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Abstract

The global financial crisis leads to a growing awareness of the need for robust dynamic investment control. This study considers a consumption–investment problem for an investor with homothetic robust utility under a quadratic security market model, in which all inflation rates, interest rates, and asset risk premiums and volatilities are stochastic and predictable. Homothetic robust utility is characterized by investors' relative risk aversion and "relative ambiguity aversion." We show that the optimal portfolio is decomposed into the sum of myopic demand, intertemporal risk hedging demand, inflation risk demand, and "intertemporal ambiguity hedging demand." We obtain a loglinear approximate analytical solution to a nonlinear partial differential equation for the indirect utility function. The coefficients for this solution are provided as a system of nonlinear algebraic equations. We also present an algorithm to solve this system numerically.

JEL classification: C61, G11
Keywords: Homothetic robust utility, Approximate analytical solution, Consumption–Investment problem, Stochastic interest rate, Stochastic volatility, Inflation.

1 Introduction

The global financial crisis, which resulted in significant losses for investors, has raised an awareness of the need for robust dynamic investment control

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accounting for the Knightian uncertainty, under which the assumed probability itself cannot be specified, in addressing the investment problem. The robust utility is proposed by Hansen and Sargent [20] and studied by Anderson, Hansen, and Sargent [2] and Hansen, Sargent, Turmuhambetova, and Williams [21]. However, the robust utility does not have homotheticity, which is a desirable property of the utility function—one that constant relative risk aversion (CRRA) utility has. Maenhout [28] modifies the robust utility so that it has homotheticity, obtaining the homothetic robust (HR) utility. Subsequently, the HR utility is applied to a wide range of financial problems, including Uppal and Wang [36], Marquering and Verbeek [30], Liu, Pang, and Wang [27], and Yi, Viens, Law, and Li [39]. We consider the dynamic consumption–investment control problem for an investor with the HR utility.

In addressing the consumption–investment problem, it is crucial to set up a realistic security market model that captures actual asset price fluctuations. Many previous studies show that inflation rates, interest rates, and asset risk premiums and volatilities are stochastic and predictable,[1] which are now considered stylized facts. Hence, it is necessary to incorporate the stylized facts into our security market model. For this purpose, we model the stylized facts based on the state vector process. This approach makes the investment opportunity set stochastic due to the variation in the state vector as in Merton [31]. In the framework of Merton [31], the dynamic consumption–investment problem is attributed to the problem of solving the second-order partial differential equation (PDE), which is derived from the Hamilton–Jacobi–Bellman (HJB) equation for the indirect utility function. In the standard CRRA utility setting, the PDE is a nonhomogeneous linear equation, and it is solvable in a semi-analytical form, as shown by Batbold, Kikuchi and Kusuda [5]. However, for the HR utility setting, not only the nonhomogeneous term but also a nonlinear term appear in the PDE, resulting in the difficulty of finding an exact solution. This study thus aims to derive an approximate analytical solution for the robust dynamic consumption–investment control problem under a security market model while accounting for all stylized facts.

With respect to such a security market model, we focus on the class of quadratic security market models independently developed by Ahn, Dittmar and Gallant [1] and Leippold and Wu [25], because they can capture all the above-mentioned stylized facts in security markets and are analytically tractable. These models are used in security pricing studies such as Chen,

[1]It is evident that interest rates are stochastic and mean-reverting. Campbell [9], Campbell and Shiller [12], Fama and French [18], Poterba and Summers [33], and Hodrick [22] posit that the risk premiums of stocks are stochastic and mean-reverting, while Bollerslev, Chou, and Kroner [6], Campbell, Lo, and MacKinlay [11], and Campbell, Lettau, Malkiel, and Xu [10] show that the return volatilities of stocks are stochastic and mean-reverting.

Filipović, and Poor [16], Kim and Singleton [24], and Filipović, Gourier, and Mancini [19]. In anaylyzing the optimal consumption–investment, Batbold *et al.* [5] assume a quadratic security market model, in which all the aforementioned processes are expressed as mean-reverting stochastic processes. They analyze a dynamic consumption–investment control problem for the CRRA utility and derive a semi-analytical solution. In their quadratic model, the state vector process is modeled by the canonical form (Dai and Singleton [17]) of the multidimensional Ornstein-Uhlenbeck process, and it is assumed that the instantaneous nominal risk-free rate, instantaneous dividend rate, and instantaneous expected inflation rate are quadratic functions of the state vector and that the market price of risk and inflation volatility are affine functions of the state vector. This model also assumes that the instantaneous nominal risk-free security, default-free bonds, default-free inflation-indexed bonds, and non-bond indices are traded. Note that since the inflation rate is assumed to be stochastic, security markets would be incomplete if default-free inflation-indexed bonds were not traded. We also note that risk-free securities are default-free inflation-indexed bonds for long-term investors, as emphasized by Campbell and Viceira [14, 15].

Robust dynamic consumption–investment control problems under both the HR utility and stochastic investment opportunity sets are analyzed in numerous studies, including Maenhout [29], Liu [26], Branger, Larsen, and Munk [7], Munk and Rubtsov [32], Yi, Li, Viens, and Zeng [38], and Batbold, Kikuchi, and Kusuda [4]. However, their security market models ignore some of the above listed stylized facts. Conversely, we analyze a dynamic robust consumption–investment control problem for an investor with the HR utility, assuming the quadratic security market model of Batbold *et al.* [5]. The HR utility is characterized by investors' relative risk aversion coefficient and "relative ambiguity aversion coefficient" In the dynamic consumption–investment control problem for investors with robust utility, we determine the worst probability in the first stage and the optimal consumption–investment control in the second one.

The main results of this study are as follows. First, we obtain the optimal portfolio decomposed into the sum of four terms. The optimal portfolio decomposed into the sum of three terms, namely, myopic demand, intertemporal hedging demand, and inflation risk demand, is shown by Brennan and Xia [8], Sangvinatsos and Wachter [34], and Batbold *et al.* [5]. The fourth term can be considered to represent the demand for insurance against the ambiguity of the changes in indirect utility due to the changes in the state process, which is called intertemporal ambiguity hedging demand. To distinguish the intertemporal hedging demand from the intertemporal ambiguity hedging demand, we call the former "intertemporal risk hedging demand"

Second, we show that in the first stage of the worst probability determination, an investor values a lower price per unit investment risk compared to the market price of the risk, depending on her/his relative ambiguity aversion, and in the second stage of consumption–investment decision-making, she/he values an even lower price per unit investment risk, depending on her/his relative risk aversion. The optimal investment is thus inversely proportional to the "relative uncertainty aversion," which is the sum of relative risk aversion and relative ambiguity aversion.

