n S x_n + \nu_n T x_n, \textit{ and}$$
$$x_{n+1} = a_n x_n + b_n S^L z_n + c_n T^M z_n + d_n \frac{1}{n} \sum_{l=0}^{n-1} T^l z_n$$

for all $n \in \mathbb{N}$, where $L, M \in \mathbb{N} \cup \{0\}$. Then, $\{x_n\}$ converges weakly to an element $\widehat{x}$ of $F(S) \cap F(T)$, where $\widehat{x} \equiv \lim_{n\to\infty} P_{F(S)\cap F(T)} x_n$.

By using (3) in Lemma 2.2, we can also obtain the following theorem for two 2-demiclosed mappings.

Theorem 4.3. *Let C be a nonempty, closed, and convex subset of H. Let $S, T : C \to C$ be quasi-nonexpansive and 2-demiclosed mappings with $F(S) \cap F(T) \neq \emptyset$. Let $P_{F(S)\cap F(T)}$ be the metric projection from H onto $F(S) \cap F(T)$. Let $\{\lambda_n\}$, $\{\mu_n\}$, $\{\nu_n\}$, $\{\xi_n\}$, and $\{\theta_n\}$ be sequences of real numbers in the interval $[0,1]$ that satisfy*

$$\lambda_n + \mu_n + \nu_n + \xi_n + \theta_n = 1 \quad \textit{for all } n \in \mathbb{N},$$
$$\varliminf_{n\to\infty} \lambda_n \mu_n > 0, \quad \varliminf_{n\to\infty} \lambda_n \nu_n > 0,$$
$$\varliminf_{n\to\infty} \lambda_n \xi_n > 0, \quad \varliminf_{n\to\infty} \lambda_n \theta_n > 0.$$

Let $a \in (0,1]$ and let $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, and $\{d_n\}$ be sequences of real numbers in the interval $[0,1]$ such that $a_n + b_n + c_n + d_n = 1$ and $a \leq b_n + c_n + d_n$ for all $n \in \mathbb{N}$. Define a sequence $\{x_n\}$ in C as follows:

$$x_1 \in C: \textit{ given},$$
$$z_n = \lambda_n x_n + \mu_n S x_n + \nu_n S^2 x_n + \xi_n T x_n + \theta_n T^2 x_n, \textit{ and}$$
$$x_{n+1} = a_n x_n + b_n S^L z_n + c_n T^M z_n + d_n \frac{1}{n} \sum_{l=0}^{n-1} T^l z_n$$

for all $n \in \mathbb{N}$, where $L, M \in \mathbb{N} \cup \{0\}$. Then, $\{x_n\}$ converges weakly to an element $\widehat{x}$ of $F(S) \cap F(T)$, where $\widehat{x} \equiv \lim_{n\to\infty} P_{F(S)\cap F(T)} x_n$.

5. Strong convergence by hybrid methods

In this section, we present Nakajo–Takahashi type strong convergence theorems for finding common fixed points of two quasi-nonexpansive mappings. Each of which is required to be demiclosed (2.4) or 2-demiclosed (2.5). The mappings are not necessarily commutative or continuous. The basic elements of the proof adopted here were developed in many works, such as [2, 11, 13, 23, 24, 35] and [47].

Theorem 5.1. *Let C be a nonempty, closed, and convex subset of H. Let $S : C \to C$ be a quasi-nonexpansive and demiclosed mapping and let $T : C \to C$ be a quasi-nonexpansive and 2-demiclosed mapping. Suppose that $F(S) \cap F(T) \neq \emptyset$. Let $\{\lambda_n\}$, $\{\mu_n\}$, $\{\nu_n\}$, and $\{\xi_n\}$ be sequences of real numbers in the interval $[0, 1]$ that satisfy*

$$\begin{aligned} \lambda_n + \mu_n + \nu_n + \xi_n = 1 \quad \text{for all } n \in \mathbb{N}, \\ \varliminf_{n\to\infty} \lambda_n\mu_n > 0, \quad \varliminf_{n\to\infty} \lambda_n\nu_n > 0, \quad \text{and} \quad \varliminf_{n\to\infty} \lambda_n\xi_n > 0. \end{aligned} \tag{5.1}$$

Let $a \in (0, 1]$ and let $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, and $\{d_n\}$ be sequences of real numbers in the interval $[0, 1]$ such that $a_n + b_n + c_n + d_n = 1$ and $a \leq b_n + c_n + d_n$ for all $n \in \mathbb{N}$. Define a sequence $\{x_n\}$ in C as follows:

$$\begin{aligned} x_1 &= x \in C : \text{ given}, \\ z_n &= \lambda_n x_n + \mu_n S x_n + \nu_n T x_n + \xi_n T^2 x_n, \\ y_n &= a_n x_n + b_n S^L z_n + c_n T^M z_n + d_n \frac{1}{n} \sum_{l=0}^{n-1} T^l z_n, \\ C_n &= \{w \in C : \|y_n - w\| \leq \|x_n - w\|\}, \\ Q_n &= \{w \in C : \langle x - x_n, x_n - w\rangle \geq 0\}, \text{ and} \\ x_{n+1} &= P_{C_n \cap Q_n} x \end{aligned} \tag{5.2}$$

for all $n \in \mathbb{N}$, where $L, M \in \mathbb{N} \cup \{0\}$. Then, $\{x_n\}$ converges strongly to a point $\widehat{x}$ of $F(S) \cap F(T)$, where $\widehat{x} = P_{F(S)\cap F(T)} x$.

Proof. First, note that because the mappings S and T are quasi-nonexpansive, $F(S) \cap F(T)$ is closed and convex. Furthermore, because we assume that $F(S) \cap F(T) \neq \emptyset$, the metric projection $P_{F(S)\cap F(T)}$ from H onto $F(S) \cap F(T)$ exists. As a preliminary consideration, we prove that (5.3)–(5.5) hold when the sequences $\{x_n\}$, $\{y_n\}$, and $\{z_n\}$ are given. As S and T are quasi-nonexpansive, the following inequality is true:

$$\|z_n - q\| \leq \|x_n - q\| \tag{5.3}$$

for all $q \in F(S) \cap F(T)$ and $n \in \mathbb{N}$. Indeed,

$$\begin{aligned}
&\|z_n - q\| \\
&= \left\|\lambda_n x_n + \mu_n S x_n + \nu_n T x_n + \xi_n T^2 x_n - q\right\| \\
&= \left\|\lambda_n (x_n - q) + \mu_n (S x_n - q) + \nu_n (T x_n - q) + \xi_n \left(T^2 x_n - q\right)\right\| \\
&\leq \lambda_n \|x_n - q\| + \mu_n \|S x_n - q\| + \nu_n \|T x_n - q\| + \xi_n \left\|T^2 x_n - q\right\| \\
&\leq \lambda_n \|x_n - q\| + \mu_n \|x_n - q\| + \nu_n \|x_n - q\| + \xi_n \|x_n - q\| \\
&= \|x_n - q\|,
\end{aligned}$$

which shows that (5.3) holds. Furthermore, it also holds that

$$\left\|\frac{1}{n}\sum_{l=0}^{n-1} T^l z_n - q\right\| \leq \|z_n - q\| \tag{5.4}$$

for all $q \in F(T)$ and $n \in \mathbb{N}$. Indeed, as T is quasi-nonexpansive,

$$\begin{aligned}
\left\|\frac{1}{n}\sum_{l=0}^{n-1} T^l z_n - q\right\| &\leq \frac{1}{n}\left\|\sum_{l=0}^{n-1} T^l z_n - nq\right\| \\
&= \frac{1}{n}\left\|\sum_{l=0}^{n-1}\left(T^l z_n - q\right)\right\| \leq \frac{1}{n}\sum_{l=0}^{n-1}\left\|T^l z_n - q\right\| \\
&\leq \frac{1}{n}\sum_{l=0}^{n-1}\|z_n - q\| = \|z_n - q\|.
\end{aligned}$$

