FIXED POINT THEOREM FOR GENERIC 2-GENERALIZD HYBRID MAPPINGS IN HILBERT SPACES

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ABSTRACT. We establish a fixed point theorem for a class of mappings called generic 2-generalized hybrid mappings in the setting of a real Hilbert space. Two examples of that class of mappings are presented herein. The mappings are not quasi-nonexpansive even though they have fixed points. One of the mappings is not continuous. The fixed point theorem proved in this article improves many previous works in the literature.

1. Introduction

Let E be a Banach space with a norm $\|\cdot\|$. For a mapping $T : C \to E$, the set of fixed points is denoted as

$$F(T) = \{x \in C : Tx = x\},$$

where C is a nonempty subset of E. The Schauder fixed point theorem [21] asserts that any continuous mapping defined on a compact and convex set has a fixed point. A mapping $T : C \to E$ is called *nonexpansive* if

$$\|Tx - Ty\| \leq \|x - y\| \quad \text{for all } x, y \in C.$$

Obviously, a nonexpansive mapping is continuous. Under the setting of a reflexive Banach space, Kirk [12] proved the existence of fixed points for nonexpansive mappings by supposing that C is weakly compact and has "normal structure"; for related results, see also Browder [3] and Göhde [4]. It is known that a nonempty, closed, and convex subset of a Hilbert space has the normal structure (see Problem 4.4.2 in Takahashi [22]). Therefore, the following theorem is derived from the Browder-Göhde-Kirk's fixed point theorems:

Theorem 1.1. *Let C be a nonempty, closed, convex, and bounded subset of a real Hilbert space H. Let T be a nonexpansive mapping from C into itself. Then, $F(T)$ is nonempty.*

In the following part of this article, we use H to denote a real Hilbert space with an inner product $\langle \cdot, \cdot \rangle$ and the induced norm $\|\cdot\|$. Theorem 1.1 is extended in various directions. The conditions required for a mapping are relaxed to uniformly include many important types of mappings. Kocourek *et al.*, in their work in 2010 [13], proposed a broad class of mappings: A mapping $T : C \to H$ is called *generalized hybrid* if there exist $\alpha, \beta \in \mathbb{R}$ such that

$$\alpha \|Tx - Ty\|^2 + (1 - \alpha) \|x - Ty\|^2 \leq \beta \|Tx - y\|^2 + (1 - \beta) \|x - y\|^2$$

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for all $x, y \in C$, where $\mathbb{R}$ is the set of real numbers. This is also known as (α, β)-*generalized hybrid mappings*. A $(1,0)$-generalized hybrid mapping is nonexpansive, and a nonexpansive mapping is a special case of the generalized hybrid mappings. Similarly, a $(2,1)$-generalized hybrid mapping is *nonspreading* [14, 15], and a $\left(\frac{3}{2}, \frac{1}{2}\right)$-generalized hybrid mapping is *hybrid* [23]. Nonspreading mappings are deduced from optimization problems. Igarashi *et al.* [6] illustrated that a nonspreading mapping is not necessarily continuous. Hence, a generalized hybrid mapping is not necessarily continuous. Kocourek *et al.* [13] proved a fixed point theorem and weak convergence theorems for finding fixed points of generalized hybrid mappings.

A mapping $T : C \to C$ is called 2-*generalized hybrid* [18] if there exist $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{R}$ such that

$$\begin{aligned}&\alpha_1 \left\|T^2x - Ty\right\|^2 + \alpha_2 \left\|Tx - Ty\right\|^2 + (1 - \alpha_1 - \alpha_2) \left\|x - Ty\right\|^2 \\ &\leq \beta_1 \left\|T^2x - y\right\|^2 + \beta_2 \left\|Tx - y\right\|^2 + (1 - \beta_1 - \beta_2) \left\|x - y\right\|^2\end{aligned}$$

for all $x, y \in C$. The class of 2-generalized hybrid mappings contains generalized hybrid mappings as a special case of $\alpha_1 = \beta_1 = 0$. Hojo *et al.* [5] provided examples of 2-generalized hybrid mappings that are not generalized hybrid. Maruyama *et al.* [18] proved a fixed point theorem for this type of mappings in Hilbert spaces; see also Alizadeh and Moradlou [1, 2] and Rouhani [19]. Many researchers have studied approximation methods for finding fixed points of 2-generalized hybrid mappings; see, for instance, [1, 2, 5, 18, 19, 20]. A mapping $T : C \to H$ that has a fixed point is *quasi-nonexpansive* if

$$\|Tx - u\| \leq \|x - u\| \quad \text{for all } x \in C \text{ and } u \in F(T).$$

A 2-generalized hybrid mapping with a fixed point is quasi-nonexpansive, which was demonstrated by Maruyama *et al.* [18]. The previous studies cited here utilized the fact that the mappings are quasi-nonexpansive to prove convergence theorems for finding fixed points.

Recently, Kondo and Takahashi [16, 17] introduced a more general class of mappings than the 2-generalized hybrid mappings. A mapping $T : C \to C$ is called *generic* 2-*generalized hybrid* if there exist $\alpha_{ij}, \beta_i, \gamma_i \in \mathbb{R}$ $(i, j = 0, 1, 2)$ such that

$$\begin{aligned}&\alpha_{00}\|x - y\|^2 + \alpha_{01}\|x - Ty\|^2 + \alpha_{02}\|x - T^2y\|^2 \\ &+\alpha_{10} \left\|Tx - y\right\|^2 + \alpha_{11} \left\|Tx - Ty\right\|^2 + \alpha_{12} \left\|Tx - T^2y\right\|^2 \\ &+\alpha_{20} \left\|T^2x - y\right\|^2 + \alpha_{21} \left\|T^2x - Ty\right\|^2 + \alpha_{22} \left\|T^2x - T^2y\right\|^2 \\ &+\beta_0 \left\|x - Tx\right\|^2 + \beta_1 \left\|Tx - T^2x\right\|^2 + \beta_2 \left\|T^2x - x\right\|^2 \\ &+\gamma_0 \left\|y - Ty\right\|^2 + \gamma_1 \left\|Ty - T^2y\right\|^2 + \gamma_2 \left\|T^2y - y\right\|^2 \leq 0\end{aligned} \tag{1.1}$$

for all $x, y \in C$. They assumed certain parameter conditions on the mapping. Under the conditions, the mapping becomes quasi-nonexpansive if it has a fixed point. Kondo and Takahashi proved various types of convergence theorems that approximate fixed points by using the fact that the mapping is quasi-nonexpansive. In addition to the convergence theorems, they also proved the existence of fixed points for generic 2-generalized hybrid mappings. However, the conditions imposed on the parameters $\alpha_{ij}, \beta_i, \gamma_i\ (\in \mathbb{R})$ were still restrictive. For fixed point theorems for another class of mappings in Hilbert spaces, see Takahashi *et al.* [24], Kawasaki and Takahashi [11], and Kawasaki and Kobayashi [10], whereas for recent contributions to fixed point theorems in settings of Banach spaces, see Kawasaki [7, 8, 9] and articles cited therein.