Third, we apply the method of Campbell and Viceira [15] and Batbold *et al.* [4] to the nonhomogeneous and nonlinear PDE for the indirect utility function and derive an approximate analytical solution. The coefficients for this solution are provided as a system of nonlinear algebraic equations. We present an algorithm to derive a numerical solution to the system of nonlinear algebraic equations. The approximate optimal portfolio is decomposed into the sum of myopic demand, inflation hedging demand, and intertemporal uncertainty hedging demand which is the sum of intertemporal risk hedging demand and intertemporal ambiguity hedging demand. We show that all types of demand are nonlinear functions of the state vector, because the inverse matrix of volatilities is its nonlinear function. The fact that the optimal portfolio is a nonlinear function of the state vector suggests that achieving the market timing effect is not as simple as rebalancing the portfolio weight of a single risky security or index based on the business cycle. Rather, this implies that the market timing effect cannot be achieved without finely rebalancing the portfolio weights among the risky securities in response to the various phases created by the multidimensional state vector. The importance of this timing effect is pointed out by Batbold *et al.* [5] in the case of the CRRA utility. Because we have access to the approximate analytical solution for the optimal portfolio, we can implement the aforementioned complex portfolio rebalancing to achieve market timing effects as long as we can precisely estimate the parameters and the latent factor process in the quadratic security market model. We can then precisely estimate them using the quasi-maximum likelihood method based on nonlinear filtering; however, we will develop this in a future study.

The remainder of this paper is organized as follows. In Section 2, we explain the quadratic security market model and the real budget constraint. In Section 3, we introduce the investor's robust consumption–investment control problem. In Section 4, we derive the PDE for indirect utility. In Section 5, we derive the approximate analytical solution. In Section 6, we summarize this study and address future research issues.

2 Quadratic Security Market Model and Real Budget Constraint

We consider frictionless US markets over the time span $[0, \infty)$. The investors' common subjective probability and information structure are modeled by a complete filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathrm{P})$, where $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,\infty)}$ is the natural filtration generated by an N-dimensional standard Brownian motion B_t. We indicate the expectation operator under P with E, and the conditional expectation operator given $\mathcal{F}_t$ with E_t.

There are markets for consumption goods and securities at every date $t \in [0, \infty)$, and the consumer price index p_t is observed. The traded securities are the instantaneously nominal risk-free security called the *money market account*; a continuum of zero-coupon bonds whose maturity dates are $(t, t+\tau^*]$, each of which has a one USD payoff at maturity; a continuum of zero-coupon inflation-indexed bonds whose maturity dates are $(t, t+\tau^*]$, each of which has a p_T USD payoff at maturity T; and J types of non-bond main indices (stock indices, REIT indices, *etc.*).[2]

At every date t, P_t, P_t^T, P_{It}^T, and S_t^j denote the USD prices of the money market account, zero-coupon bond with maturity date T, zero-coupon inflation-indexed bond with maturity date T, and j-th index, respectively. Let A' and I_N denote the transpose of A and the $N \times N$ identity matrix, respectively.

We assume the following quadratic latent factor security market model.

Assumption 1. *1. State vector process X_t satisfies the following SDE:*

$$dX_t = -\mathcal{K} X_t \, dt + I_N \, dB_t, \tag{2.1}$$

where $\mathcal{K}$ is an $N \times N$ positive lower triangular constant matrix diagonalized as

$$L = Q^{-1} \mathcal{K} Q = \begin{pmatrix} l_1 & 0 & \cdots & 0 \\ 0 & l_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \cdots & l_N \end{pmatrix}, \tag{2.2}$$

where $l_1, l_2, \cdots, l_N > 0$.

2. The market price Λ_t of risk is an affine function of the state vector, and the instantaneous nominal risk-free rate r_t is a quadratic function of the state vector.

$$\Lambda_t = \lambda + \Lambda X_t, \tag{2.3}$$

$$r_t = \rho_0 + \rho' X_t + \frac{1}{2} X_t' \mathcal{R} X_t, \tag{2.4}$$

where Λ is such that $\mathcal{K} + \Lambda$ is regular, and $\mathcal{R}$ is a positive-definite symmetric matrix.

[2]Defaultable bonds can be included into our security market model. In this case, we would model defaultable bond prices based on the quadratic modeling of intensity by Chen, Fillipović, and Poor [16] to ensure consistency with our model. However, we do not consider defaultable bonds to reduce complexity.

3. *The consumer price index p_t satisfies*

$$\frac{dp_t}{p_t} = i_t \, dt + \Lambda'_{It} dB_t, \qquad p_0 = 1, \tag{2.5}$$

where i_t and Λ'_{It} are given by

$$\begin{aligned} i_t &= \iota_0 + \iota' X_t + \frac{1}{2} X_t' \mathcal{I} X_t, & (2.6)\\ \Lambda_{It} &= \lambda_I + \Lambda_I X_t, & (2.7) \end{aligned}$$

where $\mathcal{I}$ is a positive-definite symmetric matrix such that a matrix $\bar{\mathcal{R}}$ defined by

$$\bar{\mathcal{R}} = \mathcal{R} - \mathcal{I} + \Lambda_I' \Lambda + \Lambda' \Lambda_I, \tag{2.8}$$

is positive-definite.

4. *The dividend rate of the j-th index is given by:*

$$D_t^j = \left(\delta_{0j} + \delta_j' X_t + \frac{1}{2} X_t' \Delta_j X_t \right) \exp \left(\sigma_{0j} t + \sigma_j' X_t + \frac{1}{2} X_t' \Sigma_j X_t \right), \tag{2.9}$$

where $(\delta_{0j}, \delta_j, \Delta_j)$ is such that Δ_j is a positive definite symmetric matrix and[3]

$$\delta_{0j} \geq \frac{1}{2} \delta_j' \Delta_j^{-1} \delta_j. \tag{2.10}$$

Note that $\delta_{0j} + \delta_j' X_t + \frac{1}{2} X_t' \Delta_j X_t$ is the instantaneous dividend rate.

5. *Markets are complete.*

2.1 No-Arbitrage Rate of Return on Securities

We define $\bar{\Lambda}_t$ and $\bar{r}_t$ by

$$\begin{aligned} \bar{\Lambda}_t &= \Lambda_t - \Lambda_{It}, & (2.11)\\ \bar{r}_t &= r_t - i_t + \Lambda'_{It} \Lambda_t, & (2.12) \end{aligned}$$

where $\bar{\Lambda}_t$ is the real market price of risk.

Note that the real market price of risk is an affine function of X_t and $\bar{r}_t$ is a quadratic function of X_t:

$$\begin{aligned} \bar{\Lambda}_t &= \bar{\lambda} + \bar{\Lambda} X_t, & (2.13)\\ \bar{r}_t &= \bar{\rho}_0 + \bar{\rho}' X_t + \frac{1}{2} X_t' \bar{\mathcal{R}} X_t, & (2.14) \end{aligned}$$

[3] Conditions (2.10) ensure that dividend rates are non-negative processes.

where $\bar{\mathcal{R}}$ is given by eq. (2.8) and

$$\begin{aligned}\bar{\lambda} &= \lambda - \lambda_I, && (2.15)\\ \bar{\Lambda} &= \Lambda - \Lambda_I, && (2.16)\\ \bar{\rho}_0 &= \rho_0 - \iota_0 + \lambda_I'\lambda, && (2.17)\\ \bar{\rho} &= \rho - \iota + \Lambda'\lambda_I + \Lambda_I'\lambda. && (2.18)\end{aligned}$$

Let $\tau = T - t$ denote the time to maturity of bond P_t^T. First, we show the no-arbitrage rate of return on securities.