Using (5.3) and (5.4), we obtain

$$\|y_n - q\| \leq \|x_n - q\| \tag{5.5}$$

for all $q \in F(S) \cap F(T)$ and $n \in \mathbb{N}$. Indeed, as S and T are quasi-nonexpansive,

$$\begin{aligned}
&\|y_n - q\| \qquad\qquad (5.6) \\
&= \left\|a_n x_n + b_n S^L z_n + c_n T^M z_n + d_n \frac{1}{n}\sum_{l=0}^{n-1} T^l z_n - q\right\| \\
&= \left\|a_n (x_n - q) + b_n \left(S^L z_n - q\right) + c_n \left(T^M z_n - q\right) + d_n \left(\frac{1}{n}\sum_{l=0}^{n-1} T^l z_n - q\right)\right\| \\
&= a_n \|x_n - q\| + b_n \left\|S^L z_n - q\right\| + c_n \left\|T^M z_n - q\right\| + d_n \left\|\frac{1}{n}\sum_{l=0}^{n-1} T^l z_n - q\right\| \\
&\leq a_n \|x_n - q\| + b_n \|z_n - q\| + c_n \|z_n - q\| + d_n \|z_n - q\| \\
&\leq a_n \|x_n - q\| + b_n \|x_n - q\| + c_n \|x_n - q\| + d_n \|x_n - q\| \\
&= \|x_n - q\|.
\end{aligned}$$

which demonstrates that (5.5) holds. Furthermore, it is clear from their definition that C_n and Q_n are closed and convex for all $n \in \mathbb{N}$ when x_n and $y_n \in C$ are given.

Next, we verify that the sequences $\{x_n\}$, $\{y_n\}$, and $\{z_n\}$ are properly defined. To do so, we use mathematical induction to demonstrate that $F(S) \cap F(T) \subset C_n \cap Q_n$ for all $n \in \mathbb{N}$. (i) It holds that $F(S) \cap F(T) \subset Q_1$ because $Q_1 = C$. Let $q \in F(S) \cap F(T)$. From (5.5), it holds that

$$\|y_1 - q\| \leq \|x_1 - q\|,$$

which indicates that $q \in C_1$. Thus, $F(S) \cap F(T) \subset C_1$. (ii) Assume that $F(S) \cap F(T) \subset C_k \cap Q_k$, where $k \in \mathbb{N}$. As it is assumed that $F(S) \cap F(T) \neq \emptyset$, $C_k \cap Q_k$ is also nonempty. As $C_k \cap Q_k$ is closed and convex subset in H, the metric projection $P_{C_k \cap Q_k}$ from H onto $C_k \cap Q_k$ exists. Thus, $x_{k+1} = P_{C_k \cap Q_k} x$ is defined. Consequently, z_{k+1}, $y_{k+1} (\in C)$, C_{k+1}, and $Q_{k+1} (\subset C)$ are also defined. We show that $F(S) \cap F(T) \subset C_{k+1} \cap Q_{k+1}$. The inclusion $F(S) \cap F(T) \subset C_{k+1}$ follows from (5.5). We prove that $F(S) \cap F(T) \subset Q_{k+1}$. Choose $q \in F(S) \cap F(T)$ arbitrarily. As $x_{k+1} = P_{C_k \cap Q_k} x$ and $q \in F(S) \cap F(T) \subset C_k \cap Q_k$, we have from (2.1) that $\langle x - x_{k+1}, x_{k+1} - q \rangle \geq 0$. This means that $q \in Q_{k+1}$ and thus, we have $F(S) \cap F(T) \subset C_{k+1} \cap Q_{k+1}$ as claimed. We have demonstrated that $F(S) \cap F(T) \subset C_n \cap Q_n$ for all $n \in \mathbb{N}$, and since $F(S) \cap F(T) \neq \emptyset$ is assumed, this means that $C_n \cap Q_n \neq \emptyset$ for all $n \in \mathbb{N}$. Therefore, the sequences $\{x_n\}$, $\{z_n\}$, and $\{y_n\}$ are defined successively.

From the definition of Q_n, it holds that

$$x_n = P_{Q_n} x$$

for all $n \in \mathbb{N}$. Consequently,

$$\|x - x_n\| \leq \|x - q\| \tag{5.7}$$

for all $q \in F(S) \cap F(T)$ and $n \in \mathbb{N}$. This is because $q \in F(S) \cap F(T) \subset C_n \cap Q_n \subset Q_n$. From (5.7), $\{x_n\}$ is bounded. In view of (5.5), $\{y_n\}$ is also bounded.

Next, notice that

$$\|x - x_n\| \leq \|x - x_{n+1}\| \tag{5.8}$$

for all $n \in \mathbb{N}$. Indeed, as $x_n = P_{Q_n} x$ and $x_{n+1} = P_{C_n \cap Q_n} x \in Q_n$, (5.8) holds. Because $\{x_n\}$ is bounded, so is $\{\|x - x_n\|\}$. As $\{\|x - x_n\|\}$ is monotone increasing and bounded, it is convergent in $\mathbb{R}$.

Observe that $x_n - y_n \to 0$. As $x_n = P_{Q_n} x$ and $x_{n+1} = P_{C_n \cap Q_n} x \in Q_n$, it holds from (2.2) that

$$\|x - x_n\|^2 + \|x_n - x_{n+1}\|^2 \leq \|x - x_{n+1}\|^2$$

for all $n \in \mathbb{N}$. As $\{\|x - x_n\|\}$ is convergent, it follows that $x_n - x_{n+1} \to 0$. From $x_{n+1} = P_{C_n \cap Q_n} x \in C_n$, it follows that $\|y_n - x_{n+1}\| \leq \|x_n - x_{n+1}\|$. Using $x_n - x_{n+1} \to 0$, we have $y_n - x_{n+1} \to 0$. Thus,

$$\|y_n - x_n\| \leq \|y_n - x_{n+1}\| + \|x_{n+1} - x_n\| \to 0$$

as claimed.

Next, we show

$$Sx_n - x_n \to 0, \quad Tx_n - x_n \to 0, \quad \text{and} \quad T^2x_n - x_n \to 0. \tag{5.9}$$

Let $q \in F(S) \cap F(T)$ and $n \in \mathbb{N}$. We show that

$$\begin{aligned} \|z_n - q\|^2 &\leq \|x_n - q\|^2 - \lambda_n\mu_n \|x_n - Sx_n\|^2 \\ &\quad -\lambda_n\nu_n \|x_n - Tx_n\|^2 - \lambda_n\xi_n \left\|x_n - T^2x_n\right\|^2. \end{aligned} \tag{5.10}$$

Indeed, as S and T are quasi-nonexpansive, it follows from Lemma 2.2-(2) that

$$\begin{aligned} & \|z_n - q\|^2 \\ = {} & \left\|\lambda_n x_n + \mu_n Sx_n + \nu_n Tx_n + \xi_n T^2 x_n - q\right\|^2 \\ = {} & \left\|\lambda_n (x_n - q) + \mu_n (Sx_n - q) + \nu_n (Tx_n - q) + \xi_n \left(T^2x_n - q\right)\right\|^2 \\ = {} & \lambda_n \|x_n - q\|^2 + \mu_n \|Sx_n - q\|^2 + \nu_n \|Tx_n - q\|^2 + \xi_n \left\|T^2x_n - q\right\|^2 \\ & -\lambda_n\mu_n \|x_n - Sx_n\|^2 - \lambda_n\nu_n \|x_n - Tx_n\|^2 - \lambda_n\xi_n \left\|x_n - T^2x_n\right\|^2 \\ & -\mu_n\nu_n \|Sx_n - Tx_n\|^2 - \mu_n\xi_n \left\|Sx_n - T^2x_n\right\|^2 - \nu_n\xi_n \left\|Tx_n - T^2x_n\right\|^2 \\ \leq {} & \lambda_n \|x_n - q\|^2 + \mu_n \|x_n - q\|^2 + \nu_n \|x_n - q\|^2 + \xi_n \|x_n - q\|^2 \\ & -\lambda_n\mu_n \|x_n - Sx_n\|^2 - \lambda_n\nu_n \|x_n - Tx_n\|^2 - \lambda_n\xi_n \left\|x_n - T^2x_n\right\|^2 \\ & -\mu_n\nu_n \|Sx_n - Tx_n\|^2 - \mu_n\xi_n \left\|Sx_n - T^2x_n\right\|^2 - \nu_n\xi_n \left\|Tx_n - T^2x_n\right\|^2 \\ = {} & \|x_n - q\|^2 \\ & -\lambda_n\mu_n \|x_n - Sx_n\|^2 - \lambda_n\nu_n \|x_n - Tx_n\|^2 - \lambda_n\xi_n \left\|x_n - T^2x_n\right\|^2 \\ & -\mu_n\nu_n \|Sx_n - Tx_n\|^2 - \mu_n\xi_n \left\|Sx_n - T^2x_n\right\|^2 - \nu_n\xi_n \left\|Tx_n - T^2x_n\right\|^2. \end{aligned} \tag{5.11}$$

