An International Trade Model under Risk:

Comparative Static Analysis

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Abstract The risk-free, two-sector, two-factor, constant-returns-to-scale model has long served as a standard model of international trade, with R. W. Jones and M.C. Kemp among others being its key promoters. The purpose of this chapter is to make an effort to extend the basic theorems of the standard model to cover new situations with price risk. The question to ask is whether and to what extent those results are still applicable to the world under risk. It is shown that when firms in the risky factor exhibit decreasing absolute risk aversion, the Rybczynski theorem and the Stolper-Samuelson theorem may fail to hold for some cases, whereas the factor price equalization theorem cannot carry over to the stochastic world. Besides, the implications of uniform (relative) changes in both factor endowments, and those in both (expected) commodity prices, are also carefully investigated.

Keywords standard model of International trade, decreasing absolute risk aversion, comparative static results, Rybczynski theorem, Stolper-Samuelson theorem, factor price equalization theorem

1 The Basic International Trade Model and its Possible Extension to the World of Risk and Uncertainty

In the past, a simple general equilibrium model of production has long served as an important workhorse for the evolution of many areas of economics, including international economics, public finance and economic growth. Presumably, the most famous model of this sort is the risk-free, two-sector, two-factor, constant-returns-to-scale model. In this chapter, we attempt to blend the more recent development in the economics of risk and uncertainty with some important problems which has been traditionally examined within the framework of the risk-free, two-by-two model. [1)]

We are concerned with the competitive economy in which one of the two sectors is confronted with risk as the result of price fluctuations in its product. Extending the usual, small-country assumption to the price risk milieu, we assume that firms in the risk-affected sector have no control over the distribution function of the price of their product, whose shape depends on world weather, people's taste patterns and many other natural and socio-economic factors influencing the demand-supply conditions in the world-wide market. Following Arrow and Pratt. we further assume that those risk-affected firms exhibit decreasing absolute risk aversion and seek to maximize expected utility from profits. [2)]

In deriving comparative static results from the aforementioned model under price risk, it is very effective for us to highlight the dual relationship between factor endowments and commodity outputs on the one hand and factor prices and commodity price parameters on the other. Indeed, it may prove so convenient for this purpose to make full use of the variable, input-output coefficients of both sectors instead of the production functions per se. Clearly, seeing to what extent the presence of price risk affects the dual structure of the model and thus forces the well-known, non-stochastic results to be amended should be the main task assigned to the present study. [3)]

More specifically, we will examine the possibility of extending the basic theorems of the traditional, risk-free model to cover situations with price risk. It is recalled here that, in the risk-free world, the profits of each firm must vanish at the long-run equilibrium. As will be seen below, however, even at such a long-run equilibrium, the risk-affected sector's expected profits should be positive in the case of risk aversion. Accordingly, the part of the expected price of the risk-affected output exceeding the wage rate and rental may be regarded as the per unit payment for the third hypothetical "risk factor." Existence of such "risk-bearing fee" and its responsiveness to changes in factor endowments and in commodity price parameters should require special care in exploring the comparative static properties of our stochastic model.

The following results among other things are worth mentioning here.

(i) Although the powerful factor-price equalization theorem can be established in the risk-free world, it is no longer valid — at least in its strict version — in the stochastic context.

(ii) The famous Rybcznski and Stolper-Samuelson theorems may fail to hold for certain cases to be specified below. So, this partially invalidates the Heckscher-Ohlin theorem as well

(iii) A uniform relative change in the two factor endowments affects commodity outputs unevenly, whereas a uniform relative in the two (expected) commodity prices affects factor prices unevenly.

(iv) The nice magnification effect advocated by Ronald Jones cannot carry over to the stochastic world.

The contents of this chapter are as follows. In Section 2, we will make a set of assumptions to establish our two-sector two-factor model with price risk. In Section 3, we will place the key equations describing the model on a common footing by rewriting them in terms of relative changes in variables and parameters. Section 4 will discuss the relationship between factor prices and commodity price parameters. The subsequent sections will be devoted to the comparative static properties of the model. Namely, we will explore the implications of changes in factor endowments in Section 5, those of changes in (expected) commodity prices in Section 6, and those of changes in price risk in Section 7. And final remarks will be made in Section 8.

2 Basic Assumptions and the Model with Price Risk

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Throughout this paper, we maintain the following set of basic assumptions under which our stochastic trade model is to be established. As is easily recognized, some of those assumptions are usually made for the traditional risk-free model, whereas several others are newly installed to take account of price risk.

More specifically, the following set of assumptions and notations will be maintained.

(i) Two primary factors, labor (L) and capital (C), are used in producing two distinct commodities, the uncertainty or risk-affected commodity (U) and the certainty or risk-free commodity (C). Technology in each of the two sectors exhibits constant returns to scale.

(ii) The money wage rate and the money rent for capital service are respectively denoted by w and r, whereas the market prices of the two commodities are denoted by p_U and p_C.

(iii) Firms in sector U face price risk, but firms in sector C is risk-free. Free competition prevails in all markets. The provability distribution of p_U, together with p_C, is exogenously given to the country in question; in other words, the "small country" assumption in the probabilistic sense is still maintained.

(iv) Factors are incessantly supplied and fully employed. While factors are perfectly mobile between the two sectors, factor intensity rankings never reverse even when factor prices change.

(v) On the one hand, firms in sector U, which have to make their input-output decision prior to the knowledge of the price of their output, seek to maximize expected utility from profits. It is assumed that those firms are not only risk averters but also exhibit decreasing absolute risk aversion. On the other hand, firms in sector C seek to maximize utility from profits.

(vi) All quantities should be valued in terms of money. Money is implicitly present in the economy and serves merely as the unit of measurement.

(vii) The demand side and other socio-politico- economic sides that must be introduced for a fuller description of the economy are all but neglected here.

With those assumptions in mind, let us build a two-sector, two-factor general equilibrium model with price risk. First of all, we note that the linearly homogeneous production function and the profit level in sector U are respectively given as follows.

$$U = F_U(L_U, K_U),$$

$$\Pi_U = p_U U - w L_U - r K_U .$$

The random variable p_U can be rewritten as follows.

$$p_U = \mu_U + \gamma \varepsilon ,$$

where $E[\varepsilon] = 0$ and γ is a shift parameter which is assumed to be one initially. An increase in γ leads to an increased spread of the probability distribution of p_U around the constant mean μ_U. Clearly, this may be regarded as the definition of an increase in the riskiness of p_U. [4)]

Let V_U be the utility attainable from Π_U. Assuming risk aversion on the part of U- producers, we have the following equations:

$$V_U'(\Pi_U) > 0, \quad V_U''(\Pi_U) < 0.$$

Those producers are interested in maximizing $E【V_U(\Pi_U)】$ with respect to their decision variables, L_U and K_U. This results in the following equations:

$$E【V_U'(\Pi_U)\{(\mu_U+\gamma\varepsilon)F_{LU}-w)\}】 = 0, \tag{1}$$

$$E【V_U'(\Pi_U)\{(\mu_U+\gamma\varepsilon)F_{KU}-r)\}】 = 0, \tag{2}$$

where we note

$$F_{LU} \equiv \partial F_U / \partial L_U, \; F_{KU} \equiv \partial F_U / \partial K_U.$$ [5)]

From 10.1 and 10.2, we immediately obtain the following equations:

$$w = (\mu_U - \gamma\rho) F_{LU}, \tag{3}$$

$$r = (\mu_U - \gamma\rho) F_{KU}, \tag{4}$$

where ρ is defined as follows.

$$\rho \equiv -E【V_U'(\Pi_U)\varepsilon】 / E【V_U'(\Pi_U)】. \tag{5}$$

The value of ρ is really important and requires careful interpretation. It represents the per unit psychological cost of risk bearing associated with price risk, which may reasonably be called the "risk bearing fee." As will be shown in Lemma 10.1 below, ρ is always positive whenever U- producers are risk averse. Therefore, a nice interpretation may be given to (3) and (4): At U- producer's risk affected equilibrium, the price of each factor is equal to the "net" or "effective" expected price of its marginal product. Since factor prices are positive, μ_U must be greater than $\gamma\rho$.