In this article, we establish a fixed point theorem for generic 2-generalized hybrid mappings under relaxed conditions on the parameters $\alpha_{ij}, \beta_i, \gamma_i\,(\in \mathbb{R})$ than in the previous works by Kondo and Takahashi [16, 17]. Our theorem includes cases such as

$$\lambda \|Tx - Ty\|^2 + (1-\lambda)\left\|T^2x - T^2y\right\|^2 \le \|x-y\|^2\,; \tag{1.2}$$

$$\mu \|Tx - y\|^2 + (1-\mu)\left\|T^2x - y\right\|^2 \le \|x-y\|^2\,; \tag{1.3}$$

$$\nu \|Tx - Ty\|^2 + (1-\nu)\left\|T^2x - Ty\right\|^2 \le \|x-Ty\|^2\,; \tag{1.4}$$

$$\xi \left\|Tx - T^2y\right\|^2 + (1-\xi)\left\|T^2x - T^2y\right\|^2 \le \left\|x-T^2y\right\|^2\,; \tag{1.5}$$

for all $x, y \in C$, where $\lambda, \mu, \nu, \xi \in (0, 1]$. These types of mappings are generic 2-generalized hybrid. For example, if $\alpha_{00} = -1$, $\alpha_{11} = \lambda$, $\alpha_{22} = 1 - \lambda$, and the other parameters are all 0 in (1.1), the condition (1.2) is obtained. The parameter combinations in (1.2)–(1.5) were not addressed in the previous works [16, 17]. Moreover, a mapping with any of the conditions (1.2)–(1.5) is not necessarily quasi-nonexpansive even if it has a fixed point. In Section 2, the main theorem is proved. The theorem extends many existing results, and it simultaneously guarantees the existence of fixed points in mappings characterized by the conditions (1.2)–(1.5). Some remarks concerning the main theorem are given in Section 3. In Section 4, examples for which we address in this article, but are not in previous studies are presented. One of these examples is not continuous, although it satisfies (1.5).

2. Main Result

This section presents the main theorem of this article. Although the previous work by Kondo and Takahashi [16] addressed the fixed point problem for $(\alpha_{ij}, \beta_i, \gamma_i;\ i, j = 0, 1, 2)$-generic 2-generalized hybrid mappings, we relax conditions on the parameters $(\alpha_{ij}, \beta_i, \gamma_i;\ i, j = 0, 1, 2)$ to prove a fixed point theorem. First, we prepare the following lemma. Although part (1) of Lemma 2.1 was proved by Kondo and Takahashi [16], we reproduce the proof for completeness. As part (2) is necessary to further extend Theorem 2.1, we prepare it herein.

Lemma 2.1. (1) *Let $\alpha, \beta \in \mathbb{R}$ such that $\alpha + \beta \ge 0$, and let $\{a_n\}$ and $\{b_n\}$ be sequences of nonnegative real numbers such that $a_n - b_n \to 0$. Then, it holds that* $\liminf_{n\to\infty} (\alpha a_n + \beta b_n) \ge 0$.

(2) *Let $\alpha, \beta, \gamma \in \mathbb{R}$ such that $\alpha + \beta + \gamma \ge 0$, and let $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ be sequences of nonnegative real numbers such that $a_n - b_n \to 0$, $b_n - c_n \to 0$, and $c_n - a_n \to 0$. Then, it holds that* $\liminf_{n\to\infty} (\alpha a_n + \beta b_n + \gamma c_n) \ge 0$.

Proof. (1) If $\alpha = \beta = 0$, the desired result follows. Assume, without loss of generality, that $\alpha > 0$. We prove that

$$\forall \varepsilon > 0,\ \exists n_0 \in \mathbb{N} \text{ such that } n \ge n_0 \Longrightarrow \alpha a_n + \beta b_n > -\varepsilon,$$

where $\mathbb{N}$ is the set of positive integers. Let $\varepsilon > 0$. As $a_n - b_n \to 0$, we have that for a positive real number $\varepsilon/\alpha > 0$,

$$\exists n_0 \in \mathbb{N} \text{ such that } n \ge n_0 \Longrightarrow b_n - \frac{\varepsilon}{\alpha} < a_n \left(< b_n + \frac{\varepsilon}{\alpha}\right). \tag{2.1}$$

Let $n \geq n_0$. Using $\alpha > 0$, $\alpha + \beta \geq 0$, and $b_n \geq 0$, we obtain from (2.1) that

$$\begin{aligned}\alpha a_n + \beta b_n &> \alpha \left(b_n - \frac{\varepsilon}{\alpha} \right) + \beta b_n \\ &= (\alpha + \beta) b_n - \varepsilon \geq -\varepsilon.\end{aligned}$$

This ends the proof for (1).

(2) If $\alpha + \beta = \gamma = 0$, then the desired result follows from (1). The rest of the proof is divided into two cases: (i) $\alpha + \beta > 0$, and (ii) $\gamma > 0$. Our aim is to show that

$$\forall \varepsilon > 0,\ \exists n_0 \in \mathbb{N} \text{ such that } n \geq n_0 \Longrightarrow \alpha a_n + \beta b_n + \gamma c_n > -\varepsilon. \tag{2.2}$$

(i) First, assume that $\alpha + \beta > 0$, and assume, without loss of generality, that $\alpha > 0$. As $a_n - b_n \to 0$ and $\alpha > 0$, it holds that

$$\exists n_1 \in \mathbb{N} \text{ such that } n \geq n_1 \Longrightarrow b_n - \frac{\varepsilon}{2\alpha} < a_n \left(< b_n + \frac{\varepsilon}{2\alpha} \right). \tag{2.3}$$

As $b_n - c_n \to 0$ and $\alpha + \beta > 0$, we have that

$$\exists n_2 \in \mathbb{N} \text{ such that } n \geq n_2 \Longrightarrow c_n - \frac{\varepsilon}{2(\alpha + \beta)} < b_n \left(< c_n + \frac{\varepsilon}{2(\alpha + \beta)} \right). \tag{2.4}$$

Define $n_0 \equiv \max\{n_1,\ n_2\} \in \mathbb{N}$, and let $n \geq n_0$. Then, as $\alpha > 0$, $\alpha + \beta > 0$, $\alpha + \beta + \gamma \geq 0$, and $c_n \geq 0$, we obtain from (2.3) and (2.4) that

$$\begin{aligned}&\alpha a_n + \beta b_n + \gamma c_n \\ &> \alpha \left(b_n - \frac{\varepsilon}{2\alpha} \right) + \beta b_n + \gamma c_n \\ &= (\alpha + \beta) b_n - \frac{\varepsilon}{2} + \gamma c_n \\ &> (\alpha + \beta) \left(c_n - \frac{\varepsilon}{2(\alpha + \beta)} \right) - \frac{\varepsilon}{2} + \gamma c_n \\ &= (\alpha + \beta + \gamma) c_n - \varepsilon \geq -\varepsilon.\end{aligned}$$

Therefore, (2.2) holds.