As $\mu_n\nu_n \|Sx_n - Tx_n\|^2 + \mu_n\xi_n \left\|Sx_n - T^2x_n\right\|^2 + \nu_n\xi_n \left\|Tx_n - T^2x_n\right\|^2 \geq 0$, we obtain (5.10). Using (5.4) yields

$$\begin{aligned} & \|y_n - q\|^2 \\ = {} & \left\|a_n x_n + b_n S^L z_n + c_n T^M z_n + d_n \frac{1}{n}\sum_{l=0}^{n-1} T^l z_n - q\right\|^2 \\ = {} & \left\|a_n (x_n - q) + b_n \left(S^L z_n - q\right) + c_n \left(T^M z_n - q\right) + d_n \left(\frac{1}{n}\sum_{l=0}^{n-1} T^l z_n - q\right)\right\|^2 \\ \leq {} & a_n \|x_n - q\|^2 + b_n \left\|S^L z_n - q\right\|^2 + c_n \left\|T^M z_n - q\right\|^2 + d_n \left\|\frac{1}{n}\sum_{l=0}^{n-1} T^l z_n - q\right\|^2 \\ \leq {} & a_n \|x_n - q\|^2 + b_n \|z_n - q\|^2 + c_n \|z_n - q\|^2 + d_n \|z_n - q\|^2 \\ = {} & a_n \|x_n - q\|^2 + (b_n + c_n + d_n) \|z_n - q\|^2. \end{aligned} \tag{5.12}$$

From (5.10), the following holds:

$$\begin{aligned}
&\|y_n - q\|^2 \\
\leq\ & a_n \|x_n - q\|^2 + (b_n + c_n + d_n) (\|x_n - q\|^2 - \lambda_n \mu_n \|x_n - Sx_n\|^2 \\
&-\lambda_n \nu_n \|x_n - Tx_n\|^2 - \lambda_n \xi_n \left\|x_n - T^2 x_n\right\|^2) \\
=\ & \|x_n - q\|^2 - (b_n + c_n + d_n) (\lambda_n \mu_n \|x_n - Sx_n\|^2 + \lambda_n \nu_n \|x_n - Tx_n\|^2 \\
&+\lambda_n \xi_n \left\|x_n - T^2 x_n\right\|^2).
\end{aligned}$$

The result is that

$$\begin{aligned}
&(b_n + c_n + d_n) (\lambda_n \mu_n \|x_n - Sx_n\|^2 + \lambda_n \nu_n \|x_n - Tx_n\|^2 \\
&+\lambda_n \xi_n \left\|x_n - T^2 x_n\right\|^2) \\
\leq\ & \|x_n - q\|^2 - \|y_n - q\|^2 \\
=\ & (\|x_n - q\| + \|y_n - q\|) (\|x_n - q\| - \|y_n - q\|) \\
\leq\ & (\|x_n - q\| + \|y_n - q\|) \|x_n - y_n\| .
\end{aligned}$$

As $a \leq b_n + c_n + d_n$, we obtain

$$\begin{aligned}
&a(\lambda_n \mu_n \|x_n - Sx_n\|^2 + \lambda_n \nu_n \|x_n - Tx_n\|^2 + \lambda_n \xi_n \left\|x_n - T^2 x_n\right\|^2) \\
\leq\ & (\|x_n - q\| + \|y_n - q\|) \|x_n - y_n\| .
\end{aligned}$$

As $a > 0$, $\{x_n\}$ and $\{y_n\}$ are bounded, and $x_n - y_n \to 0$, using assumptions on the parameters $\lambda_n, \mu_n, \nu_n, \xi_n$ (5.1), we obtain (5.9).

Our goal is to demonstrate that $x_n \to \widehat{x}\left(= P_{F(S)\cap F(T)} x\right)$; in other words, for any subsequence $\{x_{n_i}\}$ of $\{x_n\}$, there exists a subsequence $\left\{x_{n_j}\right\}$ of $\{x_{n_i}\}$ such that $x_{n_j} \to \widehat{x}$. Choose a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ arbitrarily. As $\{x_{n_i}\}$ is bounded, there exists a subsequence $\left\{x_{n_j}\right\}$ of $\{x_{n_i}\}$ such that $x_{n_j} \rightharpoonup v$ for some $v \in H$. As S is demiclosed (2.4) and T is 2-demiclosed (2.5), we have from (5.9) that $v \in F(S) \cap F(T)$. We prove that $x_{n_j} \to v$. As $v \in F(S) \cap F(T)$, it follows from (5.7) that

$$\begin{aligned}
\left\|x_{n_j} - v\right\|^2 &= \left\|x_{n_j} - x\right\|^2 + 2\left\langle x_{n_j} - x,\ x - v\right\rangle + \|x - v\|^2 \\
&\leq \|x - v\|^2 + 2\left\langle x_{n_j} - x,\ x - v\right\rangle + \|x - v\|^2 .
\end{aligned}$$

From $x_{n_j} \rightharpoonup v$, we obtain

$$\begin{aligned}
\left\|x_{n_j} - v\right\|^2 &\leq \|x - v\|^2 + 2\left\langle x_{n_j} - x,\ x - v\right\rangle + \|x - v\|^2 \\
&\to 2\|x - v\|^2 + 2\langle v - x,\ x - v\rangle = 0.
\end{aligned}$$

This means that $x_{n_j} \to v$ as claimed. Finally, we demonstrate that

$$v\left(= \lim_{j\to\infty} x_{n_j}\right) = \widehat{x}\left(= P_{F(S)\cap F(T)} x\right).$$

As $v \in F(S) \cap F(T)$, it suffices to show that $\|x - v\| \leq \|x - \widehat{x}\|$. Using (5.7) for $q = \widehat{x} \in F(S) \cap F(T)$, we have

$$\left\|x - x_{n_j}\right\| \leq \|x - \widehat{x}\|$$

for all $j \in \mathbb{N}$. As $x_{n_j} \to v$, it holds that $\|x - v\| \leq \|x - \widehat{x}\|$. Thus, $v = \widehat{x}$. This means that $x_n \to \widehat{x}$. This completes the proof. □

Remark 5.1. *As in* Remark 4.1, *the sequence* y_n *in* (5.2) *can be defined more generally; for instance,*

$$y_n = a_n x_n + b_n S^L y_n + c_n T^M y_n + d_n \frac{1}{n} \sum_{l=0}^{n-1} S^l y_n + e_n \frac{1}{n} \sum_{l=0}^{n-1} T^l y_n, \text{ or} \quad (5.13)$$

$$y_n = a_n x_n + b_n S^L y_n + c_n T^M y_n + d_n \frac{1}{n} \sum_{l=0}^{n-1} S^l y_n + e_n \frac{1}{n} \sum_{l=m}^{m+n-1} T^l y_n,$$

where $m \in \mathbb{N} \cup \{0\}$. *In* (5.13), $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{d_n\}$, *and* $\{e_n\}$ *are sequences of real numbers in* $[0,1]$ *such that* $a_n + b_n + c_n + d_n + e_n = 1$ *and* $a \leq b_n + c_n + d_n + e_n$ *for all* $n \in \mathbb{N}$. *For this point, carefully check* (5.6) *and* (5.12) *in the proof of* Theorem 5.1.

The following corollary is derived from Theorem 5.1.