LEMMA 1. (risk bearing fee)

If $V_U''(\Pi_U) < 0$ then $\rho > 0$.

Proof. From the definition of Π_U, we immediately find $\Pi_U - E[\Pi_U] = \gamma \varepsilon U$. Therefore, if $V_U''(\Pi_U) < 0$, we have the following relationship:

$$V_U'(\Pi_U) \lesseqgtr V_U'(E[\Pi_U]) \Leftrightarrow \gamma \gtreqless 0$$

.

This immediately implies the following:

$$V_U'(\Pi_U)\varepsilon \leqq V_U'(E[\Pi_U])\varepsilon \quad \textit{for any value of } \varepsilon, \tag{6}$$

with the equality only when $\varepsilon = 0$. Hence, taking expectations on both sides of 10.6, we obtain

$$E【 V_U'(\Pi_U)\varepsilon 】 < V_U'(E[\Pi_U]) E【\varepsilon】 = 0.$$

In the light of (5), this implies that ρ must be positive. Q.E.D.

Concerning the behavior of C- producers, they are interested in maximizing utility from profit:

$$V_C(\Pi_C) = V_C(p_C F_C(L_C, K_C) - w L_C - r K_C),$$

where V_C is the increasing utility function of Π_C and F_C the linearly homogeneous production function of (L_C, K_C). Then, the optimum conditions are given as follows:

$$w = p_C F_{LC}, \tag{7}$$

$$r = p_U F_{KC}, \tag{8}$$

where $F_{LC} \equiv \partial F_C / \partial L_C$, $F_{KC} \equiv \partial F_U / \partial K_C$. In the light of (3), (4), (7), and (8), we have the following equation.

$$\frac{F_{LU}}{F_{KU}} = \frac{w}{r} = \frac{F_{LC}}{F_{KC}} . \quad (9)$$

Therefore, as with the certainty case, the factor price ratio must be equal to the marginal product ratio at each sector .

We note that the technology of the economy can also be described by the following matrix:

$$a = \begin{bmatrix} a_{LU} & a_{LC} \\ a_{KU} & a_{KC} \end{bmatrix} .$$

where a_{ij} stands for the quantity of factor i required to produce a unit of commodity j ; namely, $a_{LU} \equiv L_U / U$, $a_{KU} \equiv K_U / U$, etc.

Let us try to reformulate our two-by-two model in terms of the technology matrix (a_{ij}) instead of the production functions F_U and F_C . First of all, the requirement that both factors be fully employed is obviously given as follows.

$$L = a_{LU}U + a_{LC}C , \quad (10)$$

$$K = a_{KU}U + a_{KC}C . \quad (11)$$

Secondly, noting that the production function F_C of the uncertainty sector is linearly homogeneous, application of the Euler theorem yields the following equation..

$$U = F_{LU} L_U + F_{KU} K_U ,$$

which leads to the following equation.

$$\mu_U - \gamma \rho = (\mu_U - \gamma \rho) F_{LU} a_{LU} + (\mu_U - \gamma \rho) F_{KU} a_{KU} .$$

In view of (3) and (4), this result in the following.

$$\mu_U = w a_{LU} + r a_{KU} + \gamma \rho . \quad (12)$$

In a similar way, we can obtain the following equation for the certainty sector.

$$p_C = w\,a_{LC} + r\,a_{KC}\ . \tag{13}$$

As is quite clear from comparison between Eqs. (10) and (11) and Eqs. (12) and (13), the present formulation based on the input-output matrix (a_{ij}) is very instructive in showing the formal similarity of the relationship between factor endowments and commodity outputs on the one hand and relationship between commodity price parameters and factor prices on the other. Such a duality feature will be proved much more deeply in carrying out comparative static analysis in the subsequent sections.

In the general case of variable coefficient, we note that each a_{ij} depends on the rate of factor prices in the following way : [6)]

$$\begin{aligned} a_{LU} &= a_{LU}(w/r), \quad a_{KU} = a_{KU}(w/r), \\ a_{LC} &= a_{LC}(w/r), \quad a_{KC} = a_{KC}(w/r).. \end{aligned} \tag{14}$$

It now a simple matter to show that the expected profits of U- producers are positive if those producers are risk averse. Indeed, in view of (12), we find the following equation.

$$\begin{aligned} \Pi_U &= (\mu_U + \gamma\,\varepsilon)\,U - w\,a_{LU}U - r\,a_{KU}U \\ &= \{(\mu_U + \gamma\,\varepsilon) - (w\,a_{LU} + r\,a_{KU})\}\;U \\ &= \{(\mu_U + \gamma\,\varepsilon) - (\mu_U - \gamma\,\rho)\}\;U \\ &= (\rho + \varepsilon)\,\gamma\;U\ . \end{aligned} \tag{15}$$

We thereby have $E\,[\Pi_U] = \gamma\,\rho\,U$, implying that expected profits are positive whenever $V_U''(\Pi_U)$ is negative (see Lemma 10.1 above). This is in marked contrast to the certainty sector in which profits are of course zero. [7)]

The production structure in the economy is thus determined by ten independent equations including (5) and (10) ~ (15). (Note that the system (14) per se contains four equations.) Our uncertainty model is expected to determine ten variables: U, C, w, r, a_{LU}, a_{KU}, a_{LC}, a_{KC}, ρ, and Π_U . Hence, we have a determinate system with L, K, μ_U, γ, and p_C being treated as parameters.

3 The Equations of Change

We are now in a position to explore the comparative static properties of the model introduced in the previous section. In so doing, it will prove quite useful to place the key equations describing our model on a common footing by rewriting them in terms of *relative or proportional changes* in variables and parameters.

In what follows, we will effectively make use of *log differentiation* rather than simple differentiation. For instance. we note that $dlog\, L = dL/L$, $dlog\, w = dw/w$, $dlog\, a_{KU} = da_{KU}/ a_{KU}$, etc. If we differentiate 10.10 and rearrange it, then we find the following equation.

$$a_{LU}\, dU + a_{LC}\, dC = dL - (da_{LU} U + da_{LC} C) .$$

Dividing both sides of this equation by L and rearranging it results in the following equation.

$$(a_{LU} U/L)(dU/U) + (a_{LC} C/L)(dC/C)$$

$$= dL/L - \text{【} (da_{LU}/a_{LU})(a_{LU} U/L) + (da_{LC}/a_{LC})(a_{LC} C/L) \text{】} ,$$

This equation can be rewritten in terms of logarithmic differential as follows.

$$\lambda_{LU}\ dlog\, U + \lambda_{LC}\ dlog\, U \qquad (16)$$

$$= dlog\, L - (\lambda_{LU}\, dlog\, a_{LU} + \lambda_{LC}\, dlog\, a_{LC}) ,$$

where $\lambda_{LU} \equiv a_{LU} U / L$ and $\lambda_{LC} \equiv a_{LC} C /L$. Clearly, the λ's refer to factor endowment's *relative* allocations in each sector. A fraction of labor force L is allocated to the uncertainty sector (λ_{LU}), and this fraction plus the fraction of labor force allocated to the certainty sector (λ_{LC}) must add to unity:

$$\lambda_{LU} + \lambda_{LC} = 1 . \qquad (17)$$

In a similar fashion, if we differentiate (11) and rearrange it, we can derive the following equation.

$$\lambda_{KU}\ dlog\, U + \lambda_{KC}\ dlog\, U \qquad (18)$$

$$= \quad dlog\, K \;-\; (\lambda_{KU}\, dlog\, a_{KU} + \lambda_{KC}\, dlog\, a_{KC}) \quad ,$$

where $\lambda_{KU} \equiv a_{KU}\, U / K$ and $\lambda_{KC} \equiv a_{KC}\, K / L$. We also note that the following equation must hold.

$$\lambda_{KU} + \lambda_{KC} = 1 \quad . \tag{19}$$

Similarly, the equilibrium price-cost equations (12) and (13) can also be rewritten as the following equations of *relative* rates of exchange.