Lemma 1. *Under Assumption 1, if there is no arbitrage, then the security return rate processes satisfy the following:*

1. *The money market account:*

$$\frac{dP_t}{P_t} = r_t\,dt, \qquad P_0 = 1. \tag{2.19}$$

2. *The default-free bond with time τ to maturity:*

$$\frac{dP_t^T}{P_t^T} = \big(r_t + (\sigma(\tau) + \Sigma(\tau)X_t)'\Lambda_t\big)\,dt + (\sigma(\tau)+\Sigma(\tau)X_t)'\,dB_t, \qquad P_T^T = 1, \tag{2.20}$$

where

$$\frac{d\Sigma(\tau)}{d\tau} = \Sigma(\tau)^2 - (\mathcal{K}+\Lambda)'\Sigma(\tau) - \Sigma(\tau)(\mathcal{K}+\Lambda) - \mathcal{R}, \qquad \Sigma(0) = 0, \tag{2.21}$$

$$\frac{d\sigma(\tau)}{d\tau} = -(\mathcal{K} + \Lambda - \Sigma(\tau))'\sigma(\tau) - (\Sigma(\tau)\lambda + \rho), \qquad \sigma(0) = 0. \tag{2.22}$$

3. *The default-free inflation-indexed bond with time τ to maturity:*

$$\begin{aligned}\frac{dP_{It}^T}{P_{It}^T} &= \Big(r_t + \Big(\sigma_I(\tau) + \lambda_I + (\Sigma_I(\tau) + \Lambda_I)X_t\Big)'\Lambda_t\Big)\,dt\\ &\quad + \Big(\sigma_I(\tau) + \lambda_I + (\Sigma_I(\tau) + \Lambda_I)X_t\Big)'\,dB_t, \qquad P_{IT}^T = p_T,\end{aligned} \tag{2.23}$$

where

$$\frac{d\Sigma_I(\tau)}{d\tau} = \Sigma_I(\tau)^2 - (\mathcal{K}+\bar{\Lambda})'\Sigma_I(\tau) - \Sigma_I(\tau)(\mathcal{K}+\bar{\Lambda}) - \bar{\mathcal{R}}, \qquad \Sigma_I(0) = 0, \tag{2.24}$$

$$\frac{d\sigma_I(\tau)}{d\tau} = -(\mathcal{K} + \bar{\Lambda} - \Sigma_I(\tau))'\sigma_I(\tau) - (\Sigma_I(\tau)\bar{\lambda} + \bar{\rho}), \qquad \sigma_I(0) = 0. \tag{2.25}$$

4. The j-th index:

$$\frac{dS_t^j + D_t^j dt}{S_t^j} = \left(r_t + (\sigma_j + \Sigma_j X_t)'\Lambda_t\right) dt + (\sigma_j + \Sigma_j X_t)' \, dB_t, \quad (2.26)$$

where

$$\Sigma_j^2 - (\mathcal{K} + \Lambda)'\Sigma_j - \Sigma_j(\mathcal{K} + \Lambda) + \Delta_j - \mathcal{R}_j = 0,^4 \quad (2.27)$$

$$\sigma_j = (\mathcal{K} + \Lambda - \Sigma_j)'^{-1}(\delta_j - \rho - \Sigma_j \lambda). \quad (2.28)$$

Proof. See Appendix A.1 in Batblod *et al.* [5]. □

Remark 1. *We show in the real budget constraint in eq. (2.30) in Lemma 2 that $\bar{r}_t$ in eq. (2.12) is the instantaneous real risk-free rate and that eq. (2.12) is a generalized Fisher equation. Therefore, $(\bar{\rho}_0, \bar{\rho}, \bar{\mathcal{R}})$ is the real rate version of $(\rho_0, \rho, \mathcal{R})$. From this, we observe that $(\Sigma_I(\tau), \sigma_I(\tau))$ in eqs. (2.24) and (2.25) is the real rate version of $(\Sigma(\tau), \sigma(\tau))$ in eqs. (2.21) and (2.22).*

2.2 Real Budget Constraint

Let Φ_t^j denote the portfolio weight on the j-th index. Regarding the default-free bond, let $\varphi_t(\tau)$ and φ_t^I denote the densities of the portfolio weights on the default-free bond and the default-free inflation-indexed bond with τ-time to maturity. We assume that the functional space of the densities of the portfolio weights on the bonds includes the set of distributions.

Let c_t denote the consumption rate and define Ψ_t as:

$$\Psi_t = \int_0^{\tau^*} \left\{\varphi_t(\tau)(\sigma(\tau)+\Sigma(\tau)X_t)+\varphi_t^I(\tau)(\sigma_I(\tau)+\Sigma_I(\tau)X_t)\right\} d\tau + \sum_{j=1}^{J} \Phi_t^j(\sigma_j+\Sigma_j X_t) - \Lambda_{It}. \quad (2.29)$$

$u_t = (c_t, \Psi_t)$ denotes a control.

Let W_t denote the real wealth process. Next, the investor's real budget constraint is expressed in the following lemma.

Lemma 2. *Under Assumption 1 and the no-arbitrage condition, given a control u_t, the budget constraint satisfies*

$$\frac{dW_t}{W_t} = \left(\bar{r}_t + \Psi_t'\bar{\Lambda}_t - \frac{c_t}{W_t}\right) dt + \Psi_t' \, dB_t. \quad (2.30)$$

Proof. See Appendix A.2 in Batbold *et al.* [5]. □

[4] Kikuchi [23] provides a sufficient condition under which the unique solution to this Riccatti algebraic equation is positive-definite.

Remark 2. *Note that the real budget constraint in eq. (2) is expressed by the instantaneous real risk-free rate, the real market price of risk, investment control, and consumption-wealth ratio. The real budget constraint stands for the instantaneous real rate of return on investment. Eq. (2.30) shows that increasing risky asset investment in the sense of Ψ_t increases the investment risk, while the real expected excess return increases in proportion to the real market price of risk. That is, the real market price of risk is interpreted as the price per unit investment risk for all investors.*

Let $\mathbb{X}_t' = (W_t, X_t')$ and let $W_0 > 0$. We call the control satisfying budget constraint (2.30) with initial state $\mathbf{X}_0 = (W_0, X_0')'$ the admissible control and denote by $\mathcal{B}(\mathbf{X}_0)$ the set of admissible controls.

3 Robust Consumption–Investment Control Problem

Here, we introduce the HR utility, and show a robust consumption–investment control problem.