Corollary 5.1 ([23]). *Let* C *be a nonempty, closed, and convex subset of* H. *Let* $S : C \to C$ *be a quasi-nonexpansive and demiclosed mapping and let* $T : C \to C$ *be a quasi-nonexpansive and* 2*-demiclosed mapping. Suppose that* $F(S) \cap F(T) \neq \emptyset$. *Let* $a, b \in \mathbb{R}$ *with* $0 < a < b < 1$ *and let* $\{\lambda_n\}$, $\{\mu_n\}$, $\{\nu_n\}$, *and* $\{\xi_n\}$ *be sequences of real numbers in the interval* $[a, b]$ *such that* $\lambda_n + \mu_n + \nu_n + \xi_n = 1$ *for all* $n \in \mathbb{N}$. *Define a sequence* $\{x_n\}$ *in* C *as follows:*

$$\begin{aligned}
x_1 &= x \in C : \text{ given}, \\
y_n &= \lambda_n x_n + \mu_n S x_n + \nu_n T x_n + \xi_n T^2 x_n, \\
C_n &= \{z \in C : \|y_n - z\| \leq \|x_n - z\|\}, \\
Q_n &= \{z \in C : \langle x - x_n, \ x_n - z \rangle \geq 0\}, \text{ and} \\
x_{n+1} &= P_{C_n \cap Q_n} x
\end{aligned}$$

for all $n \in \mathbb{N}$. *Then,* $\{x_n\}$ *converges strongly to a point* $\widehat{x}$ *of* $F(S) \cap F(T)$, *where* $\widehat{x} = P_{F(S) \cap F(T)} x$.

Proof. Letting $a_n = c_n = d_n = 0$ and $L = 0$ and applying Theorem 5.1, we obtain the desired result. □

As in Section 4, we can obtain Theorem 5.2 and 5.3. To prove Theorem 5.2, the equality (1) of Lemma 2.2 is necessary whereas for Theorem 5.3, equality (3) of Lemma 2.2 is employed. To confirm this point, inspect (5.11) in the proof of Theorem 5.1 carefully.

Theorem 5.2. *Let* C *be a nonempty, closed, and convex subset of* H. *Let* $S, T : C \to C$ *be quasi-nonexpansive and demiclosed mappings such that* $F(S) \cap F(T) \neq \emptyset$. *Let* $\{\lambda_n\}$, $\{\mu_n\}$, *and* $\{\nu_n\}$ *be sequences of real numbers* *in the interval* $[0,1]$ *that satisfy*

$$\lambda_n + \mu_n + \nu_n = 1 \quad \text{for all } n \in \mathbb{N},$$
$$\liminf_{n\to\infty} \lambda_n \mu_n > 0, \quad \text{and} \quad \liminf_{n\to\infty} \lambda_n \nu_n > 0.$$

Let $a \in (0,1]$ *and let* $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, *and* $\{d_n\}$ *be sequences of real numbers in the interval* $[0,1]$ *such that* $a_n + b_n + c_n + d_n = 1$ *and* $a \le b_n + c_n + d_n$ *for all* $n \in \mathbb{N}$. *Define a sequence* $\{x_n\}$ *in* C *as follows:*

$$\begin{aligned}
x_1 &= x \in C: \ \text{given}, \\
z_n &= \lambda_n x_n + \mu_n S x_n + \nu_n T x_n, \\
y_n &= a_n x_n + b_n S^L z_n + c_n T^M z_n + d_n \frac{1}{n} \sum_{l=0}^{n-1} T^l z_n, \\
C_n &= \{w \in C : \|y_n - w\| \le \|x_n - w\|\}, \\
Q_n &= \{w \in C : \langle x - x_n, \ x_n - w \rangle \ge 0\}, \ \text{and} \\
x_{n+1} &= P_{C_n \cap Q_n} x
\end{aligned}$$

for all $n \in \mathbb{N}$, *where* $L, M \in \mathbb{N} \cup \{0\}$. *Then,* $\{x_n\}$ *converges strongly to a point* $\widehat{x}$ *of* $F(S) \cap F(T)$, *where* $\widehat{x} = P_{F(S)\cap F(T)} x$.

Theorem 5.3. *Let* C *be a nonempty, closed, and convex subset of* H. *Let* $S, T : C \to C$ *be quasi-nonexpansive and* 2*-demiclosed mappings such that* $F(S) \cap F(T) \neq \emptyset$. *Let* $\{\lambda_n\}$, $\{\mu_n\}$, $\{\nu_n\}$, $\{\xi_n\}$, *and* $\{\theta_n\}$ *be sequences of real numbers in the interval* $[0,1]$ *that satisfy*

$$\lambda_n + \mu_n + \nu_n + \xi_n + \theta_n = 1 \quad \text{for all } n \in \mathbb{N},$$
$$\liminf_{n\to\infty} \lambda_n \mu_n > 0, \qquad \liminf_{n\to\infty} \lambda_n \nu_n > 0,$$
$$\liminf_{n\to\infty} \lambda_n \xi_n > 0, \qquad \liminf_{n\to\infty} \lambda_n \theta_n > 0.$$

Let $a \in (0,1]$ *and let* $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, *and* $\{d_n\}$ *be sequences of real numbers in the interval* $[0,1]$ *such that* $a_n + b_n + c_n + d_n = 1$ *and* $a \le b_n + c_n + d_n$ *for all* $n \in \mathbb{N}$. *Define a sequence* $\{x_n\}$ *in* C *as follows:*

$$\begin{aligned}
x_1 &= x \in C: \ \text{given}, \\
z_n &= \lambda_n x_n + \mu_n S x_n + \nu_n S^2 x_n + \xi_n T x_n + \theta_n T^2 x_n, \\
y_n &= a_n x_n + b_n S^L z_n + c_n T^M z_n + d_n \frac{1}{n} \sum_{l=0}^{n-1} T^l z_n, \\
C_n &= \{w \in C : \|y_n - w\| \le \|x_n - w\|\}, \\
Q_n &= \{w \in C : \langle x - x_n, \ x_n - w \rangle \ge 0\}, \ \text{and} \\
x_{n+1} &= P_{C_n \cap Q_n} x
\end{aligned}$$

for all $n \in \mathbb{N}$, *where* $L, M \in \mathbb{N} \cup \{0\}$. *Then,* $\{x_n\}$ *converges strongly to a point* $\widehat{x}$ *of* $F(S) \cap F(T)$, *where* $\widehat{x} = P_{F(S)\cap F(T)} x$.

6. Strong convergence by shrinking projection methods

In this section, we use Takahashi, Takeuchi, and Kubota's shrinking projection method [45] to derive strong convergence to common fixed points of two nonlinear quasi-nonexpansive mappings. The fundamentals of the proof were improved by many researchers [11, 13, 23, 24, 31, 47]. To achieve this goal, we consider nonlinear mappings with certain conditions; let C be a nonempty and closed subset of H, and consider mappings $S : C \to H$ and $T : C \to C$ that satisfy

$$Sx_n - x_n \to 0 \text{ and } x_n \to v \implies v \in F(S), \text{ and} \tag{6.1}$$

$$Tx_n - x_n \to 0,\ T^2x_n - x_n \to 0, \text{ and } x_n \to v \implies v \in F(T); \tag{6.2}$$

respectively. To fulfill condition (6.1), it is sufficient that S is demiclosed or continuous. A mapping T satisfies the condition (6.2) if it satisfies any of the following conditions: (a) T is demiclosed; (b) T is 2-demiclosed; (c) T is continuous; or (d) T satisfies the condition (6.1). Therefore, these two conditions do not greatly restrict the class of mappings to which the result applies.