$$\begin{aligned} &\theta_{LU}\, dlog\, w + \theta_{KU}\, dlog\, r + \theta_R\, dlog\, \rho \\ &\quad = dlog\, \mu_U - \theta_R\, dlog\, \gamma \\ &\quad - (\theta_{LU}\, dlog\, a_{LU} + \theta_{KU}\, dlog\, a_{KU}) \quad , \end{aligned} \tag{20}$$

$$\begin{aligned} &\theta_{LC}\, dlog\, w + \theta_{KC}\, dlog\, r \\ &= dlog\, p_C - (\theta_{LC}\, dlog\, a_{LC} + \theta_{KC}\, dlog\, a_{KC}) \quad , \end{aligned} \tag{21}$$

where $\theta_{LU} \equiv a_{LU}\, w / \mu_U$, $\theta_{KU} \equiv a_{KU}\, r / \mu_U$, $\theta_R \equiv \gamma\, \rho / \mu_U$, $\theta_{LC} \equiv a_{LC}\, w / p_C$, and $\theta_{KC} \equiv a_{KC}\, r / p_C$. The θ' s represent factor's *relative* shares in each sector. Hence, θ_{LU} and θ_{KU} respectively stand for labor's share and capital's share in the uncertainty sector, whereas θ_R shows what we may call "risk factor's share" in the same sector. Obviously, these three shares must add to unity:

$$\theta_{LU} + \theta_{KU} + \theta_R = 1 \, . \tag{22}$$

In a similar way, we can also find the following equation.

$$\theta_{LC} + \theta_{KC} = 1 \, , \tag{23}$$

which shows the zero profit condition for the certainty sector, with no risk factor being present. As far as all physical factors are employed and firms are risk averse, each θ should be positive and less than unity.

Let us recall that the technology matrix in the economy is given as follows.

$$a = \begin{bmatrix} a_{LU} & a_{LC} \\ a_{KU} & a_{KC} \end{bmatrix} \quad .$$

Since the determinant of the technology matrix is written as $A = |a|$, we find $A = a_{LU} a_{KC} - a_{LC} a_{KU}$. Obviously, the value of A is positive if the uncertainty sector is relatively more labor intensive (i.e., $a_{LU} / a_{KU} > a_{LC} / a_{KC}$), it is negative if it is relatively more capital intensive (i.e., $a_{LU} / a_{KU} < a_{LC} / a_{KC}$). Further, let us define the allocation matrix λ and the share matrix θ as follows:

$$\lambda = \begin{bmatrix} \lambda_{LU} & \lambda_{LC} \\ \lambda_{KU} & \lambda_{KC} \end{bmatrix} \quad , \qquad \theta = \begin{bmatrix} \theta_{LU} & \theta_{LC} \\ \theta_{KU} & \theta_{KC} \end{bmatrix} \quad .$$

In the light of (17), (19), (22), and (23), we find the following equations.

$$\begin{aligned} \Lambda &\equiv |\lambda| = \lambda_{LU} \lambda_{KC} - \lambda_{LC} \lambda_{KU} \\ &= \lambda_{LU} - \lambda_{KU} = \lambda_{KC} - \lambda_{LC} \end{aligned} \tag{24}$$

$$\begin{aligned} \Theta &\equiv |\theta| = \theta_{LU} \theta_{KC} - \theta_{LC} \theta_{KU} \\ &= \theta_{LU} - (1 - \theta_R) \theta_{LC} = (1 - \theta_R) \theta_{KC} - \theta_{KU} \end{aligned} \tag{25}$$

It is not a difficult job to show that Λ and Θ are both positive if the uncertainty sector is relatively more labor intensive, and negative if the it is relatively more capital intensive. [8)]

Our next step is to simplify Eqs. (16), (18), (20), and (21) by eliminating every a_{ij} together with the λ and θ weighted sums of log a_{ij}'s. To this end, what we should note first is that U- firms minimize unit cost $UC = wa_{LC} + r a_{KC}$ for a given w and r. This obviously results in $dUC = w\, da_{LU} + r\, d\, a_{KU} = 0$, which can be rearranged in the following way:

$$(w a_{LU} / \mu_U)(da_{UC} / a_{LU}) + (r a_{KU} / \mu_U)(d a_{KU} / a_{KU}) = 0 \ ,$$

Obviously, this leads to the following nice equation of logarithmic differential..

$$\theta_{LU}\ dlog\, a_{UC} + \theta_{KU}\ dlog\ a_{KU} = 0 \ . \tag{26}$$

Similarly, for C- firms , we are able to derive the following nice equation.

$$\theta_{LC}\ dlog\ a_{LC} + \theta_{KU}\ dlog\ a_{KC} = 0\ . \tag{27}$$

Secondly, we may establish the relationship between factor price changes and factor proportions. To this end, we note the definition of the elasticity of substitution between factors in each sector. For U- sector, we note the following definition:.

$$\sigma_U \equiv dlog(K_U / L_U) / dlog(w/r)\ ,$$

which can be rewritten as follows:

$$dlog\ K_U - dlog\ L_U = \sigma_U\ (dlog\ w - dlog\ r).\ ,$$

or equivalently,

$$(dlog\ K_U - dlog\ U) - (dlog\ L_U - dlog\ U)$$
$$= \sigma_U\ (dlog\ w - dlog\ r)\ .$$

In terms of logarithmic differential, this implies following:

$$log\ a_{KU} - log\ a_{LU} = \sigma_U\ (dlog\ w - dlog\ r)\ . \tag{28}$$

For C- sector, if the elasticity of substitution between factors is denoted by σ_C , then we can derive the following equation in a similar fashion.

$$log\ a_{KC} - log\ a_{LC} = \sigma_C\ (dlog\ w - dlog\ r)\ . \tag{29}$$

If we combine those two equations (28) and (29) with Eqs. (26) and (27) and Eqs. (22) and (23), we can obtain solutions for the *dlog a* 's of both sectors in the following way:

$$log\ a_{LU} = -\ [\ \theta_{KU} / (1-\theta_R)\ ]\ \sigma_U\ (dlog\ w - dlog\ r)\ , \tag{30}$$

$$log\, a_{KU} = [\theta_{LU} / (1-\theta_R)]\, \sigma_U\, (dlog\, w - dlog\, r), \quad (31)$$

$$log\, a_{LC} = -\theta_{KC}\, \sigma_C\, (dlog\, w - dlog\, r), \quad (32)$$

$$log\, a_{KC} = \theta_{LC}\, \sigma_C\, (dlog\, w - dlog\, r). \quad (33)$$

If we substitute these solutions into Eqs. (16), (17), (20) and (21), then we obtain the following set of equations:

$$\lambda_{LU}\, dlog\, U + \lambda_{LC}\, dlog\, C - \delta_L\, (dlog\, w - dlog\, r) = dlog\, L, \quad (34)$$

$$\lambda_{KU}\, dlog\, U + \lambda_{KC}\, dlog\, C - \delta_K\, (dlog\, w - dlog\, r) = dlog\, K, \quad (35)$$

$$\theta_{LU}\, dlog\, w + \theta_{KU}\, dlog\, r = dlog\, \mu_U - \theta_R\, (dlog\, \rho + dlog\, \gamma), \quad (36)$$

$$\theta_{LC}\, dlog\, w + \theta_{KC}\, dlog\, r = dlog\, p_C, \quad (37)$$

where δ_L and δ_K are the positive quantities which are newly defined as follows.

$$\delta_L \equiv \lambda_{LU}\, \theta_{KU}\, \sigma_U / (1-\theta_R) + \lambda_{LC}\, \theta_{KC}\, \sigma_C,$$

$$\delta_K \equiv \lambda_{KU}\, \theta_{LU}\, \sigma_U / (1-\theta_R) + \lambda_{KC}\, \theta_{LC}\, \sigma_C.$$

Now, from (5) above, we find the following equation.

$$E【V_U'(\Pi_U)(\rho+\varepsilon)】 = 0. \quad (38)$$

We note that total differentiation of this equation yields the following equation.