(ii) Next, assume that $\gamma > 0$. If $\alpha + \beta = 0$, then it holds from (1) that $\liminf_{n\to\infty} (\alpha a_n + \beta b_n) \geq 0$. Using $\gamma > 0$ and $c_n \geq 0$, we have the desired result (2.2) as follows:

$$\begin{aligned}&\liminf_{n\to\infty} (\alpha a_n + \beta b_n + \gamma c_n) \\ &\geq \liminf_{n\to\infty} (\alpha a_n + \beta b_n) + \liminf_{n\to\infty} \gamma c_n \\ &\geq \liminf_{n\to\infty} \gamma c_n \geq 0.\end{aligned}$$

Hence, we can assume that $\alpha + \beta < 0$. It follows from $b_n - c_n \to 0$ and $c_n - a_n \to 0$ that

$$\frac{\alpha}{\alpha + \beta} a_n + \frac{\beta}{\alpha + \beta} b_n - c_n \to 0. \tag{2.5}$$

Let $\varepsilon > 0$, and define $d_n \equiv \frac{\alpha}{\alpha+\beta} a_n + \frac{\beta}{\alpha+\beta} b_n \in \mathbb{R}$. The expression (2.5) means that $c_n - d_n \to 0$. It follows that for a positive real number $\varepsilon/2\gamma > 0$,

$$\exists n_3 \in \mathbb{N} \text{ such that } n \geq n_3 \Longrightarrow d_n - \frac{\varepsilon}{2\gamma} < c_n \left(< d_n + \frac{\varepsilon}{2\gamma} \right). \tag{2.6}$$

Since $\alpha+\beta+\gamma\geq 0$, $c_n-d_n\to 0$, and $c_n\geq 0$, we obtain

$$\exists n_4\in\mathbb{N}\text{ such that } n\geq n_4\Longrightarrow(\alpha+\beta+\gamma)\,d_n>-\frac{\varepsilon}{2}. \tag{2.7}$$

Define $n_0\equiv\max\{n_3,\ n_4\}\in\mathbb{N}$, and let $n\geq n_0$. Then, as $\gamma>0$, we obtain from (2.6) and (2.7) that

$$\begin{aligned}
&\alpha a_n+\beta b_n+\gamma c_n\\
&=(\alpha+\beta)\left(\frac{\alpha}{\alpha+\beta}a_n+\frac{\beta}{\alpha+\beta}b_n\right)+\gamma c_n\\
&=(\alpha+\beta)\,d_n+\gamma c_n\\
&>(\alpha+\beta)\,d_n+\gamma\left(d_n-\frac{\varepsilon}{2\gamma}\right)\\
&=(\alpha+\beta+\gamma)\,d_n-\frac{\varepsilon}{2}>-\varepsilon.
\end{aligned}$$

This completes the proof. □

For an $(\alpha_{ij},\beta_i,\gamma_i;\ i,j=0,1,2)$-generic 2-generalized hybrid mapping, we use the following notations:

$$\begin{aligned}
&\alpha_{i\bullet}\equiv\alpha_{i0}+\alpha_{i1}+\alpha_{i2},\qquad\alpha_{\bullet i}\equiv\alpha_{0i}+\alpha_{1i}+\alpha_{2i},\quad\text{and}\\
&\alpha_{\bullet\bullet}\equiv\sum_{i,j=0,1,2}\alpha_{ij},
\end{aligned}\tag{2.8}$$

where $i=0,1,2$. We denote by $x_n\rightharpoonup x$ a weak convergence of a sequence $\{x_n\}$ in H to a point $x\,(\in H)$. For a mapping $T:C\to C$, define

$$F^{-1}(T)\equiv\left\{x\in C:Tx=T^2x\right\}, \tag{2.9}$$

where C is a nonempty subset of H. It is obvious that $x\in F^{-1}(T)$ is equivalent with $Tx\in F(T)$. The main theorem of this article is as follows:

Theorem 2.1. *Let C be a nonempty, closed, and convex subset of H, and let $T:C\to C$ be an $(\alpha_{ij},\beta_i,\gamma_i;\ i,j=0,1,2)$-generic 2-generalized hybrid mapping. Suppose that there exists an element $z\in C$ such that the sequence $\{T^nz\}$ in C is bounded. Consider the following conditions:*

(1a) $\alpha_{\bullet\bullet}\geq 0$, $\alpha_{1\bullet}+\beta_0>0$, $\beta_1\geq 0$, $\alpha_{2\bullet}+\beta_2\geq 0$, $\gamma_0+\gamma_1\geq 0$, $\gamma_2\geq 0$;
(1b) $\alpha_{\bullet\bullet}\geq 0$, $\alpha_{1\bullet}+\beta_0\geq 0$, $\beta_1\geq 0$, $\alpha_{2\bullet}+\beta_2>0$, $\gamma_0+\gamma_1\geq 0$, $\gamma_2\geq 0$;
(1c) $\alpha_{\bullet\bullet}\geq 0$, $\alpha_{1\bullet}+\beta_0\geq 0$, $\beta_1>0$, $\alpha_{2\bullet}+\beta_2\geq 0$, $\gamma_0+\gamma_1\geq 0$, $\gamma_2\geq 0$;
(2a) $\alpha_{\bullet\bullet}\geq 0$, $\alpha_{\bullet 1}+\gamma_0>0$, $\gamma_1\geq 0$, $\alpha_{\bullet 2}+\gamma_2\geq 0$, $\beta_0+\beta_1\geq 0$, $\beta_2\geq 0$;
(2b) $\alpha_{\bullet\bullet}\geq 0$, $\alpha_{\bullet 1}+\gamma_0\geq 0$, $\gamma_1\geq 0$, $\alpha_{\bullet 2}+\gamma_2>0$, $\beta_0+\beta_1\geq 0$, $\beta_2\geq 0$;
(2c) $\alpha_{\bullet\bullet}\geq 0$, $\alpha_{\bullet 1}+\gamma_0\geq 0$, $\gamma_1>0$, $\alpha_{\bullet 2}+\gamma_2\geq 0$, $\beta_0+\beta_1\geq 0$, $\beta_2\geq 0$.

If (1a) *or* (2a) *is satisfied, then $F(T)$ is nonempty. If* (1b) *or* (2b) *is satisfied, then $F\left(T^2\right)$ is nonempty. If* (1c) *or* (2c) *is satisfied, then $F^{-1}(T)$ is nonempty, where $F^{-1}(T)$ is defined in* (2.9)*. Furthermore, T has at most one fixed point if $\alpha_{\bullet\bullet}>0$.*