Theorem 6.1. *Let C be a nonempty, closed, and convex subset of H. Let $S, T : C \to C$ be quasi-nonexpansive mappings with $F(S) \cap F(T) \neq \emptyset$. Suppose that S and T satisfy the conditions (6.1) and (6.2), respectively. Let $\{\lambda_n\}$, $\{\mu_n\}$, $\{\nu_n\}$, and $\{\xi_n\}$ be sequences of real numbers in the interval $[0, 1]$ such that*

$$\begin{gathered}\lambda_n + \mu_n + \nu_n + \xi_n = 1 \text{ for all } n \in \mathbb{N}, \\ \varliminf_{n\to\infty} \lambda_n\mu_n > 0, \quad \varliminf_{n\to\infty} \lambda_n\nu_n > 0, \quad \text{and} \quad \varliminf_{n\to\infty} \lambda_n\xi_n > 0.\end{gathered} \tag{6.3}$$

Let $a \in (0, 1]$ and let $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, and $\{d_n\}$ be sequences of real numbers in the interval $[0, 1]$ such that $a_n + b_n + c_n + d_n = 1$ and $a \leq b_n + c_n + d_n$ for all $n \in \mathbb{N}$. Let $\{u_n\}$ be a sequence in H such that $u_n \to u\,(\in H)$. Define a sequence $\{x_n\}$ in C as follows:

$$\begin{aligned} x_1 &= x \in C : \text{ given} \\ C_1 &= C, \\ z_n &= \lambda_n x_n + \mu_n S x_n + \nu_n T x_n + \xi_n T^2 x_n, \\ y_n &= a_n x_n + b_n S^L z_n + c_n T^M z_n + d_n \frac{1}{n} \sum_{l=0}^{n-1} T^l z_n, \\ C_{n+1} &= \{w \in C_n : \|y_n - w\| \leq \|x_n - w\|\}, \text{ and} \\ x_{n+1} &= P_{C_{n+1}} u_{n+1} \end{aligned} \tag{6.4}$$

for all $n \in \mathbb{N}$, where $L, M \in \mathbb{N} \cup \{0\}$. Then, $\{x_n\}$ converges strongly to a point $\widehat{u}$ of $F(S) \cap F(T)$, where $\widehat{u} = P_{F(S)\cap F(T)} u$.

Proof. At the outset, it should be noted that $F(S) \cap F(T)$ is closed and convex because S and T are quasi-nonexpansive. From the hypothesis that

$F(S)\cap F(T)\neq\emptyset$, the metric projection $P_{F(S)\cap F(T)}$ from H onto $F(S)\cap F(T)$ exists. Notice that a sequence $\{u_n\}$ in H is given. We now verify that (6.5)–(6.7) hold when sequences $\{x_n\}$, $\{y_n\}$, and $\{z_n\}$ are given. As S and T are quasi-nonexpansive, it holds that

$$\|z_n-q\|\leq\|x_n-q\| \tag{6.5}$$

for all $q\in F(S)\cap F(T)$ and $n\in\mathbb{N}$. The proof of (6.5) is the same as for (5.3), and hence, we omit it here. Furthermore, it holds that

$$\left\|\frac{1}{n}\sum_{l=0}^{n-1}T^l z_n-q\right\|\leq\|z_n-q\| \tag{6.6}$$

for all $q\in F(T)$ and $n\in\mathbb{N}$. This can be demonstrated in much the same way as (5.4). Using (6.5) and (6.6), we obtain

$$\|y_n-q\|\leq\|x_n-q\| \tag{6.7}$$

for all $q\in F(S)\cap F(T)$ and $n\in\mathbb{N}$. The proof of (6.7) is same as that of (5.5).

Next, observe that

- C_n is a closed and convex subset of C and
- $F(S)\cap F(T)\subset C_n$ for all $n\in\mathbb{N}$.

We use mathematical induction. (i) For $n=1$, the desired result is true because $C_1=C$. (ii) Assume that C_k is closed and convex and $F(S)\cap F(T)\subset C_k$, where $k\in\mathbb{N}$. As $F(S)\cap F(T)\neq\emptyset$ is assumed and $F(S)\cap F(T)\subset C_k$, we have that $C_k\neq\emptyset$. As C_k is nonempty, closed, and convex, the metric projection P_{C_k} from H onto C_k exists. Consequently, $x_k\,(\in C_k)$, z_k, $y_k\,(\in C)$, and $C_{k+1}\,(\subset C)$ are defined. As C_k is closed and convex, C_{k+1} is also closed and convex. We show that $F(S)\cap F(T)\subset C_{k+1}$. Let $q\in F(S)\cap F(T)$. From (6.7), we have $\|y_k-q\|\leq\|x_k-q\|$. This implies that $q\in C_{k+1}$. We obtain $F(S)\cap F(T)\subset C_{k+1}$ as claimed. We demonstrated that C_n is a closed and convex subset of C and $F(S)\cap F(T)\subset C_n$ for all $n\in\mathbb{N}$. As $F(S)\cap F(T)\neq\emptyset$ is assumed, $C_n\neq\emptyset$ for all $n\in\mathbb{N}$. Consequently, the sequences $\{x_n\}$, $\{z_n\}$, and $\{y_n\}$ are defined properly.

Define $\overline{u}_n=P_{C_n}u\,(\in C_n)$. As $C_n\subset C_{n-1}\subset\cdots\subset C_1=C$, $\{\overline{u}_n\}$ is a sequence in C. It holds that

$$\|u-\overline{u}_n\|\leq\|u-q\| \tag{6.8}$$

for all $q\in F(S)\cap F(T)$ and $n\in\mathbb{N}$. This is because $q\in F(S)\cap F(T)\subset C_n$ and $\overline{u}_n=P_{C_n}u$. The inequality (6.8) shows that $\{\overline{u}_n\}$ is bounded. Furthermore,

$$\|u-\overline{u}_n\|\leq\|u-\overline{u}_{n+1}\|$$

for all $n\in\mathbb{N}$ because $\overline{u}_n=P_{C_n}u$ and $\overline{u}_{n+1}=P_{C_{n+1}}u\in C_{n+1}\subset C_n$. As $\{\|u-\overline{u}_n\|\}$ is bounded and monotone increasing, it is convergent.

We show that there exists $\overline{u}\in C$ such that

$$\overline{u}_n\to\overline{u}. \tag{6.9}$$

Choose $m, n \in \mathbb{N}$ with $m \geq n$. As $\overline{u}_n = P_{C_n} u$ and $\overline{u}_m = P_{C_m} u \in C_m \subset C_n$, we have from (2.2) that

$$\|u - \overline{u}_n\|^2 + \|\overline{u}_n - \overline{u}_m\|^2 \leq \|u - \overline{u}_m\|^2 .$$

As $\{\|u - \overline{u}_n\|\}$ is convergent, it follows that $\overline{u}_n - \overline{u}_m \to 0$ as $m, n \to \infty$. This means that $\{\overline{u}_n\}$ is a Cauchy sequence in C. As C is closed in a Hilbert space H, it is complete. Thus, there exists $\overline{u} \in C$ such that $\overline{u}_n \to \overline{u}$ as claimed. Furthermore, $\{x_n\}$ has the same limit point, that is,

$$x_n \to \overline{u}. \tag{6.10}$$

Indeed, as the metric projection P_{C_n} is nonexpansive and $u_n \to u$, using (6.9), we have

$$\begin{aligned} \|x_n - \overline{u}\| &\leq \|x_n - \overline{u}_n\| + \|\overline{u}_n - \overline{u}\| \\ &= \|P_{C_n} u_n - P_{C_n} u\| + \|\overline{u}_n - \overline{u}\| \\ &\leq \|u_n - u\| + \|\overline{u}_n - \overline{u}\| \to 0 \end{aligned}$$

as $n \to \infty$. As a result from (6.10), $\{x_n\}$ is bounded. From (6.7), $\{y_n\}$ is also bounded.

As $\{x_n\}$ is convergent, $x_n - x_{n+1} \to 0$. We show that $y_n - x_{n+1} \to 0$. Indeed, from $x_{n+1} = P_{C_{n+1}} u_{n+1} \in C_{n+1}$, it follows that

$$\|y_n - x_{n+1}\| \leq \|x_n - x_{n+1}\| .$$

Using $x_n - x_{n+1} \to 0$, we have $y_n - x_{n+1} \to 0$. Hence, it results that $x_n - y_n \to 0$ from

$$\|x_n - y_n\| \leq \|x_n - x_{n+1}\| + \|x_{n+1} - y_n\| \to 0. \tag{6.11}$$

Next, observe that

$$Sx_n - x_n \to 0, \qquad Tx_n - x_n \to 0, \quad \text{and} \quad T^2 x_n - x_n \to 0. \tag{6.12}$$