3.1 Homothetic Robust Utility

An investor with the HR utility regards probability P as the most likely probability (hereafter, base probability) but cannot deny other probabilities as the real probability. Thus, the investor assumes set $\mathbb{P}$ of all equivalent probability measures[5] as probability candidates. In conformity with Girsanov's theorem, any equivalent probability measure is characterized by a measurable process ξ_t with Novikov's integrable condition as the following Radon-Nikodým derivative:

$$\frac{d\mathrm{P}^\xi}{d\mathrm{P}} = \exp\biggl(\int_0^\infty \xi_t\, dB_t - \frac{1}{2}\int_0^\infty \xi_t'\xi_t dt\biggr).$$

Therefore, she/he decides the worst probability, which minimizes her/his utility among set $\mathbb{P}$ of equivalent probability measures for every consumption plan. That is, she/he rationally determines the worst probability for every consumption plan, considering deviations from the base probabilities, as follows:

$$U(c) = \inf_{\mathrm{P}^\xi \in \mathbb{P}} \mathrm{E}^\xi \left[\int_0^\infty e^{-\beta t}\left(\frac{c_t^{1-\gamma}}{1-\gamma} + \frac{(1-\gamma)U_t^\xi(c)}{2\theta}\,\xi_t'\xi_t\right) dt\right], \tag{3.1}$$ [6]

[5]A probability measure $\tilde{\mathrm{P}}$ is said to be an equivalent probability measure of P if and only if $\mathrm{P}(A) = 0 \Leftrightarrow \tilde{\mathrm{P}}(A) = 0$.

[6]This representation of the HR utility utilizes the following expression of the discounted relative entropy process by Skiadas [35].

$$\mathcal{E}_t^\xi = \frac{1}{2}\mathrm{E}_t^\xi\left[\int_t^\infty e^{-\beta(s-t)}\xi_s'\xi_s ds\right].$$

where E^{ξ} is the expectation under P^{ξ}, β is the subjective discount rate, γ is the relative risk aversion coefficient, θ is a positive constant, and U_t^{ξ} is the utility process defined recursively as follows:

$$U_t^{\xi}(c) = \mathrm{E}_t^{\xi}\left[\int_t^{\infty} e^{-\beta(s-t)}\left(\frac{c_s^{1-\gamma}}{1-\gamma} + \frac{(1-\gamma)U_s^{\xi}(c)}{2\theta}\,\xi_s'\xi_s\right) ds\right]. \tag{3.2}$$

Remark 3. *In eq. (3.1), as $\theta \searrow 0$, worst probability ξ^* converges to zero, that is, P^{ξ^*} converges to* P *and the HR utility degenerates into the CRRA utility. The HR utility can be interpreted as a generalization of the CRRA utility, which has been considered the standard utility in economics, to an environment of Knightian uncertainty, while retaining homotheticity.*

We call θ the relative ambiguity aversion coefficient.

3.2 Robust Consumption–Investment Problem

Assumption 2. *The investor maximizes the following HR utility over an infinite time horizon under budget constraint (2.30):*

The investor's indirect utility function is recursively defined by

$$J^{\xi}(\mathbb{X}_t) = \mathrm{E}_t^{\xi}\left[\int_t^{\infty} e^{-\beta(s-t)}\left\{\frac{c_s^{1-\gamma}}{1-\gamma} + \frac{(1-\gamma)J^{\xi}(\mathbb{X}_s)}{2\theta}\,\xi_s'\xi_s\right\} ds\right]. \tag{3.3}$$

The investor's consumption–investment problem and the value function are defined by

$$V(\mathbb{X}_0) = \sup_{u\in\mathcal{B}(\mathbb{X}_0)} \inf_{\mathrm{P}^{\xi}\in\mathbb{P}} J^{\xi}(\mathbb{X}_0). \tag{3.4}$$

4 Worst Probability and Optimal Control

Here, we derive the worst probability and optimal control and then the PDE for indirect utility.

4.1 Worst Probability

As the standard Brownian motion under P^{ξ} is given by

$$B_t^{\xi} = B_t - \int_0^t \xi_s\, ds,$$

the SDE (2.1) for the state vector under P^{ξ} is rewritten as

$$d\mathbb{X}_t = \left(\begin{pmatrix} W_t(\bar{r}_t + \Psi_t'\bar{\Lambda}_t) - c_t \\ -\mathcal{K}X_t \end{pmatrix} + \begin{pmatrix} W_t\Psi_t' \\ I_N \end{pmatrix}\xi_t\right) dt + \begin{pmatrix} W_t\Psi_t' \\ I_N \end{pmatrix} dB_t^{\xi}. \tag{4.1}$$

A nonrecurisuve representation is provided by Balter and Horvath [3].

Therefore, the HJB equation for problem (3.4) is expressed as

$$\sup_{u\in\mathcal{B}(\mathbb{X}_0)} \inf_{\mathrm{P}^\xi\in\mathbb{P}} \left\{ \begin{pmatrix} W_t\left(\bar{r}_t+\Psi_t'\bar{\Lambda}_t\right)-c_t \\ -\mathcal{K}X_t \end{pmatrix}' \begin{pmatrix} J^\xi_W \\ J^\xi_X \end{pmatrix} + \frac{1}{2}\operatorname{tr}\left[\begin{pmatrix} W_t\Psi_t' \\ I_N \end{pmatrix}\begin{pmatrix} W_t\Psi_t' \\ I_N \end{pmatrix}'\begin{pmatrix} J^\xi_{WW} & J^\xi_{WX} \\ J^\xi_{XW} & J^\xi_{XX} \end{pmatrix}\right] \right.$$
$$\left. -\beta J^\xi + \frac{c_t^{1-\gamma}}{1-\gamma} + \frac{(1-\gamma)J^\xi}{2\theta}\xi_t'\xi_t + \xi_t'\begin{pmatrix} W_t\Psi_t' \\ I_N \end{pmatrix}'\begin{pmatrix} J^\xi_W \\ J^\xi_X \end{pmatrix}\right\} = 0, \quad (4.2)$$

$$\text{s.t.} \quad \lim_{T\to\infty} \mathrm{E}[e^{-\beta T}J^\xi(\mathbb{X}_T)] = 0.$$

It is straghtforward to see that worst probability P^{ξ^*} satisfies

$$\xi_t^* = -\frac{\theta}{(1-\gamma)J^*}\begin{pmatrix} W_t\Psi_t' \\ I_N \end{pmatrix}'\begin{pmatrix} J^*_W \\ J^*_X \end{pmatrix}, \quad (4.3)$$

where J^* is the abbreviation of the indirect utility J^{ξ^*} under the worst probability.