Proof. **Case (1abc)**. Suppose that $\alpha_{\bullet\bullet}\geq 0$, $\gamma_0+\gamma_1\geq 0$, and $\gamma_2\geq 0$. Define

$$S_nz\equiv\frac{1}{n}\sum_{k=0}^{n-1}T^kz$$

for each $n \in \mathbb{N}$. Then, $\{S_n z\}$ is a sequence in C since C is convex. As $\{T^n z\}$ is bounded, so is $\{S_n z\}$. There exist a subsequence $\{S_{n_i} z\}$ of $\{S_n z\}$ and $v \in H$ such that $S_{n_i} z \rightharpoonup v$. Note that since C is closed and convex, it is weakly closed. Since $\{S_{n_i} z\}$ is a sequence in C, $S_{n_i} z \rightharpoonup v$, and C is weakly closed, it holds that $v \in C$. Therefore, Tv and $T^2 v \, (\in C)$ exist. As T is $(\alpha_{ij}, \beta_i, \gamma_i; i, j = 0, 1, 2)$-generic 2-generalized hybrid, we have from (1.1) that

$$\begin{aligned}
&\alpha_{00}\|v - T^k z\|^2 + \alpha_{01}\left\|v - T^{k+1} z\right\|^2 + \alpha_{02}\left\|v - T^{k+2} z\right\|^2 \\
&+\alpha_{10}\left\|Tv - T^k z\right\|^2 + \alpha_{11}\left\|Tv - T^{k+1} z\right\|^2 + \alpha_{12}\left\|Tv - T^{k+2} z\right\|^2 \\
&+\alpha_{20}\|T^2 v - T^k z\|^2 + \alpha_{21}\left\|T^2 v - T^{k+1} z\right\|^2 + \alpha_{22}\left\|T^2 v - T^{k+2} z\right\|^2 \\
&+\beta_0\|v - Tv\|^2 + \beta_1\left\|Tv - T^2 v\right\|^2 + \beta_2\left\|T^2 v - v\right\|^2 \\
&+\gamma_0\left\|T^k z - T^{k+1} z\right\|^2 + \gamma_1\left\|T^{k+1} z - T^{k+2} z\right\|^2 + \gamma_2\left\|T^{k+2} z - T^k z\right\|^2 \leq 0
\end{aligned}$$

for all $k \in \mathbb{N} \cup \{0\}$. As $\gamma_2 \geq 0$, subtracting $\gamma_2\left\|T^{k+2} z - T^k z\right\|^2 (\geq 0)$ from the left-hand side, we have that

$$\begin{aligned}
&\alpha_{00}\left\|v - T^k z\right\|^2 + \alpha_{01}\left\|v - T^{k+1} z\right\|^2 + \alpha_{02}\left\|v - T^{k+2} z\right\|^2 \\
&+\alpha_{10}\left(\|Tv - v\|^2 + 2\left\langle Tv - v,\ v - T^k z\right\rangle + \left\|v - T^k z\right\|^2\right) \\
&+\alpha_{11}\left(\|Tv - v\|^2 + 2\left\langle Tv - v,\ v - T^{k+1} z\right\rangle + \left\|v - T^{k+1} z\right\|^2\right) \\
&+\alpha_{12}\left(\|Tv - v\|^2 + 2\left\langle Tv - v,\ v - T^{k+2} z\right\rangle + \left\|v - T^{k+2} z\right\|^2\right) \\
&+\alpha_{20}\left(\left\|T^2 v - v\right\|^2 + 2\left\langle T^2 v - v,\ v - T^k z\right\rangle + \left\|v - T^k z\right\|^2\right) \\
&+\alpha_{21}\left(\left\|T^2 v - v\right\|^2 + 2\left\langle T^2 v - v,\ v - T^{k+1} z\right\rangle + \left\|v - T^{k+1} z\right\|^2\right) \\
&+\alpha_{22}\left(\left\|T^2 v - v\right\|^2 + 2\left\langle T^2 v - v,\ v - T^{k+2} z\right\rangle + \left\|v - T^{k+2} z\right\|^2\right) \\
&+\beta_0\|v - Tv\|^2 + \beta_1\left\|Tv - T^2 v\right\|^2 + \beta_2\left\|T^2 v - v\right\|^2 \\
&+\gamma_0\left\|T^k z - T^{k+1} z\right\|^2 + \gamma_1\left\|T^{k+1} z - T^{k+2} z\right\|^2 \leq 0.
\end{aligned}$$

From this, the following holds:

$$\begin{aligned}
&\alpha_{\bullet 0}\left\|v - T^k z\right\|^2 + \alpha_{\bullet 1}\left\|v - T^{k+1} z\right\|^2 + \alpha_{\bullet 2}\left\|v - T^{k+2} z\right\|^2 \\
&+2\alpha_{10}\left\langle Tv - v,\ v - T^k z\right\rangle + 2\alpha_{11}\left\langle Tv - v,\ v - T^{k+1} z\right\rangle \\
&+2\alpha_{12}\left\langle Tv - v,\ v - T^{k+2} z\right\rangle \\
&+2\alpha_{20}\left\langle T^2 v - v,\ v - T^k z\right\rangle + 2\alpha_{21}\left\langle T^2 v - v,\ v - T^{k+1} z\right\rangle \\
&+2\alpha_{22}\left\langle T^2 v - v,\ v - T^{k+2} z\right\rangle \\
&+(\alpha_{1\bullet} + \beta_0)\|Tv - v\|^2 + \beta_1\left\|Tv - T^2 v\right\|^2 + (\alpha_{2\bullet} + \beta_2)\left\|T^2 v - v\right\|^2 \\
&+\gamma_0\left\|T^k z - T^{k+1} z\right\|^2 + \gamma_1\left\|T^{k+1} z - T^{k+2} z\right\|^2 \leq 0.
\end{aligned}$$

This yields that

$$\begin{aligned}
&\alpha_{\bullet\bullet}\left\|v-T^kz\right\|^2+\alpha_{\bullet 1}\left(\left\|v-T^{k+1}z\right\|^2-\left\|v-T^kz\right\|^2\right)\\
&+\alpha_{\bullet 2}\left(\left\|v-T^{k+2}z\right\|^2-\left\|v-T^kz\right\|^2\right)\\
&+2\left\langle Tv-v,\ \alpha_{1\bullet}v-\alpha_{10}T^kz-\alpha_{11}T^{k+1}z-\alpha_{12}T^{k+2}z\right\rangle\\
&+2\left\langle T^2v-v,\ \alpha_{2\bullet}v-\alpha_{20}T^kz-\alpha_{21}T^{k+1}z-\alpha_{22}T^{k+2}z\right\rangle\\
&+(\alpha_{1\bullet}+\beta_0)\left\|Tv-v\right\|^2+\beta_1\left\|Tv-T^2v\right\|^2+(\alpha_{2\bullet}+\beta_2)\left\|T^2v-v\right\|^2\\
&+\gamma_0\left\|T^kz-T^{k+1}z\right\|^2+\gamma_1\left\|T^{k+1}z-T^{k+2}z\right\|^2\leq 0.
\end{aligned}$$

Since $\alpha_{\bullet\bullet}\geq 0$, subtracting $\alpha_{\bullet\bullet}\left\|v-T^kz\right\|^2$ (≥ 0) from the left-hand side yields

$$\begin{aligned}
&\alpha_{\bullet 1}\left(\left\|v-T^{k+1}z\right\|^2-\left\|v-T^kz\right\|^2\right)+\alpha_{\bullet 2}\left(\left\|v-T^{k+2}z\right\|^2-\left\|v-T^kz\right\|^2\right)\\
&+2\left\langle Tv-v,\ \alpha_{1\bullet}v-\left\{\alpha_{1\bullet}T^kz+\alpha_{11}\left(T^{k+1}z-T^kz\right)+\alpha_{12}\left(T^{k+2}z-T^kz\right)\right\}\right\rangle\\
&+2\left\langle T^2v-v,\ \alpha_{2\bullet}v-\left\{\alpha_{2\bullet}T^kz+\alpha_{21}\left(T^{k+1}z-T^kz\right)+\alpha_{22}\left(T^{k+2}z-T^kz\right)\right\}\right\rangle\\
&+(\alpha_{1\bullet}+\beta_0)\left\|Tv-v\right\|^2+\beta_1\left\|Tv-T^2v\right\|^2+(\alpha_{2\bullet}+\beta_2)\left\|T^2v-v\right\|^2\\
&+\gamma_0\left\|T^kz-T^{k+1}z\right\|^2+\gamma_1\left\|T^{k+1}z-T^{k+2}z\right\|^2\leq 0.
\end{aligned}\tag{2.10}$$