To show this, we verify that

$$\begin{aligned} \|z_n - q\|^2 \quad \leq \quad & \|x_n - q\|^2 - \lambda_n \mu_n \|x_n - Sx_n\|^2 \\ & - \lambda_n \nu_n \|x_n - Tx_n\|^2 - \lambda_n \xi_n \left\|x_n - T^2 x_n\right\|^2 \end{aligned} \tag{6.13}$$

for all $q \in F(S) \cap F(T)$ and $n \in \mathbb{N}$. Using (2) in Lemma 2.2 and the hypotheses that S and T are quasi-nonexpansive, we have the following:

$$\begin{aligned} & \|z_n - q\|^2 \\ = \; & \left\|\lambda_n (x_n - q) + \mu_n (Sx_n - q) + \nu_n (Tx_n - q) + \xi_n \left(T^2 x_n - q\right)\right\|^2 \\ = \; & \lambda_n \|x_n - q\|^2 + \mu_n \|Sx_n - q\|^2 + \nu_n \|Tx_n - q\|^2 + \xi_n \left\|T^2 x_n - q\right\|^2 \\ & - \lambda_n \mu_n \|x_n - Sx_n\|^2 - \lambda_n \nu_n \|x_n - Tx_n\|^2 - \lambda_n \xi_n \left\|x_n - T^2 x_n\right\|^2 \\ & - \mu_n \nu_n \|Sx_n - Tx_n\|^2 - \mu_n \xi_n \left\|Sx_n - T^2 x_n\right\|^2 - \nu_n \xi_n \left\|Tx_n - T^2 x_n\right\|^2 \end{aligned} \tag{6.14}$$

$$
\begin{aligned}
&\leq \lambda_n \|x_n - q\|^2 + \mu_n \|x_n - q\|^2 + \nu_n \|x_n - q\|^2 + \xi_n \|x_n - q\|^2 \\
&\quad -\lambda_n \mu_n \|x_n - Sx_n\|^2 - \lambda_n \nu_n \|x_n - Tx_n\|^2 - \lambda_n \xi_n \left\|x_n - T^2 x_n\right\|^2 \\
&= \|x_n - q\|^2 \\
&\quad -\lambda_n \mu_n \|x_n - Sx_n\|^2 - \lambda_n \nu_n \|x_n - Tx_n\|^2 - \lambda_n \xi_n \left\|x_n - T^2 x_n\right\|^2 .
\end{aligned}
$$

Thus, (6.13) holds. In view of (6.6), we have

$$
\begin{aligned}
&\|y_n - q\|^2 \\
&= \left\| a_n (x_n - q) + b_n \left(S^L z_n - q\right) + c_n \left(T^M z_n - q\right) + d_n \left(\frac{1}{n} \sum_{l=0}^{n-1} T^l z_n - q \right) \right\|^2 \\
&\leq a_n \|x_n - q\|^2 + b_n \left\|S^L z_n - q\right\|^2 + c_n \left\|T^M z_n - q\right\|^2 + d_n \left\| \frac{1}{n} \sum_{l=0}^{n-1} T^l z_n - q \right\|^2 \\
&\leq a_n \|x_n - q\|^2 + b_n \|z_n - q\|^2 + c_n \|z_n - q\|^2 + d_n \|z_n - q\|^2 \\
&= a_n \|x_n - q\|^2 + (b_n + c_n + d_n) \|z_n - q\|^2 .
\end{aligned}
$$

Using (6.13) yields

$$
\begin{aligned}
&\|y_n - q\|^2 \\
&\leq a_n \|x_n - q\|^2 + (b_n + c_n + d_n) (\|x_n - q\|^2 - \lambda_n \mu_n \|x_n - Sx_n\|^2 \\
&\quad -\lambda_n \nu_n \|x_n - Tx_n\|^2 - \lambda_n \xi_n \left\|x_n - T^2 x_n\right\|^2) \\
&= \|x_n - q\|^2 - (b_n + c_n + d_n) (\lambda_n \mu_n \|x_n - Sx_n\|^2 + \lambda_n \nu_n \|x_n - Tx_n\|^2 \\
&\quad -\lambda_n \xi_n \left\|x_n - T^2 x_n\right\|^2).
\end{aligned}
$$

Therefore, we have

$$
\begin{aligned}
&(b_n + c_n + d_n) (\lambda_n \mu_n \|x_n - Sx_n\|^2 + \lambda_n \nu_n \|x_n - Tx_n\|^2 \\
&\quad +\lambda_n \xi_n \left\|x_n - T^2 x_n\right\|^2) \\
&\leq \|x_n - q\|^2 - \|y_n - q\|^2 \\
&= (\|x_n - q\| + \|y_n - q\|) (\|x_n - q\| - \|y_n - q\|) \\
&\leq (\|x_n - q\| + \|y_n - q\|) \|x_n - y_n\| .
\end{aligned}
$$

Because $a \leq b_n + c_n + d_n$ is assumed, we have

$$
\begin{aligned}
&a(\lambda_n \mu_n \|x_n - Sx_n\|^2 + \lambda_n \nu_n \|x_n - Tx_n\|^2 + \lambda_n \xi_n \left\|x_n - T^2 x_n\right\|^2) \\
&\leq (\|x_n - q\| + \|y_n - q\|) \|x_n - y_n\| .
\end{aligned}
$$

Note that $a > 0$, $\{x_n\}$ and $\{y_n\}$ are bounded, and $x_n - y_n \to 0$. In addition, as the parameters $\lambda_n, \mu_n, \nu_n, \xi_n$ satisfy (6.3), we obtain (6.12) as claimed. Since the mappings S and T satisfy the properties (6.1) and (6.2), respectively, it follows from (6.10) and (6.12) that $\overline{u} \in F(S) \cap F(T)$.

As a final step, we prove that

$$
\overline{u} \left(= \lim_{n \to \infty} \overline{u}_n = \lim_{n \to \infty} x_n \right) = \widehat{u} \left(= P_{F(S) \cap F(T)} u\right) .
$$

As $\overline{u} \in F(S) \cap F(T)$ and $\widehat{u} = P_{F(S)\cap F(T)}u$, it is sufficient to prove that $\|u - \overline{u}\| \leq \|u - \widehat{u}\|$. As $\widehat{u} \in F(S) \cap F(T)$, it holds from (6.8) that

$$\|u - \overline{u}_n\| \leq \|u - \widehat{u}\|$$

for all $n \in \mathbb{N}$. From (6.9), we obtain $\|u - \overline{u}\| \leq \|u - \widehat{u}\|$, and thus, $\overline{u} = \widehat{u}$. From (6.10), $x_n \to \overline{u} = \widehat{u}$. This completes the proof. □

Remark 6.1. *As* Remarks 4.1 *and* 5.1, y_n *in* (6.4) *can be replaced by various types of sequences such as* (5.13).

From Theorem 6.1, the following result is obtained:

Corollary 6.1 ([23])**.** *Let C be a nonempty, closed, and convex subset of H. Let $S, T : C \to C$ be quasi-nonexpansive mappings with $F(S) \cap F(T) \neq \emptyset$. Suppose that S and T satisfy the conditions* (6.1) *and* (6.2)*, respectively. Let $a, b \in \mathbb{R}$ with $0 < a < b < 1$ and let $\{\lambda_n\}$, $\{\mu_n\}$, $\{\nu_n\}$, and $\{\xi_n\}$ be sequences of real numbers in the interval $[a, b]$ such that $\lambda_n + \mu_n + \nu_n + \xi_n = 1$ for all $n \in \mathbb{N}$. Let $\{u_n\}$ be a sequence in H such that $u_n \to u\,(\in H)$. Define a sequence $\{x_n\}$ in C as follows:*

$$\begin{aligned} x_1 &= x \in C: \ \textit{given}, \\ C_1 &= C, \\ y_n &= \lambda_n x_n + \mu_n S x_n + \nu_n T x_n + \xi_n T^2 x_n, \\ C_{n+1} &= \{z \in C_n : \|y_n - z\| \leq \|x_n - z\|\}, \ \textit{and} \\ x_{n+1} &= P_{C_{n+1}} u_{n+1} \end{aligned}$$

for all $n \in \mathbb{N}$. Then, $\{x_n\}$ converges strongly to a point $\widehat{u}$ of $F(S) \cap F(T)$, where $\widehat{u} = P_{F(S)\cap F(T)}u$.

Proof. Letting $a_n = c_n = d_n = 0$ and $L = 0$ in Theorem 6.1, we obtain the desired result. □

As in the previous two sections, the following two theorems can be proved by using (1) and (3) in Lemma 2.2, respectively. For details, inspect (6.14) in the proof of Theorem 6.1.