$$E【V_U''(\Pi_U)(\rho+\varepsilon)\, d\Pi_U】 + E【V_U'(\Pi_U)】\, d\rho = 0. \quad (39)$$

Since $\Pi_U = (\rho + \varepsilon)\gamma U$ from (15), its total differentiation yields the following:

$$d\Pi_U = (\rho + \varepsilon)(\gamma dU + U d\gamma) + \gamma U d\rho \ .$$

By inserting this equation into (39) and rearranging it, we can obtain the following results :.

$$-\alpha \ dlog\, U + \beta\, dlog\ \rho = \alpha\, dlog\ \gamma \ , \tag{40}$$

where α and β are the newly introduced quantities to be defined as follows:

$$\alpha \equiv -E【V_U''(\Pi_U)(\rho + \varepsilon)^2】\gamma U \ , \tag{41}$$

$$\beta \equiv -E【V_U''(\Pi_U)(\rho + \varepsilon)】\gamma U + E【V_U'(\Pi_U)】\rho \ . \tag{42}$$

Obviously, when U - firms are risk averse, α must be positive. The assumption that they exhibit decreasing absolute risk aversion is strong enough to determine the sign of β . To this end, let us define the absolute risk aversion function in the following way: [9)]

$$R_U(\Pi_U) \equiv - V_U''(\Pi_C) / V_U'(\Pi_C) \ . \tag{43}$$

Then, we can derive the following result.

LEMMA 2 (the value of β)

If $R_U'(\Pi_U) < 0$ then we find $\beta > 0$.

Proof. First of all, let us rewrite 10.38 down below.

$$E【V_U'(\Pi_U)(\rho + \varepsilon)】 = 0 \ .$$

Now, let us focus on a particular ε^* such that $V_U'(\Pi_U{}^*)(\rho + \varepsilon^*) = 0$, where $\Pi_U{}^* = (\mu_U + \gamma\varepsilon^*)U - w a_{LU} U - r a_{KU} U$.

Since $\Pi_U - \Pi_U^* = \gamma(\varepsilon - \varepsilon^*)\, U$, we should have the following relation:

$$\Pi_U \gtreqless \Pi_U^* \quad \Leftrightarrow \quad \varepsilon \gtreqless \varepsilon^* \quad (= -\rho\ )$$
$$\Leftrightarrow \quad \rho + \varepsilon \gtreqless 0 .$$

.

If $R_U'(\Pi_U) < 0$, then we must have the following relation:

$$-\ V_U''(\Pi_C) / V_U'(\Pi_C) \equiv R_U'(\Pi_U) \lesseqgtr R_U'(\Pi_U^*)$$

$$\Leftrightarrow \quad \Pi_U \gtreqless \Pi_U^*$$

Accordingly, we should have the following result .

$$-\ 【 V_U''(\Pi_C) / V_U'(\Pi_C) 】 (\rho + \varepsilon)$$
$$\leqq R_U'(\Pi_U^*)(\rho + \varepsilon) \qquad \textit{for any value of } \varepsilon ,$$

which obviously implies the following :

$$V_U''(\Pi_C)(\rho + \varepsilon)$$
$$\geqq -\ R_U'(\Pi_U^*)\ V_U'(\Pi_C)(\rho + \varepsilon) \qquad \textit{for any value of } \varepsilon .$$

If we take expected values both side of this equation and note 10.38 above, we find

$$E\ 【 V_U''(\Pi_C)(\rho + \varepsilon) 】$$
$$\geqq -\ R_U'(\Pi_U^*)\quad E 【 V_U'(\Pi_C)(\rho + \varepsilon) 】$$
$$= 0 .$$

Clearly, this ensures that β is positive. Q.E.D.

The system of equations of *relative or proportional change* for our uncertainty model is exhibited in (34) ~ (37) and (40). In matrix form, it can neatly be rewritten as follows.

$$\begin{bmatrix} \lambda_{LU} & \lambda_{LC} & 0 & -\delta_L & \delta_L \\ \lambda_{KU} & \lambda_{KC} & 0 & \delta_K & -\delta_K \\ -\alpha & 0 & \beta & 0 & 0 \\ 0 & 0 & \theta_R & \theta_{LU} & \theta_{KU} \\ 0 & 0 & 0 & \theta_{LC} & \theta_{KC} \end{bmatrix} \begin{bmatrix} d\log U \\ d\log C \\ d\log \rho \\ d\log w \\ d\log r \end{bmatrix} .$$

$$= \begin{bmatrix} d\log L \\ d\log K \\ \alpha d\log \gamma \\ d\log \mu_U - \theta_R d\log \gamma \\ d\log p_C \end{bmatrix} . \qquad (44)$$

The subsystem consisting of the first two equations in the system (44) represents the commodity output - factor endowment relationship and the subsystem consisting of the last two equations the factor - price commodity price parameter relationship. The third equation, which links *dlog* ρ to *dlog* γ , tells us how the imposing presence of price risk contributes to make those two subsystems tightly connected and thus the dual feature shared by any two-by-two model more complicated than in the traditional, risk-free case.

In the special case in which U - firms are just risk neutral, the quantities θ_R, α and β all vanish, so that (44) is simply reduced to the following:

$$\begin{bmatrix} \lambda_{LU} & \lambda_{LC} & -\delta_L & \delta_L \\ \lambda_{KU} & \lambda_{KC} & \delta_K & -\delta_K \\ 0 & 0 & \theta_{LU} & \theta_{KU} \\ 0 & 0 & 0 & \theta_{LC} \end{bmatrix} \begin{bmatrix} d\log U \\ d\log C \\ d\log w \\ d\log r \end{bmatrix}$$

$$= \begin{bmatrix} d\log L \\ d\log K \\ d\log \mu_U \\ d\log p_C \end{bmatrix} . \qquad (45)$$

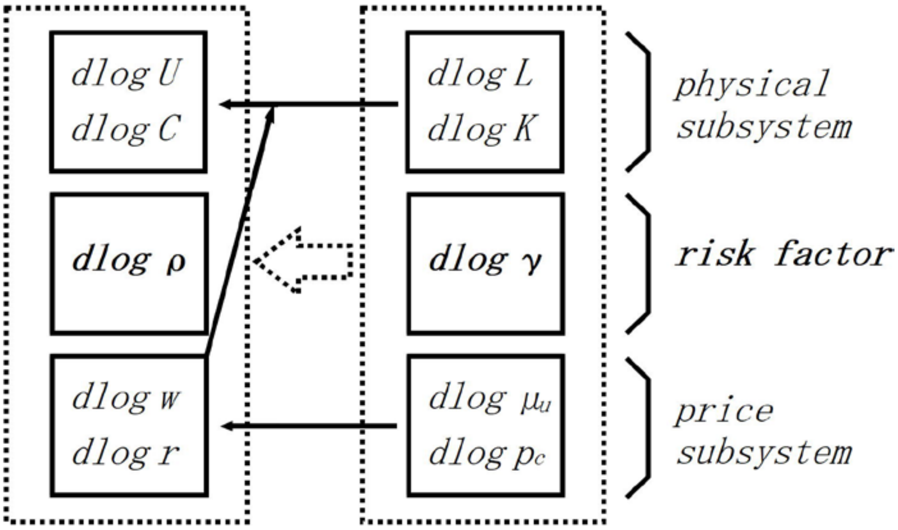
As can easily be expected, the risk-neutral system (45) has essentially the same structure as the risk-free system with respect to the variables, parameters, and coefficients: the only difference is that the relative change in *expected* price of the uncertainty sector is now present as a variable. [10)]

In this special system, the "financial" subsystem containing the last two equations may be entirely separated from the "physical" subsystem containing the first two equations in the following sense. The former subsystem itself complete enough to determine the values *dlog w* and *dlog r* for given *dlog* μ_U and *dlog* p_C. As was seen above, however, such a simple separation of the total system into the two smaller subsystems is destined to vanish once "risk averse players" come on the market stage. As everybody experiences in his daily life, risk aversion really matters and should not be neglected.