Remark 4. *Real budget constraint (2.30) under the worst probability is rewritten as*

$$\frac{dW_t}{W_t} = \left\{\bar{r}_t + \Psi_t'\left(\bar{\Lambda}_t - \frac{\theta}{(1-\gamma)J^*}\begin{pmatrix} W_t\Psi_t' \\ I_N \end{pmatrix}'\begin{pmatrix} J^*_W \\ J^*_X \end{pmatrix}\right) - \frac{c_t}{W_t}\right\}dt + \Psi_t'\,dB_t^{\xi^*}. \quad (4.4)$$

In eq. (4.4), the terms in parentheses that are subject to the inner product of investment control Ψ_t can be interpreted as the price per unit investment risk under the worst probability for an investor with relative ambiguity aversion θ. As shown by Remark 2, the price per unit investment risk when ambiguity is not considered is the market price of risk common to all investors, whereas the price per unit investment risk under the worst probability for investors with the HR utility depends on the relative ambiguity aversion and varies across investors. Eq. (4.4) shows that a more ambiguity averse investor values a lower price per unit investment risk than the market price of risk under the worst probability, meaning she/he values a lower real expected excess rate of return on the wealth process under the worst probability.

Substituting P^* into HJB equation (4.2) yields

$$\sup_{u\in\mathcal{B}(X_0)} \left[\begin{pmatrix} W_t\left(\bar{r}_t+\Psi_t'\bar{\Lambda}_t\right)-c_t \\ -\mathcal{K}X_t \end{pmatrix}' \begin{pmatrix} J^*_W \\ J^*_X \end{pmatrix} + \frac{1}{2}\operatorname{tr}\left[\begin{pmatrix} W_t\Psi_t' \\ I_N \end{pmatrix}\begin{pmatrix} W_t\Psi_t' \\ I_N \end{pmatrix}'\begin{pmatrix} J^*_{WW} & J^*_{WX} \\ J^*_{XW} & J^*_{XX} \end{pmatrix}\right] \right.$$
$$\left. -\beta J^* + \frac{c^{1-\gamma}}{1-\gamma} - \frac{\theta}{2(1-\gamma)J^*}\begin{pmatrix} J^*_W \\ J^*_X \end{pmatrix}'\begin{pmatrix} W_t\Psi_t' \\ I_N \end{pmatrix}\begin{pmatrix} W_t\Psi_t' \\ I_N \end{pmatrix}'\begin{pmatrix} J^*_W \\ J^*_X \end{pmatrix}\right] = 0. \quad (4.5)$$

4.2 Optimal Control under the Worst Probability

It is easy to see that optimal control $u_t^* = (c_t^*, \Psi_t^*)$ in HJB equation (4.5) satisfies

$$
\begin{aligned}
c_t^* &= J_W^{-\frac{1}{\gamma}}, &(4.6)\\
\Psi_t^* &= \mathcal{T}_t \left(\bar{\Lambda}_t + \frac{J_{XW}}{J_W} + \frac{\theta}{\gamma - 1} \frac{J_X}{J} \right), &(4.7)
\end{aligned}
$$

where T_t is given by

$$
\mathcal{T}_t = \left(- \frac{W_t^* J_{WW}}{J_W} + \theta \frac{W_t^* J_W}{(1-\gamma)J} \right)^{-1}. \quad (4.8)
$$

When $N = 4$, we consider the example of an investor investing in a 10-year default-free bond $P_t(10)$, a 10-year default-free inflation-indexed bond $P_{It}(10)$, a market-capitalization-weighted stock index S_t^1, and a market-capitalization-weighted REIT index S_t^2, in addition to the money market account. We use the following notation:

$$
\Phi_t = \begin{pmatrix} \Phi_t(10) \\ \Phi_{It}(10) \\ \Phi_t^1 \\ \Phi_t^2 \end{pmatrix}, \qquad \Sigma_t(X_t) = \begin{pmatrix} (\sigma(10) + \Sigma(10)X_t)' \\ (\sigma_I(10) + \Sigma_I(10)X_t)' \\ (\sigma_1 + \Sigma_1 X_t)' \\ (\sigma_2 + \Sigma_2 X_t)' \end{pmatrix}. \quad (4.9)
$$

Then, since eq. (2.29) leads to $\Psi_t = \Sigma_t(X_t)'\Phi_t - \Lambda_{It}$, it follows from eq. (4.7) that optimal portfolio weights Φ_t^* on risky securities are given by

$$
\begin{aligned}
\Phi_t^* = {} & \mathcal{T}_t \Sigma_t(X_t)'^{-1} \bar{\Lambda}_t + \mathcal{T}_t \Sigma_t(X_t)'^{-1} \frac{\partial}{\partial X_t} \log J_W \\
& + \frac{\theta}{\gamma - 1} \mathcal{T}_t \Sigma_t(X_t)'^{-1} \frac{\partial}{\partial X_t} \log((1-\gamma)J) + \Sigma_t(X_t)'^{-1} \Lambda_{It}. \quad (4.10)
\end{aligned}
$$

Remark 5. *The optimal portfolio is decomposed into the sum of four terms. The first term myopically pursues the market price of risk, which is the reward for investing in risky assets, without considering the risk of the changes in the indirect utility due to state process changes, and is called myopic demand. The derivative in the second term is the rate of the increase in the marginal indirect utility per unit of increase in the state process. Considering that the marginal indirect utility is a decreasing function, an increase in marginal indirect utility implies a decrease in indirect utility; thus, the second term represents the demand for insurance against the risk of changes in indirect utility due to state process changes, and is called intertemporal hedging demand.*[7] *The fourth term insures inflation risk, and we call it inflation hedging demand. The optimal portfolio decomposed into the sum of*

[7] This interpretation has already been made by Wachter [37].

these three terms is presented by Brennan and Xia [8], Sangvinatsos and Wachter [34], and Batbold et al. [5]. Considering that γ is assumed to be greater than one, the derivative in the third term is the rate of the decrease in indirect utility per unit of increase in the state process. As the third term is proportional to the relative ambiguity aversion coefficient, it can be interpreted as representing the demand for insurance against the ambiguity of the changes in indirect utility due to state process changes, and we call it intertemporal ambiguity hedging demand.

4.3 PDE for the Indirect Utility Function

The consumption-related terms in HJB equation (4.5) are computed as

$$-c_t^* J_W + \frac{c_t^{*1-\gamma}}{1-\gamma} = \frac{\gamma}{1-\gamma} J_W^{1-\frac{1}{\gamma}}. \tag{4.11}$$

The investment-related terms in HJB equation (4.5) are computed as

$$\begin{aligned} W_t^* J_W \bar{\Lambda}_t' \Psi_t^* &+ \frac{1}{2} \operatorname{tr}\left[\begin{pmatrix} W_t^*(\Psi_t^*)' \\ I_N \end{pmatrix} \begin{pmatrix} W_t^*(\Psi_t^*)' \\ I_N \end{pmatrix}' \begin{pmatrix} J_{WW} & J_{WX} \\ J_{XW} & J_{XX} \end{pmatrix} \right] \\ &- \frac{\theta}{2(1-\gamma)J} \begin{pmatrix} J_W \\ J_X \end{pmatrix}' \begin{pmatrix} W_t^*(\Psi_t^*)' \\ I_N \end{pmatrix} \begin{pmatrix} W_t^*(\Psi_t^*)' \\ I_N \end{pmatrix}' \begin{pmatrix} J_W \\ J_X \end{pmatrix} \\ &= \frac{1}{2} \operatorname{tr}\left[J_{XX}\right] - \frac{\theta}{2(1-\gamma)J} J_X' J_X - \frac{\pi_t' \pi_t}{2 W_t^{*2} \left(J_{WW} - \frac{\theta J_W^2}{(1-\gamma)J} \right)}, \end{aligned} \tag{4.12}$$

where

$$\pi_t = -W_t^* J_W \left(\bar{\Lambda}_t + \frac{J_{XW}}{J_W} + \frac{\theta}{\gamma - 1} \frac{J_X}{J} \right). \tag{4.13}$$