Theorem 6.2. *Let C be a nonempty, closed, and convex subset of H. Let $S, T : C \to C$ be quasi-nonexpansive mappings with $F(S) \cap F(T) \neq \emptyset$. Suppose that S and T satisfy the condition* (6.1)*. Let $\{\lambda_n\}$, $\{\mu_n\}$, and $\{\nu_n\}$ be sequences of real numbers in the interval $[0, 1]$ such that*

$$\begin{aligned} &\lambda_n + \mu_n + \nu_n = 1 \ \ \textit{for all } n \in \mathbb{N}, \\ &\varliminf_{n\to\infty} \lambda_n \mu_n > 0, \ \ \textit{and} \ \ \varliminf_{n\to\infty} \lambda_n \nu_n > 0. \end{aligned}$$

Let $a \in (0, 1]$ and let $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, and $\{d_n\}$ be sequences of real numbers in the interval $[0, 1]$ such that $a_n + b_n + c_n + d_n = 1$ and $a \leq b_n + c_n + d_n$

for all $n \in \mathbb{N}$*. Let* $\{u_n\}$ *be a sequence in* H *such that* $u_n \to u\,(\in H)$*. Define a sequence* $\{x_n\}$ *in* C *as follows:*

$$\begin{aligned} x_1 &= x \in C: \ \textit{given} \\ C_1 &= C, \\ z_n &= \lambda_n x_n + \mu_n S x_n + \nu_n T x_n, \\ y_n &= a_n x_n + b_n S^L z_n + c_n T^M z_n + d_n \frac{1}{n} \sum_{l=0}^{n-1} T^l z_n, \\ C_{n+1} &= \{w \in C_n : \|y_n - w\| \le \|x_n - w\|\}, \ \textit{and} \\ x_{n+1} &= P_{C_{n+1}} u_{n+1} \end{aligned}$$

for all $n \in \mathbb{N}$*, where* $L, M \in \mathbb{N} \cup \{0\}$*. Then,* $\{x_n\}$ *converges strongly to a point* $\widehat{u}$ *of* $F(S) \cap F(T)$*, where* $\widehat{u} = P_{F(S)\cap F(T)} u$*.*

Theorem 6.3. *Let* C *be a nonempty, closed, and convex subset of* H*. Let* $S, T : C \to C$ *be quasi-nonexpansive mappings with* $F(S) \cap F(T) \neq \emptyset$*. Suppose that* S *and* T *satisfy the condition* (6.2)*. Let* $\{\lambda_n\}$*,* $\{\mu_n\}$*,* $\{\nu_n\}$*,* $\{\xi_n\}$*, and* $\{\theta_n\}$ *be sequences of real numbers in the interval* $[0,1]$ *such that*

$$\begin{gathered} \lambda_n + \mu_n + \nu_n + \xi_n + \theta_n = 1 \ \ \textit{for all } n \in \mathbb{N}, \\ \varliminf_{n\to\infty} \lambda_n \mu_n > 0, \qquad \varliminf_{n\to\infty} \lambda_n \nu_n > 0, \\ \varliminf_{n\to\infty} \lambda_n \xi_n > 0, \qquad \varliminf_{n\to\infty} \lambda_n \theta_n > 0. \end{gathered}$$

Let $a \in (0,1]$ *and let* $\{a_n\}$*,* $\{b_n\}$*,* $\{c_n\}$*, and* $\{d_n\}$ *be sequences of real numbers in the interval* $[0,1]$ *such that* $a_n + b_n + c_n + d_n = 1$ *and* $a \le b_n + c_n + d_n$ *for all* $n \in \mathbb{N}$*. Let* $\{u_n\}$ *be a sequence in* H *such that* $u_n \to u\,(\in H)$*. Define a sequence* $\{x_n\}$ *in* C *as follows:*

$$\begin{aligned} x_1 &= x \in C: \ \textit{given} \\ C_1 &= C, \\ z_n &= \lambda_n x_n + \mu_n S x_n + \nu_n S^2 x_n + \xi_n T x_n + \theta_n T^2 x_n, \\ y_n &= a_n x_n + b_n S^L z_n + c_n T^M z_n + d_n \frac{1}{n} \sum_{l=0}^{n-1} T^l z_n, \\ C_{n+1} &= \{w \in C_n : \|y_n - w\| \le \|x_n - w\|\}, \ \textit{and} \\ x_{n+1} &= P_{C_{n+1}} u_{n+1} \end{aligned}$$

for all $n \in \mathbb{N}$*, where* $L, M \in \mathbb{N} \cup \{0\}$*. Then,* $\{x_n\}$ *converges strongly to a point* $\widehat{u}$ *of* $F(S) \cap F(T)$*, where* $\widehat{u} = P_{F(S)\cap F(T)} u$*.*

7. Concluding Remarks

This paper establishes weak and strong convergence theorems for finding common fixed points of two quasi-nonexpansive mappings using Ishikawa iteration. For the main theorems, the mappings are required to be demiclosed or, more generally, 2-demiclosed. Some remarks are given below. Firstly, all theorems in this paper apply to generalized hybrid mappings because generalized hybrid mappings are demiclosed, and to nonexpansive mappings because nonexpansive mappings are generalized hybrid. Secondly, employing the shrinking projection method allows one to relax the conditions imposed on the mappings further to (6.1) or (6.2), as shown in Section 6. Thirdly, in all the presented theorems, the mappings are not required to be continuous or commutative. Examples of demiclosed or 2-demiclosed mappings that are not continuous are given in Section 3. Fourthly, as was pointed out in remarks, the construction of the convergent sequences can be generalized beyond the statements of the main theorems. Fifthly, all results in this paper can be extended to cases of finitely many demiclosed or 2-demiclosed mappings. Finally, weak and strong convergence theorems can also be obtained for m-demiclosed mappings.

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References

[1] S. Alizadeh and F. Moradlou, *A weak convergence theorem for 2-generalized hybrid mappings*, ROMAI Journal **11**.1 (2015).

[2] S. Alizadeh and F. Moradlou, *Weak and strong convergence theorems for m-generalized hybrid mappings in Hilbert spaces*, Topol. Methods Nonlinear Anal. **46**.1 (2015), 315-328.

[3] S. Atsushiba, *Nonlinear ergodic theorems for normally 2-generalized hybrid sequences*, J. Nonlinear Convex Anal. **20** (2019), 251-264.

[4] S. Atsushiba, *Fixed point property for normally 2-generalized hybrid mappings*, J. Nonlinear Var. Anal. **3**.2 (2019), 149-158.

[5] S. Atsushiba and W. Takahashi, *Approximating common fixed points of two nonexpansive mappings in Banach spaces*, Bull. Austral. Math. Soc. **57** (1998), 117-127.

[6] V. Berinde, *On the convergence of the Ishikawa iteration in the calss of quasi-contractive operators*, Acta Math. Univ. Comenianae **73**.1 (2004), 119-126.

[7] V. Berinde, *Iterative Approximation of Fixed Points*, second ed., in: Lecture Notes in Mathematics, vol. 1912, Springer, Berlin, (2007).

[8] P. Cholamjiak, and W. Cholamjiak, *Fixed point theorems for hybrid multivalued mappings in Hilbert spaces*, J. Fixed Point Theory Appl. **18**, 673–688 (2016).

[9] W. Cholamjiak, S. Suantai, and Y. J. Cho, *Fixed points for nonspreading-type multivalued mappings: existence and convergence results*, Ann. Acad. Rom. Sci. Ser. Math. Apll **10**.2 (2018), 838-844.

[10] M. Hojo, *Attractive point and mean convergence theorems for normally generalized hybrid mappings in Hilbert spaces*, J. Nonlinear Convex Anal. **18** (2017), 2209–2120.

[11] M. Hojo, A. Kondo and W. Takahashi, *Weak and strong convergence theorems for commutative normally 2-generalized hybrid mappings in Hilbert spaces*, Linear Nonlinear Anal. **4** (2018), 117-134.

[12] M. Hojo, S. Takahashi and W. Takahashi, *Attractive point and ergodic theorems for two nonlinear mappings in Hilbert spaces*, Linear Nonlinear Anal. **3** (2017), 275–286.