Take $n\in\mathbb{N}$ and fix it momentarily. Summing the inequalities in (2.10) with respect to k from 0 to $n-1$, and dividing by n, we have

$$\begin{aligned}
&\frac{1}{n}\alpha_{\bullet 1}\left(\left\|v-T^nz\right\|^2-\left\|v-z\right\|^2\right)\\
&+\frac{1}{n}\alpha_{\bullet 2}\left(\left\|v-T^{n+1}z\right\|^2+\left\|v-T^nz\right\|^2-\left\|v-Tz\right\|^2-\left\|v-z\right\|^2\right)\\
&+2\langle Tv-v,\ \alpha_{1\bullet}v-\{\alpha_{1\bullet}S_nz+\frac{1}{n}\alpha_{11}\left(T^nz-z\right)\\
&+\frac{1}{n}\alpha_{12}\left(T^{n+1}z+T^nz-Tz-z\right)\}\rangle\\
&+2\langle T^2v-v,\ \alpha_{2\bullet}v-\{\alpha_{2\bullet}S_nz+\frac{1}{n}\alpha_{21}\left(T^nz-z\right)\\
&+\frac{1}{n}\alpha_{22}\left(T^{n+1}z+T^nz-Tz-z\right)\}\rangle\\
&+(\alpha_{1\bullet}+\beta_0)\left\|Tv-v\right\|^2+\beta_1\left\|Tv-T^2v\right\|^2+(\alpha_{2\bullet}+\beta_2)\left\|T^2v-v\right\|^2\\
&+\gamma_0\frac{1}{n}\sum_{k=0}^{n-1}\left\|T^kz-T^{k+1}z\right\|^2+\gamma_1\frac{1}{n}\sum_{k=0}^{n-1}\left\|T^{k+1}z-T^{k+2}z\right\|^2\leq 0.
\end{aligned}\tag{2.11}$$

Since $\{T^nz\}$ is bounded, it holds that

$$\frac{1}{n}\sum_{k=0}^{n-1}\left\|T^kz-T^{k+1}z\right\|^2-\frac{1}{n}\sum_{k=0}^{n-1}\left\|T^{k+1}z-T^{k+2}z\right\|^2\to 0\text{ as }n\to\infty.$$

Since $\gamma_0+\gamma_1\geq 0$, we have from Lemma 2.1-(1) that

$$\liminf_{n\to\infty}\left(\gamma_0\frac{1}{n}\sum_{k=0}^{n-1}\left\|T^kz-T^{k+1}z\right\|^2+\gamma_1\frac{1}{n}\sum_{k=0}^{n-1}\left\|T^{k+1}z-T^{k+2}z\right\|^2\right)\geq 0.$$

Replacing n with n_i, and taking the lim inf as $i \to \infty$ in (2.11), we obtain

$$2\alpha_{1\bullet} \langle Tv - v,\ v - v\rangle + 2\alpha_{2\bullet} \left\langle T^2 v - v,\ v - v\right\rangle + (\alpha_{1\bullet} + \beta_0) \|Tv - v\|^2 + \beta_1 \left\|Tv - T^2 v\right\|^2 + (\alpha_{2\bullet} + \beta_2) \left\|T^2 v - v\right\|^2 \le 0.$$

Therefore,

$$(\alpha_{1\bullet} + \beta_0) \|Tv - v\|^2 + \beta_1 \left\|Tv - T^2 v\right\|^2 + (\alpha_{2\bullet} + \beta_2) \left\|T^2 v - v\right\|^2 \le 0. \tag{2.12}$$

(a) Assume that $\alpha_{1\bullet} + \beta_0 > 0$, $\beta_1 \ge 0$, and $\alpha_{2\bullet} + \beta_2 \ge 0$. Since $\beta_1 \ge 0$ and $\alpha_{2\bullet} + \beta_2 \ge 0$, subtracting

$$\beta_1 \left\|Tv - T^2 v\right\|^2 + (\alpha_{2\bullet} + \beta_2) \left\|T^2 v - v\right\|^2 (\ge 0)$$

from the left-hand side of (2.12) yields

$$(\alpha_{1\bullet} + \beta_0) \|Tv - v\|^2 \le 0.$$

Dividing by $\alpha_{1\bullet} + \beta_0 (> 0)$, we obtain $\|Tv - v\|^2 \le 0$. This means that $Tv = v$. Therefore, $F(T)$ is nonempty. (b) Assume that $\alpha_{1\bullet} + \beta_0 \ge 0$, $\beta_1 \ge 0$, and $\alpha_{2\bullet} + \beta_2 > 0$. Since $\alpha_{1\bullet} + \beta_0 \ge 0$ and $\beta_1 \ge 0$, we have from (2.12) that

$$(\alpha_{2\bullet} + \beta_2) \left\|T^2 v - v\right\|^2 \le 0.$$

Since $\alpha_{2\bullet} + \beta_2 > 0$, it holds that $F\left(T^2\right) \ne \emptyset$. (c) Similarly, if $\alpha_{1\bullet} + \beta_0 \ge 0$, $\beta_1 > 0$, and $\alpha_{2\bullet} + \beta_2 \ge 0$, then $Tv = T^2 v$. This means that $F^{-1}(T) \ne \emptyset$. This completes the proof for Case (1abc).

Case (2abc). In much the same way as the proof for Case (1abc), we can obtain the desired result.