By substituting optimal control (4.6) and (4.7) into HJB equation (4.5) and using eqs. (4.11) and (4.12), the following PDE for J is obtained:

$$\begin{aligned} \frac{1}{2}\operatorname{tr}\left[J_{XX}\right] &- \frac{\theta}{2(1-\gamma)J} J_X' J_X - \frac{\pi_t' \pi_t}{2 W_t^{*2} \left(J_{WW} - \frac{\theta J_W^2}{(1-\gamma)J} \right)} \\ &+ W_t^* \bar{r}_t J_W - X_t' \mathcal{K}' J_X + \frac{\gamma}{1-\gamma} J_W^{1-\frac{1}{\gamma}} - \beta J = 0. \end{aligned} \tag{4.14}$$

From the above PDE, we conjecture that the indirect utility function takes the following form:

$$J(\mathbb{X}_t) = \frac{W_t^{1-\gamma}}{1-\gamma} \big(G(X_t)\big)^{\gamma}. \tag{4.15}$$

By inserting eq. (4.7) and the partial derivatives of J into the PDE (4.14), we obtain the following proposition.

Proposition 1. *Under Assumptions 1 and 2 and the no-arbitrage condition, the indirect utility function, optimal consumption, and optimal investment for problem (3.4) satisfy eqs. (4.15), (4.16), and (4.17), respectively. $G(t, X_t)$ constituting the indirect utility function is a solution of PDE (4.18).*

$$c_t^* = \frac{W_t^*}{G(X_t)}, \tag{4.16}$$

$$\Psi_t^* = \frac{1}{\gamma+\theta}\bar{\Lambda}_t + \left(1 - \frac{1}{\gamma+\theta}\right)\frac{\gamma}{\gamma-1}\frac{G_X(X_t)}{G(X_t)}, \tag{4.17}$$

$$\frac{1}{2}\,\mathrm{tr}\left[\frac{G_{XX}}{G}\right] + \frac{\theta}{2(\gamma-1)(\gamma+\theta)}\frac{G_X'}{G}\frac{G_X}{G} + \left(-\mathcal{K}X_t + \frac{\gamma+\theta-1}{\gamma+\theta}\bar{\Lambda}_t\right)'\frac{G_X}{G} + \frac{1}{G} - \frac{\gamma-1}{2\gamma(\gamma+\theta)}\bar{\Lambda}_t'\bar{\Lambda}_t - \frac{\gamma-1}{\gamma}\bar{r}_t - \frac{\beta}{\gamma} = 0. \tag{4.18}$$

Proof. See Appendix A.1. □

As shown in the proof of Proposition 1, $\mathcal{T}_t$ in eq. (4.8) is expressed as

$$\mathcal{T}_t = \frac{1}{\gamma+\theta}. \tag{4.19}$$

The reciprocal of the relative risk aversion is called relative risk tolerance. In the following, we call the sum of the relative risk aversion and the relative ambiguity aversion "relative uncertainty aversion" and the reciprocal of the relative uncertainty aversion "relative uncertainty tolerance." Eq. (4.19) shows that $\mathcal{T}_t$ is the relative uncertainty tolerance.

Remark 6. *In Remark 4, we show that a more ambiguity averse investor values a lower price per unit investment risk than the market price of risk under the worst probability; thus, the investor values a lower real expected excess rate of return on the wealth process under the worst probability. Eq. (4.19) shows that, in the first stage of worst probability determination, an investor with the HR utility values a lower price per unit investment risk than the market price of the risk, depending on her/his relative ambiguity aversion, and in the second stage of consumption–investment decision-making, values an even lower price per unit investment risk, depending on her/his relative risk aversion.*

Consider the optimal portfolio for the same example in Section 4.2. As shown in the proof of Proposition 1, the optimal portfolio in (4.10) is rewritten as

$$\Phi_t^* = \frac{1}{\gamma+\theta}\Sigma_t(X_t)'^{-1}\bar{\Lambda}_t + \frac{\gamma}{\gamma+\theta}\Sigma_t(X_t)'^{-1}\frac{G_X}{G} + \frac{\gamma}{\gamma+\theta}\frac{\theta}{\gamma-1}\Sigma_t(X_t)'^{-1}\frac{G_X}{G} + \Sigma_t(X_t)'^{-1}\Lambda_{It}. \tag{4.20}$$

The intertemporal risk hedging demand and the intertemporal ambiguity hedging demand are combined into

$$\Phi_t^* = \frac{1}{\gamma+\theta}\Sigma_t(X_t)'^{-1}\bar{\Lambda}_t + \left(1 - \frac{1}{\gamma+\theta}\right)\frac{\gamma}{\gamma-1}\Sigma_t(X_t)'^{-1}\frac{G_X}{G} + \Sigma_t(X_t)'^{-1}\Lambda_{It}. \tag{4.21}$$

Hereafter, we refer to the second term in the above equation as "intertemporal uncertainty hedging demand."

5 Approximate Analytical Solution

Here, we apply the loglinear approximation method developed by Campbell and Viceira [15] and Batbold *et al.* [4], and then obtain an approximate analytical solution.

5.1 Loglinear Approximation

In PDE (4.18), we have not only the nonhomogeneous term $\frac{1}{G}$, but also the nonlinear term $\frac{\theta}{2(\gamma-1)(\gamma+\theta)}\frac{G'_X}{G}\frac{G_X}{G}$. Campbell and Viceira [15] note that, in a nonhomogeneous linear PDE, the nonhomogeneous term is equal to the stable consumption–wealth ratio. They perform the loglinear approximation of the nonhomogeneous term and derive an approximate analytical solution. They perform the loglinear approximation around $\mathrm{E}[\log \frac{c_t^*}{W_t^*}]$, but the expected value depends on the time variable. Batbold *et al.* [4] use a loglinear approximation around $\lim_{t\to\infty}\mathrm{E}[\log \frac{c_t^*}{W_t^*}]$. In this study, we follow the loglinear approximation of Batbold *et al.* [4] as follows:

$$\frac{1}{G(X_t)} \approx g_0 - g_1 \log G(X_t), \tag{5.1}$$

where

$$g_0 = g_1(1 - \log g_1), \tag{5.2}$$

$$g_1 = \exp\Big(\lim_{t\to\infty}\mathrm{E}\Big[\log\Big(\frac{c_t^*}{W_t^*}\Big)\Big]\Big). \tag{5.3}$$