[13] M. Hojo, W. Takahashi and I. Termwuttipong, *Strong convergence theorems for 2-generalized hybrid mappings in Hilbert spaces*, Nonlinear Anal. **75** (2012), 2166-2176.

[14] S. Iemoto and W. Takahashi, *Approximating common fixed points of nonexpansive mappings and nonspreading mappings in a Hilbert space*, Nonlinear Anal. **71** (2009), 2082-2089.
[15] T. Igarashi, W. Takahashi and K. Tanaka, *Weak convergence theorems for nonspreading mappings and equilibrium problems*, in Nonlinear Analysis and Optimization (S. Akashi, W. Takahashi and T. Tanaka Eds.), Yokohama Publishers, Yokohama, 2008, 75–85.
[16] S. Ishikawa, *Fixed points by a new iteration method*, Proc. Amer. Math. Soc. **44** (1974), 147-150.
[17] S. Itoh and W. Takahashi, *The common fixed point theory of singlevalued mappings and multivalued mappings*, Pacific J. Math. **79** (1978), 493–508.
[18] Y. Kimura, W. Takahashi and M. Toyoda, *Convergence to common fixed points of a finite family of nonexpansive mappings*, Arch. Math. **84** (2005): 350-363.
[19] P. Kocourek, W. Takahashi, J. C. Yao, *Fixed point theorems and weak convergence theorems for generalized hybrid mappings in Hilbert Spaces*, Taiwanese J. Math. **14** (2010), 2497-2511.
[20] F. Kohsaka, *Existence and approximation of common fixed points of two hybrid mappings in Hilbert spaces*, J. Nonlinear Convex Anal. **16** (2015), 2193-2205.
[21] F. Kohsaka and W. Takahashi, *Existence and approximation of fixed points of firmly nonexpansive-type mappings in Banach spaces*, SIAM J. Optim. **19** (2008), 824–835.
[22] F. Kohsaka and W. Takahashi, *Fixed point theorems for a class of nonlinear mappings related to maximal monotone operators in Banach spaces*, Arch. Math. **91** (2008), 166–177.
[23] A. Kondo, *Strong approximation using hybrid methods to find common fixed points of noncommutative nonlinear mappings in Hilbert spaces*, J. Nonlinear Convex Anal. **23** (2022), 33-58.
[24] A. Kondo, *Mean convergence theorems using hybrid methods to find common fixed points of noncommutative nonlinear mappings in Hilbert spaces*, J. Appl. Math. Comput. **68**, (2022), 217–248.
[25] A. Kondo, *Ishikawa's type mean convergence theorems for finding common fixed points of nonlinear mappings in Hilbert spaces*, submitted.
[26] A. Kondo and W. Takahashi, *Attractive point and weak convergence theorems for normally N-generalized hybrid mappings in Hilbert spaces*, Linear Nonlinear Anal. **3** (2017), 297-310.
[27] A. Kondo and W. Takahashi, *Strong convergence theorems of Halpern's type for normally 2-generalized hybrid mappings in Hilbert spaces*, J. Nonlinear Convex Anal. **19** (2018), 617–631.
[28] A. Kondo and W. Takahashi, *Approximation of a common attractive point of noncommutative normally 2-generalized hybrid mappings in Hilbert spaces*, Linear Nonlinear Anal. **5** (2019), 279–297.
[29] A. Kondo and W. Takahashi, *Weak convergence theorems to common attractive points of normally 2-generalized hybrid mappings with errors*, J. Nonlinear Convex Anal. **21** (2020), 2549-2570.
[30] A. Kondo and W. Takahashi, *Strong convergence theorems for finding common attractive points of normally 2-generalized hybrid mappings and applications*, Linear Nonlinear Anal. **6** (2020), 421-438.
[31] L. Ma, X. Gao, and H. Zhou, *Three kinds of hybrid algorithms and their numerical realizations for finite family of quasi-nonexpansive mappings*, J. Nonlinear Funct. Anal. **2019** (2019), Article ID 8.
[32] W. R. Mann, *Mean value methods in iteration*, Proc. Amer. Math. Soc. **4** (1953), 506-510.
[33] B. Panyanak, *Mann and Ishikawa iterative processes for multivalued mappings in Banach spaces*, Comput. Math. with Appl. **54**.6 (2007), 872-877.

[34] T. Maruyama, W. Takahashi and M. Yao, *Fixed point and mean ergodic theorems for new nonlinear mappings in Hilbert spaces*, J. Nonlinear Convex Anal. **12** (2011), 185-197.
[35] K. Nakajo and W. Takahashi, *Strong convergence theorems for nonexpansive mappings and nonexpansive semigroups*, J. Math. Anal. Appl. **279** (2003), 372-378.
[36] B. D. Rouhani, *Ergodic theorems for hybrid sequences in a Hilbert space with applications*, J. Math. Anal. Appl. **409**.1 (2014), 205-211.
[37] B. D. Rouhani, *Ergodic and fixed point theorems for sequences and nonlinear mappings in a Hilbert space*, Demonstr. Math. **51**.1 (2018), 27-36.
[38] P. Sadeewong, T. Saleewong, P. Kumam, Y. J. Cho, *The Modified Viscosity Iteration with m-Generalized Hybrid Mappings and (a, b)-Monotone Mappings for Equilibrium Problems*, Thai J. Math. **16**.1 (2018), 243-265.
[39] T. Shimizu and W. Takahashi, *Strong convergence to common fixed points of families of nonexpansive mappings*, J. Math. Anal. Appl. **211** (1997), 71-83.
[40] K. Shimoji and W. Takahashi, *Strong convergence to common fixed points of infinite nonexpansive mappings and applications*, Taiwanese J. Math. **5** (2001), 387-404.
[41] W. Takahashi, *Nonlinear Functional Analysis. Fixed Point Theory and its Applications*, Yokohama Publishes, (2000).
[42] W. Takahashi, *Introduction to Nonlinear and Convex Analysis*, Yokohama Publishes, (2009).
[43] W. Takahashi, *Fixed point theorems for new nonlinear mappings in a Hilbert space*, J. Nonlinear Convex Anal. **11** (2010), 79-88.
[44] W. Takahashi, *Weak and strong convergence theorems for noncommutative two generalized hybrid mappings in Hilbert spaces*, J. Nonlinear Convex Anal., **19** (2018), 867-880.
[45] W. Takahashi, Y. Takeuchi and R. Kubota, *Strong convergence theorems by hybrid methods for families of nonexpansive mappings in Hilbert spaces*, J. Math. Anal. Appl. **341** (2008), 276–286.
[46] W. Takahashi and M. Toyoda, *Weak convergence theorems for nonexpansive mappings and monotone mappings*, J. Optim. Theory Appl., **118** (2003), 417–428.
[47] W. Takahashi, C. F. Wen and J. C. Yao, *Strong convergence theorems by hybrid methods for noncommutative normally 2-generalized hybrid mappings in Hilbert spaces*, Applied Anal. Optim., **3** (2019), 43-56.
[48] W. Takahashi, N. C. Wong, J. C. Yao, *Attractive point and weak convergence theorems for new generalized hybrid mappings in Hilbert spaces*, J. Nonlinear Convex Anal., **13** (2012), 745-757.
[49] W. Takahashi and J. C. Yao, *Fixed Point Theorems and Ergodic Theorems for Nonlinear Mappings in Hilbert Spaces*, Taiwanese J. Math., **15** (2011), 457-472.
[50] K. K. Tan and H. K. Xu, *Approximating fixed points of non-expansive mappings by the Ishikawa iteration process*, J. Math. Anal. Appl. **178** (1993), 301-301.
[51] K. Wongyai and T. Thianwan, *Projection Type Ishikawa Iteration with Perturbations for Common Fixed Points of Two Nonself Generalized Asymptotically Quasi-Nonexpansive Mappings*, Thai J. Math. **17**.3 (2019), 843-859.
[52] F. Yan, Y. Su, and Q. Feng, *Convergence of Ishikawa'method for generalized hybrid mappings*, Commun. Korean Math. Soc. **28**.1 (2013), 135-141.
[53] H. Zegeye and N. Shahzad, *Convergence of Mann's type iteration method for generalized asymptotically nonexpansive mappings*, Comput. Math. with Appl. **62**.11 (2011), 4007-4014.

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