In order to understand the structure of the model more transparent, we believe that a more visual approach would be very instructive.. Let us take a close look at Fig. 1. Then , on the left , we can see a group of log differentials of *variables* —— *dlog U, dlog C, dlog* ρ , *dlog w* , and *dlog r* . On the right, we can see another group of log differentials of *parameters* —— *dlog L, dlog K, dlog* γ , *dlog* μ_U, and *dlog* p_C . The question of how and to what extent those two groups of log differentials are interlocked with each other is visually solvable in Fig. 1. It is also worthy of reference to note that the model per se consists of the three subsystems —— a *physical subsystem* related to the first two log differentials, a *price subsystem* related to the last two log differentials, and a still another *risk factor* placed between those two subsystems.

For convenience, the simple case of risk neutrality is illustrated by solid lines, and the more complicated case of risk aversion by dotted lines. As can easily be seen in Fig. 1, in the simple first case, the price subsystem is independent of the physical subsystem. Indeed, the values of *dlog w* and *dlog r* are dependent only on *dlog* μ_U, and *dlog* p_C, thereby being determined exclusively within the price subsystem. In the complicated second case, however, a completely different situation would emerge. Then, as can be seen by a mixture of dotted lines, the aforementioned decomposition of the total system into the two subsystems would be no longer feasible, with the result that all the variables are dependent on all the parameters. No doubt, such decomposability would well-explain the fundamental difficulty of comparative static analysis in the general case of risk aversion.

Fig. 1 The relationship between variables and parameters:
risk neutrality versus risk aversion



Remark. The three solid line arrows taken together (➡) show the relationship between variables and parameters in the specific case of risk neutrality. In the more general case of risk aversion, however, such clear relationship breaks down and should be changed to a much vaguer one indicated by the one dotted line arrow (⇒) .

4 The Relationship between Factor Prices and Commodity Price Parameters : Reexamining the Validity of the Factor-Price Equalization Theorem

One of the most salient features of the non-stochastic two-sector, two-factor model is that under some reasonable conditions, factor prices are dependent only on commodity prices and on no other factors, which is particularly well-known as the factor-price equalization theorem. We will show in this section, however, that such a nice relationship between factor prices and commodity prices cannot automatically be carried over to the world with price risk, except for the very special case in which firms in the uncertainty sector are just risk neutral. In short, the presence of risk in the model is likely to throw the established order into disturbance.

In order to make the system (44) more manageable, let us attempt to reduce the number of variables from five to three. Making use of the last two equations in (44) (or equivalently, (36) and (37)) , we may solve for $dlog\, w$ and $dlog\, r$ to derive the following equations.

$$dlog\, w = (1/\Theta)\,[\,-\theta_{KC}\theta_R\, dlog\, \rho + \theta_{KC}\, dlog\, \mu_U - \theta_{KC}\theta_R\, dlog\, \gamma - \theta_{KU}\, dlog\, p_C\,] \quad , \tag{46}$$

$$dlog\, r = (1/\Theta)\,[\,\theta_{LC}\theta_R\, dlog\, \rho - \theta_{LC}\, dlog\, \mu_U + \theta_{LC}\theta_R\, dlog\, \gamma + \theta_{LU}\, dlog\, p_C\,] \quad . \tag{47}$$

In view of 10.22 and 10.23, we thereby obtain the following equation.

$$dlog\, w - dlog\, r = (1/\Theta)\,[\,-\theta_R\, dlog\, \rho + dlog\, \mu_U - \theta_R\, dlog\, \gamma - (1-\theta_R)\, dlog\, p_C\,] \quad . \tag{48}$$

As is quite clear from the above equations, the way in which factor prices are affected by changes in commodity price parameters should be dependent on in which direction and how much the risk aversion fee (ρ) responds to those changes. Note that $dlog\, \rho$ is linked to both $dlog\, L$ and $dlog\, K$ through both $dlog\, U$ and $dlog\, C$ (see the physical subsystem in (44)). This demonstrates the general dependence of factor price changes in the factor endowment. Therefore, the famous factor price equalization theorem no longer holds in our stochastic model, except for the special risk-neutral case in which θ_R, the risk factor's share, disappears from the stage. [11)]

Substituting for $dlog\, w$ and $dlog\, r$ in (46) and (47) into the first three equations in the system (44) and rearranging them, we obtain the following matrix equation:

$$\begin{bmatrix} \lambda_{LU} & \lambda_{LC} & \delta_L\theta_R/\Theta \\ \lambda_{KU} & \lambda_{KC} & -\delta_L\theta_R/\Theta \\ -\alpha & 0 & \beta \end{bmatrix} \begin{bmatrix} d\log U \\ d\log C \\ d\log\rho \end{bmatrix} = \tag{49}$$

$$\begin{bmatrix} d\log L + (\delta_L/\Theta)d\log\mu_U - (\delta_L\theta_R/\Theta)d\log\gamma - (\delta_L(1-\theta_R)/\Theta)d\log p_C \\ d\log K - (\delta_K/\Theta)d\log\mu_U + (\delta_K\theta_R/\Theta)d\log\gamma + (\delta_K(1-\theta_R)/\Theta)d\log p_C \\ \alpha\, d\log\gamma \end{bmatrix}$$

Given changes in parameters, the values of *dlog* U, *dlog* C and *dlog* ρ are simultaneously determined in the reduced system 10.49. And then, the values of *dlog* w and *dlog* r are consequently determined by substituting for this *dlog* ρ into (46) and (47). Let D be the determinant of the coefficient matrix in (49), namely,

$$D \equiv \begin{vmatrix} \lambda_{LU} & \lambda_{LC} & \delta_L\theta_R/\Theta \\ \lambda_{KU} & \lambda_{KC} & -\delta_L\theta_R/\Theta \\ -\alpha & 0 & \beta \end{vmatrix}$$

Then we find the value of D as follows.

$$D = \beta\Lambda + (\alpha\,\theta_R/\Theta)(\delta_L\lambda_{KC} + \delta_K\lambda_{LC}). \tag{50}$$

It is noted that D is positive if the uncertainty sector is relatively more labor intensive, and negative if it is relatively more capital intensive.

5 Changes in Factor Endowment: Reexamining the Validity of the Rybczynski Theorem

In this and following sections, we would like to investigate the comparative static properties of the uncertainty model described above more deeply by focusing on a change in one parameter only. First of all, we are interested in seeing what happens to the equilibrium variables of the system when the labor or capital endowment changes. Specifically, we wish to know whether or not the following theorem of Rybczynski remains unscathed by the introduction of price risk: An increase in the endowment of any specific factor leads to an expansion in the output of whichever sector is relatively more intensive in its use of that factor, and to a contraction of the output of the other factor.

We are anxious to know whether and to what extent the Rybczymski theorem remains valid by the presence of price risk. In this connection, we will state and prove the following result.

THOREM 1 (Rybczynski Theorem to be Reexamined)

(1) Suppose $dlog\,L > 0$. Then we have the following results.

$dlog\,U > 0$, $dlog\,C \gtreqless 0$, $dlog\,\rho > 0$, $dlog\,w < 0$, $dlog\,r > 0$

if the uncertainty sector is relatively more labor intensive ;

$dlog\,U < 0$, $dlog\,C > 0$, $dlog\,\rho < 0$, $dlog\,w < 0$, $dlog\,r > 0$

if it is relatively more capital intensive.

(2) Suppose $dlog\,L > 0$. Then we have the following results.

$dlog\,U < 0$, $dlog\,C > 0$, $dlog\,\rho < 0$, $dlog\,w > 0$, $dlog\,r < 0$

if the uncertainty sector is relatively more labor intensive;

$dlog\,U > 0$, $dlog\,C \gtreqless 0$, $dlog\,\rho > 0$, $dlog\,w > 0$, $dlog\,r < 0$

if it is relatively more capital intensive.