Finally, we prove the uniqueness of a fixed point of T. Assume that $\alpha_{\bullet\bullet} > 0$. Let $u, v \in F(T)$. As T is a $(\alpha_{ij}, \beta_i, \gamma_i;\ i, j = 0, 1, 2)$-generic 2-generalized hybrid mapping, we have from (1.1) that

$$\begin{aligned}
&\alpha_{00} \|u - v\|^2 + \alpha_{01} \|u - Tv\|^2 + \alpha_{02} \left\|u - T^2 v\right\|^2 \\
&+\alpha_{10} \|Tu - v\|^2 + \alpha_{11} \|Tu - Tv\|^2 + \alpha_{12} \left\|Tu - T^2 v\right\|^2 \\
&+\alpha_{20} \left\|T^2 u - v\right\|^2 + \alpha_{21} \left\|T^2 u - Tv\right\|^2 + \alpha_{22} \left\|T^2 u - T^2 v\right\|^2 \\
&+\beta_0 \|u - Tu\|^2 + \beta_1 \left\|Tu - T^2 u\right\|^2 + \beta_2 \left\|T^2 u - u\right\|^2 \\
&+\gamma_0 \|v - Tv\|^2 + \gamma_1 \left\|Tv - T^2 v\right\|^2 + \gamma_2 \left\|T^2 v - v\right\|^2 \le 0.
\end{aligned}$$

Since $u, v \in F(T) \subset F\left(T^2\right)$, it holds that $u = Tu = T^2 u$ and $v = Tv = T^2 v$. Thus, we obtain $\alpha_{\bullet\bullet} \|u - v\|^2 \le 0$. Since $\alpha_{\bullet\bullet} > 0$ is assumed, we obtain $u = v$. The proof is completed. □

3. Remarks

In this section, some remarks are given concerning Theorem 2.1 established in the previous section. Let T be a mapping from C into itself, where C is a nonempty subset of H. First, by analogy from (2.9), define

$$F^{-l}(T) \equiv \left\{x \in C : T^l x = T^{l+1} x\right\} \tag{3.1}$$

where $l \in \mathbb{N}$. Obviously, $x \in F^{-l}(T)$ is equivalent to $T^l x \in F(T)$. It is easy to verify that

$$F(T) = \cap_{l=1}^{\infty} F\left(T^l\right) \subset F^{-1}(T) \subset F^{-2}(T) \subset F^{-3}(T) \subset \cdots \text{ and} \tag{3.2}$$

$$F\left(T^2\right) = \cap_{l=1}^{\infty} F\left(T^{2l}\right). \tag{3.3}$$

Thus, if (1a) or (2a) in Theorem 2.1 is satisfied, it follows from (3.2) that

$$\left(\cap_{l=1}^{\infty} F\left(T^l\right)\right) \cap \left(\cap_{l=1}^{\infty} F^{-l}(T)\right) \neq \emptyset$$

since $F(T)$ is nonempty in such cases. Similarly, if (1b) or (2b) is satisfied, then we have from (3.3) that $\cap_{l=1}^{\infty} F\left(T^{2l}\right) \neq \emptyset$. If (1c) or (2c) is satisfied, then it holds from (3.2) that $\cap_{l=1}^{\infty} F^{-l}(T) \neq \emptyset$. We have obtained the following corollary:

Corollary 3.1. *Let C be a nonempty, closed, and convex subset of H, and let $T : C \to C$ be an $(\alpha_{ij}, \beta_i, \gamma_i;\ i, j = 0, 1, 2)$-generic 2-generalized hybrid mapping. Suppose that there exists an element $z \in C$ such that the sequence $\{T^n z\}$ in C is bounded. Consider the conditions* (1a)–(2c) *in Theorem* 2.1*. If* (1a) *or* (2a) *is satisfied, then $\left(\cap_{l=1}^{\infty} F\left(T^l\right)\right) \cap \left(\cap_{l=1}^{\infty} F^{-l}(T)\right)$ is nonempty, where $F^{-l}(T)$ is defined in* (3.1)*. If* (1b) *or* (2b) *is satisfied, then $\cap_{l=1}^{\infty} F\left(T^{2l}\right)$ is nonempty. If* (1c) *or* (2c) *is satisfied, then $\cap_{l=1}^{\infty} F^{-l}(T)$ is nonempty.*

Second, Theorem 2.1 can be generalized to the case of a generic L-generalized hybrid mapping, where $L \in \mathbb{N}$. Although we have proved Theorem 2.1 by using Lemma 2.1-(1), the theorem with the case of a generic 3-generalized hybrid mapping can be proved by using Lemma 2.1-(2).

Third, we compare Theorem 2.1 with the previous result in Kondo and Takahashi [16]:

Theorem 3.1 ([16])**.** *Let C be a nonempty, closed, and convex subset of H, and let $T : C \to C$ be an $(\alpha_{ij}, \beta_i, \gamma_i;\ i, j = 0, 1, 2)$-generic 2-generalized hybrid mapping. Suppose that T satisfies one of the following conditions:*

(1A) $\alpha_{0\bullet} + \alpha_{1\bullet} \geq 0$, $\alpha_{2\bullet} \geq 0$, $\alpha_{1\bullet} + \beta_0 > 0$, $\beta_1, \beta_2 \geq 0$, $\gamma_0 + \gamma_1 \geq 0$, $\gamma_2 \geq 0$;

(2A) $\alpha_{\bullet 0} + \alpha_{\bullet 1} \geq 0$, $\alpha_{\bullet 2} \geq 0$, $\alpha_{\bullet 1} + \gamma_0 > 0$, $\gamma_1, \gamma_2 \geq 0$, $\beta_0 + \beta_1 \geq 0$, $\beta_2 \geq 0$.

If there exists an element $x \in C$ such that the sequence $\{T^n z\}$ in C is bounded, then $F(T)$ is nonempty. Furthermore, a generic 2-generalized hybrid mapping T has at most one fixed point if $\alpha_{\bullet\bullet} > 0$.

In the statement of Theorem 3.1, the notations $\alpha_{i\bullet}$, $\alpha_{\bullet i}$, and $\alpha_{\bullet\bullet}$ are defined in (2.8). It is apparent that (1A) (resp. (2A)) in Theorem 3.1 is more restrictive than (1a) (resp. (2a)) in Theorem 2.1. The condition (1A) requires $\alpha_{0\bullet} + \alpha_{1\bullet} \geq 0$ and $\alpha_{2\bullet} \geq 0$, whereas the corresponding part of (1a) is $\alpha_{\bullet\bullet} \geq 0$. According to [16], under the condition (1A) or (2A), T is quasi-nonexpansive if it has a fixed point, whereas under the condition (1a) or (2a), that is not guaranteed. To illustrate the difference, consider cases (i)–(iv) that satisfy the condition (1a) and (2a), but do not satisfy (1A) nor (2A).

Let T be an $(\alpha_{ij}, \beta_i, \gamma_i;\ i, j = 0, 1, 2)$-generic 2-generalized hybrid mapping that is characterized by (1.1). (i) Substituting $\alpha_{00} = -1$, $\alpha_{11} = \lambda$, $\alpha_{22} = 1 - \lambda$, and the other parameters all equal to 0 into (1.1), we obtain

$$\lambda \|Tx - Ty\|^2 + (1 - \lambda) \left\|T^2 x - T^2 y\right\|^2 \leq \|x - y\|^2 \quad \text{for all } x, y \in C, \tag{3.4}$$

where $\lambda \in (0,1]$. (ii) Letting $\alpha_{00} = -1$, $\alpha_{10} = \mu$, $\alpha_{20} = 1-\mu$, and the other parameters all equal to 0, we have that

$$\mu \|Tx - y\|^2 + (1-\mu) \left\|T^2x - y\right\|^2 \leq \|x-y\|^2 \quad \text{for all } x, y \in C, \tag{3.5}$$

where $\mu \in (0,1]$. (iii) Letting $\alpha_{00} = -1$, $\alpha_{10} = \nu$, $\alpha_{20} = 1-\nu$, and the other parameters all equal to 0, we have that

$$\nu \|Tx - Ty\|^2 + (1-\nu) \left\|T^2x - Ty\right\|^2 \leq \|x-Ty\|^2 \quad \text{for all } x, y \in C, \tag{3.6}$$

where $\nu \in (0,1]$. (iv) Letting $\alpha_{00} = -1$, $\alpha_{10} = \xi$, $\alpha_{20} = 1-\xi$, and the other parameters all equal to 0, we have that

$$\xi \left\|Tx - T^2y\right\|^2 + (1-\xi) \left\|T^2x - T^2y\right\|^2 \leq \left\|x-T^2y\right\|^2 \quad \text{for all } x, y \in C, \tag{3.7}$$

where $\xi \in (0,1]$. The parameter combinations satisfy (1a) and (2a) in Theorem 2.1, but do not satisfy (1A) nor (2A) in Theorem 3.1. In the next section, we present examples of mappings, each of which satisfies (3.4) or (3.7).