In PDE (4.18), approximating nonhomogeneous term $\frac{1}{G}$ by eq. (5.1) and inserting eqs. (2.11) and (2.12) into $\bar{\Lambda}_t$ and $\bar{r}_t$, respectively, leads to the following approximate PDE:

$$\frac{1}{2}\operatorname{tr}\left[\frac{G_{XX}}{G}\right]+\frac{\theta}{2(\gamma-1)(\gamma+\theta)}\frac{G_X'}{G}\frac{G_X}{G}$$
$$-\left\{\mathcal{K}X_t+\frac{\gamma+\theta-1}{\gamma+\theta}\left(\bar{\lambda}+\bar{\Lambda}X_t\right)\right\}'\frac{G_X}{G}-g_1\log G$$
$$+g_0-\frac{\gamma-1}{2\gamma(\gamma+\theta)}(\bar{\lambda}+\bar{\Lambda}X_t)'(\bar{\lambda}+\bar{\Lambda}X_t)-\frac{\gamma-1}{\gamma}\left(\bar{\rho}_0+\bar{\rho}'X_t+\frac{1}{2}X_t'\bar{\mathcal{R}}X_t\right)-\frac{\beta}{\gamma}=0. \tag{5.4}$$

An analytical solution to PDE (5.4) is expressed as:

$$G(X_t)=\exp\left(\bar{a}+a'X_t+\frac{1}{2}X_t'AX_t\right), \tag{5.5}$$

where $A(\tau)$ is a symmetric matrix.

Then,

$$g_1=\exp\left(-\lim_{t\to\infty}\mathrm{E}\left[\log G(X_t)\right]\right)=\exp\left(\lim_{t\to\infty}\left[-\bar{a}-a'\mathrm{E}[X_t]-\frac{1}{2}\mathrm{E}[X_t'AX_t]\right]\right), \tag{5.6}$$

is calculated as in the following lemma.

Lemma 3. *Under Assumptions 1 and 2, g_1 is represented by the following equation with $(\bar{a},a,A)$:*

$$g_1=\exp\left(-\bar{a}-\frac{1}{2}\left(\operatorname{tr}\left[(Q^{-1})'MQ^{-1}\right]\right)\right), \tag{5.7}$$

where M is a matrix such that

$$M_{ij}=\frac{1}{l_i+l_j}(Q'AQ)_{ij}.$$

where M_{ij} and $(Q'AQ)_{ij}$ are the (i,j)-th element of M and $Q'AQ$, respectively.

Proof. See Appendix A.2. □

5.2 Approximate Analytical Solution

By substituting G and its derivatives into PDE (5.4) and by paying attention to $A'=A$ and

$$X'\left(\mathcal{K}+\frac{\gamma+\theta-1}{\gamma+\theta}\bar{\Lambda}\right)'AX=X'A\left(\mathcal{K}+\frac{\gamma+\theta-1}{\gamma+\theta}\bar{\Lambda}\right)X,$$

we obtain

$$\frac{1}{2}\operatorname{tr}\left[aa' + A + aX_t'A + AX_ta' + AX_tX_t'A\right] + \frac{\theta}{2(\gamma-1)(\gamma+\theta)}\left(a' + X_t'A\right)\left(a + AX_t\right)$$
$$+\left\{-\frac{\gamma+\theta-1}{\gamma+\theta}\bar{\lambda} - \left(\mathcal{K} + \frac{\gamma+\theta-1}{\gamma+\theta}\bar{\Lambda}\right)X_t\right\}' a - \frac{\gamma+\theta-1}{\gamma+\theta}\bar{\lambda}'AX_t$$
$$-\frac{1}{2}X_t'\left(\mathcal{K} + \frac{\gamma+\theta-1}{\gamma+\theta}\bar{\Lambda}\right)' AX_t - \frac{1}{2}X_t'A\left(\mathcal{K} + \frac{\gamma+\theta-1}{\gamma+\theta}\bar{\Lambda}\right)X_t$$
$$+g_1(1-\log g_1) - g_1\left(\bar{a} + a'X_t + \frac{1}{2}X_t'AX_t\right) - \frac{\gamma-1}{2\gamma(\gamma+\theta)}\left(\bar{\lambda}_0'\bar{\lambda} + 2\bar{\lambda}_0'\bar{\Lambda}X_t + X_t'\bar{\Lambda}'\bar{\Lambda}X_t\right)$$
$$-\frac{\gamma-1}{\gamma}\left(\bar{\rho}_0 + \bar{\rho}'X_t + \frac{1}{2}X_t'\bar{\mathcal{R}}X_t\right) - \frac{\beta}{\gamma} = 0. \quad (5.8)$$

As the equation above is identical on X, the following system of algebraic equations for $(\bar{a}, a, A)$ is derived:

$$\frac{\gamma(\gamma+\theta-1)}{(\gamma-1)(\gamma+\theta)}A^2 - \left(\mathcal{K} + \frac{\gamma+\theta-1}{\gamma+\theta}\bar{\Lambda}\right)' A - A\left(\mathcal{K} + \frac{\gamma+\theta-1}{\gamma+\theta}\bar{\Lambda}\right) - g_1A - \frac{\gamma-1}{\gamma(\gamma+\theta)}\bar{\Lambda}'\bar{\Lambda} = 0, \quad (5.9)$$

$$\frac{\gamma(\gamma+\theta-1)}{(\gamma-1)(\gamma+\theta)}Aa - \mathcal{K}'a - \frac{\gamma+\theta-1}{\gamma+\theta}(A\bar{\lambda} + \bar{\Lambda}'a) - g_1a - \frac{\gamma-1}{\gamma(\gamma+\theta)}\bar{\Lambda}'\bar{\lambda} - \frac{\gamma-1}{\gamma}\bar{\rho} = 0, \quad (5.10)$$

$$\frac{1}{2}\operatorname{tr}[A] + \frac{\gamma(\gamma+\theta-1)}{2(\gamma-1)(\gamma+\theta)}a'a + \left(-\frac{\gamma+\theta-1}{\gamma+\theta}\bar{\lambda}\right)' a$$
$$+ g_1(1 - \bar{a} - \log g_1) - \frac{\gamma-1}{2\gamma(\gamma+\theta)}\lambda'\lambda - \frac{\gamma-1}{\gamma}\bar{\rho}_0 - \frac{\beta}{\gamma} = 0, \quad (5.11)$$

where g_1 is expressed by eq. (5.7).

The optimal control when the solution to PDE (4.18) is approximated by the solution to approximate PDE (5.4) is called the approximate optimal control, denoted by $(\tilde{c}^*, \tilde{\Psi}^*)$. Then, we obtain the following proposition.

Proposition 2. *Under Assumptions 1 and 2 and the no-arbitrage condition, the optimal approximate consumption and the optimal approximate investment for problem (3.4) satisfy eqs. (5.12) and (5.13), respectively.*

$$\tilde{c}_t^* = W_t \exp\left[-\left(\bar{a} + a'X_t + \frac{1}{2}X_t'AX_t\right)\right], \quad (5.12)$$
$$\tilde{\Psi}_t^* = \frac{1}{\gamma+\theta}(\bar{\lambda} + \bar{\Lambda}X_t) + \left(1 - \frac{1}{\gamma+\theta}\right)\frac{\gamma}{\gamma-1}(a + AX_t), \quad (5.13)$$

where the set of coefficients $(\bar{a}, a, A)$ is a solution to the system of nonlinear algebraic equations (5.9)–(5.11).