Proof. Let all the parameters but $dlog\,L$ be zero in (49). Then we have the following matrix equation.

$$\begin{bmatrix} \lambda_{LU} & \lambda_{LC} & \delta_L\theta_R/\Theta \\ \lambda_{KU} & \lambda_{KC} & -\delta_L\theta_R/\Theta \\ -\alpha & 0 & \beta \end{bmatrix} \begin{bmatrix} d\log U \\ d\log C \\ d\log\rho \end{bmatrix} = \begin{bmatrix} d\log L \\ 0 \\ 0 \end{bmatrix} \qquad (51)$$

If we solve for the variables, it is not hard to derive the following equations.

$$dlog\,U = (\beta\,\lambda_{KC}/D)\,dlog\,L, \qquad (52)$$

$$dlog\,C = -\left[(\beta\,\lambda_{KU}\Theta - \alpha\,\delta_K\theta_R)/D\Theta\right]dlog\,L, \qquad (53)$$

$$dlog\,\rho = (\alpha\,\lambda_{KC}/D)\,dlog\,L. \qquad (54)$$

By virtue of (46) and (47), we find the following equations.

$$dlog\, w \;=\; -\,(\theta_{KC}\,\theta_R\,/\,\Theta)\;\; dlog\;\rho \tag{55}$$

$$dlog\, r \;=\; (\theta_{LC}\,\theta_R\,/\,\Theta)\;\; dlog\;\rho \tag{56}$$

Keeping in mind that Θ and D are positive (or negative) if the uncertainty sector ia relatively more labor (or capital) intensive, Property (1) immediately follows from (52) ~ (56). Note that the sign of $dlog\, C$ is still indeterminate, depending the sign of the quantity $(\beta\,\lambda_{KU}\,\Theta\,-\alpha\,\delta_K\,\theta_R)$.

Now, focusing on $dlog\, K$, the proof of Property (2) proceeds in a similar way and is omitted. For a later discussion, we are only content to write the following formula:

$$dlog\;\rho \;=\; (\alpha\,\lambda_{LC}\,/\,D)\; dlog\, K. \tag{57}$$

Q.E.D.

Remarkably, Theorem 1 has several interesting implications in comparison with the non-stochastic results.

(i) First of all, it is noted that the Rybczynski theorem may fail to hold for the certainty sector although it does hold for the uncertainty sector, provided that firms in the latter sector exhibit decreasing absolute risk aversion. More precisely, when the uncertainty sector is relatively more labor (or capital) intensive, an increase in the labor endowment (or the capital endowment) does not necessarily result in a decrease in the output of the certainty sector. Surely, this looks a very delicate result. Since there is an intimate link between the Heckscher-Ohlin theorem and the Rybczynski theorem, this also implies the partial invalidity of the former theorem. [12)]

(ii) Secondly, the risk bearing fee varies in either direction in response to an increase in the factor endowment, depending on the factor intensity between the two sectors. This explains why the Rybczynski theorem may become partially invalid in the case of price risk. In fact, from the first two equations in (51) , we find the following equations.

$$\lambda_{LU}\;dlog\, U \;+\; \lambda_{LC}\;dlog\;C \;=\; dlog\, L-(\delta_L\,\theta_R\,/\,\Theta)\;dlog\;\rho$$

$$\lambda_{KU}\, dlog\, U + \lambda_{KC}\, dlog\, C = (\delta_K \theta_R / \Theta)\, dlog\, \rho$$

Therefore, in case the uncertainty sector is relatively more labor intensive (i.e., Θ is positive), the increase in risk bearing fee has a double effect in that it has an effect of decreasing L *and* another effect of increasing K. Taking those two effects into consideration, increasing ρ definitely shows the possibility that *both sectors simultaneously expand* as the result of the initial increase in L. In the light of 10.57, an analogous argument can effectively proceed for the implications of an increase in K. [13)]

(iii) Thirdly, an increase in the endowment of any factor must cause the price of that factor to decline and the price of the other factor to rise, whatever the factor intensity between the two factors. This result can be well-compared with the non-stochastic case in which factor prices are not affected at all by changes in factor endowments. In contrast to the risk-free world, the physical subsystem and the price subsystem are no longer separated in the risk-affected world: in plain English, the risk bearing fee may serve as an "unlucky trouble maker" rather than a "lucky go-between."

So far, we have analyzed the compact of a change in only one parameter on equilibrium values. We now turn to a bit more complicated question, namely, the question of the impact of a *uniform relative change in the two parameters.*

THEOREM 2 (the Impact of a Uniform Relative Change in Both L and K)

Suppose $dlog\, L = dlog\, K > 0$. Then we have the following results.

(1) $dlog\, C > dlog\, U > 0$;

(2) $dlog\, \rho > 0$;

(3) $dlog\, w < 0$ **and** $dlog\, r > 0$;

if the uncertainty sector is relatively more labor intensive.

$dlog\, w > 0$ **and** $dlog\, r < 0$

if it is relatively more capital intensive.

Proof. For convenience, let us put $dlog\, L = dlog\, K = dlog\, E$. Then, if we make use of (49) together with (46) and (47), it is a rather straightforward job to derive the following results.

$$dlog\, U = \frac{\beta \Lambda}{D}\, dlog\, E \quad ;$$

$$dlog\, C = \frac{\beta \Lambda \Theta + \alpha \theta_R (\delta_L + \delta_K)}{D\Theta}\, dlog\, E \quad ;$$

$$dlog\, U - dlog\, C = -\frac{\alpha \theta_R (\delta_L + \delta_K)\, ;}{D\Theta}\, dlog\, E \quad ;$$

$$dlog\, \rho = \frac{\alpha \Lambda}{D}\, dlog\, E\, ;$$

$$dlog\, w = -\frac{\alpha \theta_{KL} \theta_R \Lambda}{D\Theta}\, dlog\, E \quad ;$$

$$dlog\, r = \frac{\alpha \theta_{KC} \theta_R \Lambda}{D\Theta}\, dlog\, E \quad .$$

Using these formulas, the desired results follow immediately. Q.E.D.

We say that a uniform relative change in the two factor endowments has a "neutral effect" if it does not alter the composition of outputs and the ratio of factor prices. Theorem 2 shows that as contrasted with the non-stochastic, constant-returns-scale economy, such neutral effect of factor endowments is no longer valid in our stochastic framework.

Let us suppose that both endowments expand at the same rate. Then, the uncertainty output grows at a lower rate than the certainty output. Besides, the price of whichever factor is used relatively more intensively in the uncertainty sector decreases and the price of the other sector increases. The reason why the uniform growth of both endowments produces such uneven effects on commodity outputs and factor prices is that it forces the risk bearing fee to increase, which in turn affects the uncertainty sector relatively more unfavorably. [14)]

In short, in the world of risk and uncertainty, people tend to exhibit risk aversion. Then, the burden of risk bearing is likely to play a trick on the economic stage.

6 Changes in (Expected) Commodity Price: The Stolper-Samuelson Theorem to be Reexamined

This section will consider the implications of a change in the expected price of the uncertainty commodity and of a change in the price of the certainty commodity. The famous theorem of Stolper and Samuelson, the one that is dual to the Rybczynski theorem, says that an increase in the price of any commodity results in an increase in the price of whichever factor is used relatively more intensively in the production of that commodity, and in a decline in the price of the other factor.. We will be especially interested in the extent to which the Stolper-Samuelson theorem is carried over to the price risk world.

THEOREM 3 (The Stopler-Samuelson Theorem to be Reexamined)

(1) Suppose $dlog\,\mu_U > 0$. Then we have the following results.

$dlog\,U > 0,\ dlog\,C < 0,\ dlog\,\rho > 0,\ dlog\,w > 0,\ dlog\,r < 0$

if the uncertainty sector is relatively more labor intensive ;

$dlog\,U > 0,\ dlog\,C < 0,\ dlog\,\rho < 0,\ dlog\,w < 0,\ dlog\,r > 0$

if it is relatively more capital intensive.

(2) Suppose $dlog\,p_C > 0$. Then we have the following results.

$dlog\,U < 0,\ dlog\,C > 0,\ dlog\,\rho < 0,\ dlog\,w \lesseqgtr 0,\ dlog\,r > 0$

if the uncertainty sector is relatively more labor intensive;

$dlog\,U < 0,\ dlog\,C > 0,\ dlog\,\rho < 0,\ dlog\,w > 0,\ dlog\,r \lesseqgtr 0$

if it is relatively more capital intensive.