Although a mapping that satisfies either of the conditions (3.4)–(3.7) is a special case of the generic 2-generalized hybrid mapping that is addressed in Theorem 2.1, we present a proof of a fixed point theorem concerning a general version of a mapping characterized by (3.4) as an exercise, because the conditions in Theorem 2.1 seem to be a bit complicated.

Theorem 3.2. *Let C be a nonempty, closed, and convex subset of H. Let $L \in \mathbb{N}$, and let $\lambda_1, \cdots, \lambda_L \in [0,1]$ that satisfies $\sum_{l=1}^{L} \lambda_l = 1$. Let T be a mapping from C into itself such that*

$$\sum_{l=1}^{L} \lambda_l \left\|T^l x - T^l y\right\|^2 \leq \|x-y\|^2 \tag{3.8}$$

for all $x, y \in C$. Suppose that there exists an element $z \in C$ such that $\{T^n z\}$ is a bounded sequence in C. If $\lambda_l > 0$, then $F\left(T^l\right)$ is nonempty, where $l = 1, \cdots, L$.

Proof. For brevity, we present a proof for the case of $L = 3$. Define

$$S_n^0 z = \frac{1}{n}\sum_{k=0}^{n-1} T^k z, \qquad S_n^1 z = \frac{1}{n}\sum_{k=1}^{n} T^k z,$$
$$S_n^2 z = \frac{1}{n}\sum_{k=2}^{n+1} T^k z, \qquad S_n^3 z = \frac{1}{n}\sum_{k=3}^{n+2} T^k z.$$

As C is convex, $\left\{S_n^0 z\right\}$, $\left\{S_n^1 z\right\}$, $\left\{S_n^2 z\right\}$, and $\left\{S_n^3 z\right\}$ are sequences in C. As $\{T^n z\}$ is bounded, so is $\left\{S_n^0 z\right\}$. Hence, there exists a subsequence $\left\{S_{n_i}^0 z\right\}$ of $\left\{S_n^0 z\right\}$ such that $S_{n_i}^0 z \rightharpoonup v$ for some $v \in H$. As C is closed and convex, it is weakly closed. As $\left\{S_{n_i}^0 z\right\} \subset C$ and $S_{n_i}^0 z \rightharpoonup v$, we have $v \in C$. Hence, $T^l v \,(\in C)$ exists for $l = 1, \cdots, L$.

Next, observe that

$$S_{n_i}^1 z \rightharpoonup v, \qquad S_{n_i}^2 z \rightharpoonup v, \qquad S_{n_i}^3 z \rightharpoonup v \tag{3.9}$$

as $i \to \infty$. It can be verified as follows: First, note that

$$S_n^0 z - S_n^1 z \to 0, \qquad S_n^0 z - S_n^2 z \to 0, \qquad S_n^0 z - S_n^3 z \to 0. \tag{3.10}$$

Indeed, it holds that

$$\begin{aligned}\left\|S_n^0 z - S_n^3 z\right\| &= \left\|\frac{1}{n}\sum_{k=0}^{n-1} T^k z - \frac{1}{n}\sum_{k=3}^{n+2} T^k z\right\| \\ &= \frac{1}{n}\left\|z + Tz + T^2 z - T^n z - T^{n+1} z - T^{n+2} z\right\|.\end{aligned}$$

As $\{T^n z\}$ is bounded, we obtain $S_n^0 z - S_n^3 z \to 0$. Similarly, the other parts of (3.10) can be demonstrated. As $S_{n_i}^0 z \rightharpoonup v$, we have from (3.10) that $S_{n_i}^1 z \rightharpoonup v$, $S_{n_i}^2 z \rightharpoonup v$, and $S_{n_i}^3 z \rightharpoonup v$ as claimed.

From (3.8) with $L = 3$, it holds that

$$\lambda_1 \left\|T^{k+1} z - Tv\right\|^2 + \lambda_2 \left\|T^{k+2} z - T^2 v\right\|^2 + \lambda_3 \left\|T^{k+3} z - T^3 v\right\|^2 \le \left\|T^k z - v\right\|^2$$

for all $k \in \mathbb{N} \cup \{0\}$. This yields

$$\begin{aligned}&\lambda_1 \left(\left\|T^{k+1} z - v\right\|^2 + 2\left\langle T^{k+1} z - v,\ v - Tv\right\rangle + \|v - Tv\|^2\right) \\ &+\lambda_2 \left(\left\|T^{k+2} z - v\right\|^2 + 2\left\langle T^{k+2} z - v,\ v - T^2 v\right\rangle + \left\|v - T^2 v\right\|^2\right) \\ &+\lambda_3 \left(\left\|T^{k+3} z - v\right\|^2 + 2\left\langle T^{k+3} z - v,\ v - T^3 v\right\rangle + \left\|v - T^3 v\right\|^2\right) \\ &\le \left\|T^k z - v\right\|^2.\end{aligned}$$

As $\lambda_1 + \lambda_2 + \lambda_3 = 1$, it follows that

$$\begin{aligned}&\lambda_1 \left(\left\|T^{k+1} z - v\right\|^2 - \left\|T^k z - v\right\|^2\right) + \lambda_1 \|v - Tv\|^2 \\ &+\lambda_2 \left(\left\|T^{k+2} z - v\right\|^2 - \left\|T^k z - v\right\|^2\right) + \lambda_2 \left\|v - T^2 v\right\|^2 \\ &+\lambda_3 \left(\left\|T^{k+3} z - v\right\|^2 - \left\|T^k z - v\right\|^2\right) + \lambda_3 \left\|v - T^3 v\right\|^2 \\ &+2\lambda_1 \left\langle T^{k+1} z - v,\ v - Tv\right\rangle + 2\lambda_2 \left\langle T^{k+2} z - v,\ v - T^2 v\right\rangle \\ &+2\lambda_3 \left\langle T^{k+3} z - v,\ v - T^3 v\right\rangle \le 0.\end{aligned}$$