Proof. Taking advantage of (49), the proof of this theorem is almost parallel to that of Theorem 10.1 and is omitted here. We only record the following formulas.

(1) For a change in $dlog\,\mu_U$, we note the following:

$$dlog\ \rho = \frac{\alpha(\delta_L \lambda_{KC} + \delta_K \lambda_{LC})}{D\Theta} dlog\ \mu_U , \tag{58}$$

$$dlog\ \mu_U - \theta_R\ dlog\ \rho = \frac{\beta \Lambda}{D} dlog\ \mu_U . \tag{59}$$

(2) For a change in $dlog\ p_C$, we note the following:

$$dlog\ \rho = -\frac{\alpha(1-\theta_R)(\delta_L \lambda_{KC} + \delta_K \lambda_{LC})}{D\Theta} dlog\ p_C , \tag{60}$$

$$dlog\ r - dlog\ p_C = \frac{\beta(1-\theta_R)\lambda_{LC}\ \Lambda}{D\Theta} dlog\ p_C , \tag{61}$$

$$.\ dlog\ p_C - dlog\ w = \frac{\beta(1-\theta_R)\lambda_{KC}\ \Lambda\ .}{D\Theta} dlog\ p_C\ . \tag{62}$$

Q.E.D.

In what follows, we can give some interesting interpretations to Theorem 2.

(i) First of all, we see that although the Stolper-Samuelson Theorem is valid for the case with changes in μ_U, it may fail to be valid for the case with changes in p_C. . In fact, an increase in p_C does not always lead to a decline in w, namely, the price of the factor L that is less intensively used in C - production. The key to such a non-symmetric result lies in the fact that the trouble-making ρ is present in the stochastic world, and that it rises when μ_U rises, but falls when p_C rises.

(ii) To see this point more sharply, on the one hand, we let all the parameters but $dlog\ \mu_U$ be zero in the last equation in (44) above and rearrange them slightly to obtain the following:

$$\theta_{LU}\ dlog\,w + \theta_{KU}\ dlog\,r = dlog\,\mu_U - \theta_R\ dlog\ \rho\ ,$$

$$\theta_{LC}\ dlog\,w + \theta_{KC}\ dlog\,r = 0\ .$$

In the light of (59), the term ($dlog\,\mu_U - \theta_R\ dlog\ \rho$) is positive whenever $dlog\,\mu_U$ is so, whatever the intersecting factor intensity condition (note here that the quantities D, Λ, and Θ are either all positive or all negative). Consequently, the effect of the positive response of $dlog\ \rho$ to $dlog\,\mu_U$ is to make a positive $dlog\,\mu_U$ merely smaller, but not negative, with the result that the Stotper-Samuelson result continues to hold in this case.

(iii) On the other hand, we let all the parameters but $dlog\,p_C$ be zero in the last two equations in (44). Then, we find the following:

$$\theta_{LU}\ dlog\,w + \theta_{KU}\ dlog\,r = -\theta_R\ dlog\ \rho\ ,$$

$$\theta_{LC}\ dlog\,w + \theta_{KC}\ dlog\,r = dlog\ p_C\ .$$

Therefore, a decreasing ρ accompanied with an increasing p_C has the same effect on w and r, , as would an increasing μ_U. This tells us that there is a possibility that contrary to the Stolper-Samuelson theorem, both w and r go in the same direction as p_C . [15)]

(iv) In short, the Stolper-Samuelson theorem is true, nice and strong in the risk-free world. When any kind of risk enters the world, however, we would have a entirely different picture. Then, the risk factor could possibly play as a "trouble maker." No matter how annoying it may be, it is the reality. Surely, we have to face up to it . .

THEOREM 4 (The Impact of a Uniform Relative Change in both μ_U and p_C)

Suppose $dlog\ \mu_U = dlog\,p_C > 0$. Then we have the following results.

(1) $dlog\,U > 0$, $dlog\,C < 0$;

(2) $dlog\ \rho > 0$;

(3) ***dlog w*** $>$ **0 ,** ***dlog r*** $\gtreqless$ **0 ,** ***dlog w*** $>$ ***dlog r***

if the uncertainty sector is relatively more labor intensive.

dlog w $\gtreqless$ **0 ,** ***dlog r*** $>$ **0 ,** ***dlog r*** $>$ ***dlog w***

if it is relatively more capital intensive.

Proof. The proof of this theorem is quite analogous to that of Theorem 2 and is omitted. Q.E.D.

We say that a uniform relative change in both μ_U and p_C has a "neutral effect" if it does not influence the output ration and the factor price ratio. As a dual of Theorem 2, Theorem 4 demonstrates such a neutral effect of (expected) commodity prices does not hold in the presence of price uncertainty, which is in marked contrast with the risk-free case. More specifically, we can derive following results.

(i) Suppose that both μ_U and p_C rise at the same rate. Then, first of all, the uncertainty output U must increase but the certainty output C must decrease.

(ii) Secondly, the price of whichever factor is more intensively used in the production of U output must rise at a faster rate than the price of the other factor, with the possibility that the latter price might even fall.

(iii) Note that risk bearing fee is conspicuously present in the risk aversion case and positively responsive to the uniform commodity price change. This is the reason why the uniform change affects both physical and financial variables unevenly as stated above. [16)]

7 Changes in Price Risk: Getting into the New World of Uncertainty

We turn to the final important problem of this final chapter. Now, we are concerned with the question of the implications of a change in price uncertainty for the equilibrium values of the model. We feel as if we were getting into the New World of Risk and Uncertainty. Hopefully, our approach will shed a new light on the old duality feature of the international economy.

THEOREM 5 (Changes in Price Risk)

Suppose ***dlog*** $\gamma > 0$**. Then we have the following results.**

(1) $dlog\,U < 0$, $dlog\,C > 0$, $dlog\,(\rho + \gamma) > 0$, $dlog\,w < 0$, $dlog\,r > 0$

if the uncertainty sector is relatively more labor intensive.

(2) $dlog\,U < 0$, $dlog\,C > 0$, $dlog\,(\rho + \gamma) > 0$, $dlog\,w > 0$, $dlog\,r < 0$

if it is relatively more labor intensive.

Proof. As was mentioned above, we start with the system (49). Now, limiting our attention to $dlog\ \gamma$, the proof is rather straightforward. We only write the following formulas.

$$. \quad dlog\ \rho = -\frac{\alpha\ \{\theta_R(\delta_L \lambda_{KC} + \delta_K \lambda_{LC}) - \Lambda\Theta\}}{D\Theta}\ dlog\ \gamma\ , \qquad (63)$$

$$dlog\ \rho + dlog\ \gamma = \frac{(\alpha + \beta)\ \Lambda}{D}\ dlog\ \gamma\ , \qquad (64)$$

$$dlog\,w = -\frac{\theta_{KC}\,\theta_R}{\Theta}\ (dlog\ \rho + dlog\ \gamma) \qquad (65)$$

$$dlog\,r = \frac{\theta_{LC}\,\theta_R}{\Theta}\ (dlog\ \rho + dlog\ \gamma) \qquad (66)$$

Q.E.D.

Theorem 5 is quite interesting in several aspects.

(i) U - sector may be relatively more labor intensive or more capital intensive) than C - sector. In spite of the difference of the factor intensity condition between the two sectors, an increase in price risk must lead to a contraction of U - sector and to an expansion of C - sector. This agrees with common sense.

(ii) Note that when the price risk measured by γ goes up, the bearing fee ρ may go in either direction. However, as can clearly be understood, the sign of the "combined risk burden" , which is represented by the sum of the "subjective risk burden"

($dlog\,\rho$) and the "objective risk burden" (dlog Υ) , must definitely be positive.

(iii) The above-mentioned result plays a crucial role in determining the signs of $dlog\, w$ and $dlog\, r$, as is seen in (65) and (66). In fact., the price risk increase results in a decline in the price of whichever factor is relatively more intensively used, and as a linking reaction, a rise in the price of the other factor,

(iv) Now, comparison of Theorems 3 and 5 shows that as may naturally be expected, the implications of a change in μ_U and those of a change in γ are antithetical in the case of risk aversion.