Let $n \in \mathbb{N}$. Summing these expressions with respect to $k = 0, 1, \cdots, n-1$ and dividing by n, we obtain

$$\begin{aligned}&\frac{\lambda_1}{n}\left(\|T^n z - v\|^2 - \|z - v\|^2\right) \\ &+\frac{\lambda_2}{n}\left(\left\|T^{n+1} z - v\right\|^2 + \|T^n z - v\|^2 - \|Tz - v\|^2 - \|z - v\|^2\right) \\ &+\frac{\lambda_3}{n}(\left\|T^{n+2} z - v\right\|^2 + \left\|T^{n+1} z - v\right\|^2 + \|T^n z - v\|^2 \\ &- \left\|T^2 z - v\right\|^2 - \|Tz - v\|^2 - \|z - v\|^2) \\ &+\lambda_1 \|v - Tv\|^2 + \lambda_2 \left\|v - T^2 v\right\|^2 + \lambda_3 \left\|v - T^3 v\right\|^2 \\ &+2\lambda_1 \left\langle S_n^1 z - v,\ v - Tv\right\rangle + 2\lambda_2 \left\langle S_n^2 z - v,\ v - T^2 v\right\rangle \\ &+2\lambda_3 \left\langle S_n^3 z - v,\ v - T^3 v\right\rangle \le 0.\end{aligned}$$

Note that $\{T^n z\}$ is bounded. Replacing n by n_i, and taking the limit as $i \to \infty$, we have from (3.9) that

$$\lambda_1 \|v - Tv\|^2 + \lambda_2 \left\|v - T^2 v\right\|^2 + \lambda_3 \left\|v - T^3 v\right\|^2 \le 0. \tag{3.11}$$

Assume that $\lambda_1 > 0$. As $\lambda_2, \lambda_3 \geq 0$, subtracting

$$\lambda_2 \left\|v - T^2 v\right\|^2 + \lambda_3 \left\|v - T^3 v\right\|^2 (\geq 0)$$

from the left-hand side of (3.11) yields

$$\lambda_1 \|v - Tv\|^2 \leq 0.$$

Dividing by $\lambda_1 (> 0)$, we obtain $\|v - Tv\|^2 \leq 0$. This means $v = Tv$. Thus, $F(T) \neq \emptyset$. Similarly, if $\lambda_l > 0$, then $F\left(T^l\right) \neq \emptyset$ for $l = 2, 3$. This completes the proof. □

Theorem 3.2 with $L = 2$ corresponds to (3.4), whereas it includes Theorem 1.1 as a special case of $L = 1$.

4. Examples

This section presents examples of generic 2-generalized hybrid mappings to illustrate the difference of parameter conditions between those in Theorem 2.1 and Theorem 3.1. Theorem 2.1 is the main result of this article, whereas Theorem 3.1 is in a previous work by Kondo and Takahashi [16]. Mappings characterized by the conditions (3.4)–(3.7) are appropriate for that aim. We present two examples. First, letting $\lambda = 1/2$ in (3.4) yields

$$\|Tx - Ty\|^2 + \left\|T^2 x - T^2 y\right\|^2 \leq 2\|x - y\|^2 \quad \text{for all } x, y \in C. \tag{4.1}$$

The following mapping satisfies (4.1).

Example 1. We consider the case of $H = C = \mathbb{R}$. Mapping $T : \mathbb{R} \to \mathbb{R}$ is defined as follows:

$$Tx = \begin{cases} 0 & \text{if } x \geq 0, \\ -\sqrt{2}x & \text{if } x < 0. \end{cases} \tag{4.2}$$

It is easily ascertained that T is not quasi-nonexpansive, although it has a fixed point $0 (\in \mathbb{R})$. Therefore, T is not in the class of mappings addressed in previous studies cited in this article.

We verify that the function T defined by (4.2) satisfies the condition (4.1). Let $x, y \in \mathbb{R}$. (i) If $x, y \geq 0$, then $Tx = Ty = T^2 x = T^2 y = 0$. Thus, the condition (4.1) is met. (ii) If $x, y < 0$, then it holds that $Tx = -\sqrt{2}x$, $Ty = -\sqrt{2}y$, and $T^2 x = T^2 y = 0$. Therefore, it follows that

$$\begin{aligned} &\|Tx - Ty\|^2 + \left\|T^2 x - T^2 y\right\|^2 - 2\|x - y\|^2 \\ &= \left(-\sqrt{2}x + \sqrt{2}y\right)^2 - 2(x - y)^2 = 0. \end{aligned}$$

This indicates that the condition (4.1) is satisfied. (iii) If $x \geq 0$ and $y < 0$, then $Ty = -\sqrt{2}y$ and $Tx = T^2 x = T^2 y = 0$. Thus, it holds that

$$\begin{aligned} &\|Tx - Ty\|^2 + \left\|T^2 x - T^2 y\right\|^2 - 2\|x - y\|^2 \\ &= \|Ty\|^2 - 2\|x - y\|^2 \\ &= 2y^2 - 2(x - y)^2 \\ &= -2x(x - 2y) \leq 0. \end{aligned}$$

Thus, the condition (4.1) follows. From the above, the mapping defined by (4.2) satisfies (4.1) as claimed. □

Next, we present an example that satisfies (3.7). Letting $\xi = 1/2$ in (3.7), we have

$$\left\|Tx - T^2y\right\|^2 + \left\|T^2x - T^2y\right\|^2 \leq 2\left\|x - T^2y\right\|^2 \quad \text{for all } x, y \in C. \tag{4.3}$$

Although the mapping in Example 1 also satisfies (4.3), we present another example that is not continuous. The mapping in Example 2 is a variant of those in Igarashi *et al.* [6] or Hojo *et al.* [5].

Example 2. Let H be a Hilbert space, and set $C = H$. Define a nonlinear mapping $T : H \to H$ as follows:

$$Tx = \begin{cases} \frac{2x}{\|x\|} & \text{if } \sqrt{2} < \|x\| \text{ and } \|x\| \neq 2, \\ 0 & \text{if } \|x\| \leq \sqrt{2} \text{ or } \|x\| = 2. \end{cases} \tag{4.4}$$

It is easy to verify that T is not quasi-nonexpansive, although it has a fixed point $0\,(\in H)$.

We show that the mapping T defined by (4.4) satisfies the condition (4.3). Let $x, y \in C = H$. It can be immediately recognized that $T^2x = T^2y = 0$. We consider two cases according to $\|x\|$. (i) If $\|x\| \leq \sqrt{2}$ or $\|x\| = 2$, then $Tx = 0$. Thus, the condition (4.3) is satisfied. (ii) If $\sqrt{2} < \|x\|$ and $\|x\| \neq 2$, then $Tx = 2x/\|x\|$. Therefore, it follows that

$$\begin{aligned} &\left\|Tx - T^2y\right\|^2 + \left\|T^2x - T^2y\right\|^2 - 2\left\|x - T^2y\right\|^2 \\ &= \|Tx\|^2 - 2\|x\|^2 \\ &< 4 - 2 \times \left(\sqrt{2}\right)^2 = 0. \end{aligned}$$

This implies that (4.3) is satisfied. From the above, the mapping defined by (4.4) satisfies (4.3) as claimed. □

Finally, note that the mappings addressed in this article are not necessarily quasi-nonexpansive even if they have a fixed point, as illustrated by Examples 1 and 2. Therefore, convergence theorems for finding fixed points are difficult to prove along the elements of the previous studies cited in this article.

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