As the saying goes, a wise man keeps clear of danger. The reality, however, is different from the saying, and many men are not so wise. We have to exert all our energy for filling in such a gap between the ideal and the reality.

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8 A Unification of Three Streams of Economic Theories: General Equilibrium Theory, Theory of International Trade, and Theory of Risk and Uncertainty

This chapter, from the beginning to the end, has been an ambitious attempt to unify the three main three streams of economic theories into one. The first stream of economic theory is *general equilibrium theory,* which has been developed a great deal by a group of shining superstars including Lionel W. McKenzie, Kenneth J. Arrow, and Gerald Debreu. By combining the initials of those three surnames, we may be allowed to call this theory "the M-A-D theory." We are not quite sure of how "mad" it really is. Ay least, we could safely say that it is one of the most beautiful theories economists have ever produced over hundred years. [17)]

The second stream is *theory of international trade*, which has greatly prompted by a group of smart brains including Ronald W. Jones, Murray C. Kemp, and J. Bhagwati. By combining the initials of those three surnames, we may be allowed to say this theory "the J-M-B theory." We are not quite certain of how "smart" it truly is. At least, we could say that it is one of the smartest theories international economists ever generated for so many years. [18)]

The third stream is *theory of risk and uncertainty*, which has a very long history but, to our regret, has been treated rather lightly until recently. It is in the 1970s and afterward that a modern approach to risk and uncertainty came out into the economics profession, with superstars being Kenneth J Arrow, George A. Akerlof, Michael Spence, and George J. Stiglitz. Since the initials of those four superstars are either "A" or "S", we may be allowed to call this theory "the AA-SS theory". We are not so confident of how "great" it certainly is. We do believe, however, that it constitutes one of the most outstanding theories over so many decades, or even so many centuries. [19)]

To tell the truth, it is not an easy job to combine those three streams of economic theories into one. As the saying goes, however, life is a challenge, and an adventure as well. We believe that although this is the right way to go, it could be only the first step to go. Perhaps, there would be a variety of ways to proceed. There remain so many unsolved problems, requiring future research. In the world of uncertainty, there should be one thing certain: that is, where there is a will, there is a way. We sincerely hope that our work in this book should be continued to the next generations, thus shedding a new light for further development of the theories of games, decisions, and markets.

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Footnotes

1) For the standard treatment of the non-stochastic, two-by-two model, see Caves & Jones (1965). Jones (1965), Bhagwati J (1964) , Komiya & Amano (1972), Takayama (1972), Feenstra (2004), Ohyama (2011) , Kemp (1969) , Kemp (1976), and Krugman & Obstfeld (2007).

2) Empirical analysis has shown that many countries suffer from high price fluctuations in their exporting commodities. See Coppock (1962), Komiya & Amano (1972), Krugman & Obstfeld (2007), and Ohyama (2011).

3) The basic characteristics of the two-by-two equilibrium model under uncertainty was investigated by Batra (1974, 1975a, 1975b). His interest was mainly centered around *production* uncertainty rather than *price* uncertainty. Unfortunately, his formation of the model was fairly tedious and obscured the duality feature of the model. Although Mayer (1976) developed an interesting model similar to our model here, he was merely content to work with the *absolute* rate of change in a variable such as dK. Following the Rochester tradition led by Jones (1965) , Caves & Jones (1973), Komiya & Amano (1972), Takayama (1972), and Ohyama (2021), a more powerful mathematical tool of the *relative* rate of change such as $d\log K$ $(= dK / K)$ was efficiently employed in our paper.

4) It is assumed here that p_U is positive for any value of ε and that $\Phi(\varepsilon)$ takes on positive values for any ε. Therefore, both $E[p_U]$ and $Var[p_U]$ should be positive. For such mean-preserving spread type of shift in the distribution function , see Sandmo (1971) and Sakai (1977). .

5) The second-order condition for a regular maximum requires that the Hessian matrix of $E[V_U(\Pi_U)]$ with respect to L_U and K_U be negative definite. This condition is weaker than the condition that the Hessian matrix of $F_U(L_U, K_U)$ be negative definite, provided U- firms are risk averse. For this point, see Sakai (1977), p. 33.

6) Linear homogeneity of the production function $F_U(L_U, K_U) = U$ implies $F_U(a_{LU}, a_{KU}) = 1$. Besides, since the first derivatives F_{LU} and F_{KU} are homogeneous of degree zero, it follows from eq. 10.9 that $F_{LU}(a_{LU}, a_{KU}) / F_{LU}(a_{LU}, a_{KU}) = w/r$. Hence, provided F_U is well-

behaved, these two equations yield $a_{LU} = a_{LU}(w/r)$ and $a_{KU} = a_{KU}(w/r)$. Similarly, we can obtain $a_{LC} = a_{LC}(w/r)$ and $a_{KC} = a_{KC}(w/r)$.

7) For this point, see Sandmo (1971).

8) This relation among the signs of A, Λ, and Θ is now well-established in the international trade literature. For example, see Jones (1965), Takayama (1972), and Ohyama (2021).

9) The absolute risk aversion function of the form $R_U(\Pi_U) \equiv - V_U''(\Pi_C) / V_U'(\Pi_C)$ was

first introduced by Pratt (1964) and Arrow (1970)..

10) For the risk-free system of equations of relative change, see Jones (1965), for instance.

11) Even in the risk-free case, factor prices may fail to be equalized when there are more factors than commodities. The merit and demerit of three-factor models in theory and history were systematically discussed by Jones (1971). In the light of such well-known result, the non-equalization of factor prices in the case of risk aversion could easily be understood if the risk bearing fee (ρ) is interpreted as the payment for a *third factor to be counted.* Also see Samuelson (1949, 1953-54).

12) This point was also noticed by Das (1977).

13) It can be shown that $dlog\, L - (\delta_L \theta_R / \Theta)\, dlog\, \rho > 0$ if $dlog\, L > 0$ (see eqs. 10.54 and 10.50). Therefore, the response of $dlog\, \rho$ to $dlog\, L$ makes a positive $dlog\, L$ smaller in the risk-free case, but not negative.

14) As a corollary of Theorem 10.2, we see that there exists a sufficiently small positive value ξ such that $dlog\, L = dlog\, K + \xi$, $dlog\, U < dlog\, C$ and the U - sector is relatively more labor intensive. Therefore, the magnification effect of factor supplies on commodity outputs cannot carry over to the case of risk aversion. It is recalled here that the magnification effect was first noticed by Jones (1965) for the risk-free case.

15) Even so, w and r increase *at different rates*, depending on the factor intensity between the two sectors. As can readily be seen by eqs. 10.61 and 19,62 above, w rises at a faster rate than r (i.e., the wage-rental ratio rises) if the U - sector is relatively more labor intensive, and conversely, if it is relatively more capital intensive.

16) Theorem 10.4 clearly implies that the magnification effect a la Jones (1965) of commodity prices on factor prices cannot be extended to the world with price uncertainty. In a sense, the introduction of uncertainty into the model may play as a sort of "trouble maker."

17) Personally speaking, Yasuhiro Sakai was once McKenzie's student at the University of Rochester. McKenzie's teaching style was very unique in that he sometimes pondered for a while with white chalk on his lips. His triumphant face after struggling to prove the existence theorem of general equilibrium by means of the powerful Kakutani fixed point theorem was an unforgettable episode in Sakai's life. So, McKenzie was rightly nicknamed "Professor Fixed Point" by many students.

18) Dr. Makoto Tawada, who received his Ph.D. degree at Australian National University, is one of Sakai's best friends in Regional Economic Association. Tawada used to tell Sakai that Jones, Kemp, and Bhagwati jointly deserve Nobel Prize in Economic Science. And Kemp himself has once told Sakai that McKenzie should be awarded the Nobel Economic Prize.

19) It was Oskar Morgenstern who recommended Sakai to do research in the theory of risk and uncertainty. By chance, Sakai's initial is no less than "S" , one of the lucky letters for risk researchers.