Remark 7. *The system of nonlinear algebraic eqs. (5.9)–(5.11) is rewritten as*

$$\begin{aligned}&\frac{\gamma(\gamma+\theta-1)}{(\gamma-1)(\gamma+\theta)}A^2\\&\quad-\left(\mathcal{K}+\frac{\gamma+\theta-1}{\gamma+\theta}\bar{\Lambda}+\frac{1}{2}g_1 I_N\right)' A - A\left(\mathcal{K}+\frac{\gamma+\theta-1}{\gamma+\theta}\bar{\Lambda}+\frac{1}{2}g_1 I_N\right)\\&\qquad\qquad-\frac{\gamma-1}{\gamma(\gamma+\theta)}\bar{\Lambda}'\bar{\Lambda}=0, \qquad (5.14)\end{aligned}$$

$$\begin{aligned}a=&\left(\frac{\gamma(\gamma+\theta-1)}{\gamma-1}A-(\gamma+\theta)(\mathcal{K}+g_1 I_N)-(\gamma+\theta-1)\bar{\Lambda}'\right)^{-1}\\&\times\left((\gamma+\theta-1)A\bar{\lambda}+\frac{\gamma-1}{\gamma}\left(\bar{\Lambda}'\bar{\lambda}+(\gamma+\theta)\bar{\rho}\right)\right) \qquad (5.15)\end{aligned}$$

$$\begin{aligned}\bar{a}=&\frac{1}{g_1}\Bigg(\frac{1}{2}\operatorname{tr}[A]+\frac{\gamma(\gamma+\theta-1)}{2(\gamma-1)(\gamma+\theta)}a'a+\left(-\frac{\gamma+\theta-1}{\gamma+\theta}\bar{\lambda}\right)' a\\&+g_1(1-\log g_1)-\frac{\gamma-1}{2\gamma(\gamma+\theta)}\lambda'\lambda-\frac{\gamma-1}{\gamma}\bar{\rho}_0-\frac{\beta}{\gamma}\Bigg). \qquad (5.16)\end{aligned}$$

Assume that g_1 is constant and let

$$C=\mathcal{K}+\frac{\gamma+\theta-1}{\gamma+\theta}\bar{\Lambda}+\frac{1}{2}g_1 I_N.$$

Then, eq. (5.14) becomes a Riccatti matrix algebraic equation for A because $\bar{\Lambda}'\bar{\Lambda}$ is positive definite. Note that there exists a unique positive semi-definite matrix solution because pair (C, I_N) is controllable. We can derive a numerical solution for the system of algebraic equations shown above by using the following algorithm:

1. *Set ϵ to be a sufficiently small positive number.*
2. *Set $g_1^{(0)}=1$ and set $k=1$.*
3. *Substitute $g_1^{(k-1)}$ into g_1 in eq. (5.14). Then, we can derive a numerical solution because eq. (5.14) becomes a Riccatti matrix algebraic equation for A. Denote the numerical equation by A_k.*
4. *Substitute $g_1^{(k-1)}$ and A_k into g_1 and A in eq. (5.15), respectively. Then, we obtain a solution a, which we denote by a_k.*
5. *Substitute $g_1^{(k-1)}$ and (a_k, A_k) into g_1 and (a, A) in eq. (5.16), respectively. Then, we obtain a solution $\bar{a}$, which we denote by $\bar{a}_k$.*

6. *Set* $g_1^{(k)} = g_1(\bar{a}_k, A_k)$.

7. *If* $|g_1^{(k)} - g_1^{(k-1)}| < \epsilon$, *then* $(\bar{a}_k, a_k, A_k)$ *is a numerical solution to the system of algebraic eqs. (5.9)–(5.11).*

8. *Otherwise, increase* k *by* 1 *and repeat Steps 3–7.*

5.3 Approximate Optimal Portfolio

Finally, we show the optimal portfolio choice for the example in Section 4.2. Approximate optimal portfolio $\tilde{\Phi}_t^*$ is given by

$$\begin{aligned}\tilde{\Phi}_t^* &= \frac{1}{\gamma+\theta}\Sigma_t(X_t)'^{-1}\Big(\bar{\lambda} + \bar{\Lambda}X_t\Big) \\ &\quad + \left(1 - \frac{1}{\gamma+\theta}\right)\frac{\gamma}{\gamma-1}\Sigma_t(X_t)'^{-1}\Big(a + AX_t\Big) + \Sigma_t(X_t)'^{-1}\Big(\lambda_I + \Lambda_I X_t\Big).\end{aligned} \tag{5.17}$$

The optimal portfolio choice for the money market account is $1 - (\tilde{\Phi}_t^*(10) + \tilde{\Phi}_{It}^*(10) + \tilde{\Phi}_t^{*1} + \tilde{\Phi}_t^{*2})$.

The optimal portfolio choice is decomposed into the sum of myopic demand, intertemporal uncertainty hedging demand, and inflation hedging demand. In eq. (5.17), all types of demand are nonlinear functions of the state vector, because the inverse matrix of volatilities is its nonlinear function. The fact that the optimal portfolio is a nonlinear function of the state vector suggests that achieving the market timing effect is not as simple as rebalancing the portfolio weight of a single risky security or index based on the business cycle. Rather, this implies that the market timing effect cannot be achieved without finely rebalancing the portfolio weights among the risky securities in response to the various phases created by the multidimensional state vector. The importance of this timing effect is pointed out by Batbold *et al.* [5] in the optimal portfolio for the CRRA utility. Because we obtain the approximate analytical solution, we can implement the above complex portfolio rebalancing to achieve market timing effects as long as we precisely estimate the parameters and latent factor process in our quadratic security market model. We can precisely estimate them using the quasi-maximum likelihood method based on nonlinear filtering, but we leave this for a future study.

6 Conclusions and Future Research Scope

In this study, we analyze a robust consumption–investment control problem for an investor with the HR utility under a quadratic security market model. Under this model, all inflation rates, interest rates, and asset risk premiums and volatilities are stochastic and predictable. We obtain an optimal portfolio decomposed into the sum of four types of demand, that is, myopic demand, intertemporal risk hedging demand, inflation hedging, and intertemporal ambiguity hedging demand, which is proportional to the investor's relative ambiguity aversion. We show that optimal investment is inversely proportional to the relative uncertainty aversion, which is the sum of relative risk aversion and relative ambiguity aversion. We apply the log-linear approximation method of Campbell and Viceira [15] and Batbold *et al.* [4] to the nonhomogeneous nonlinear PDE for the investor's indirect utility function, and derive an approximate analytical solution, in which the set of parameters is represented as a solution to a system of nonlinear algebraic equations. We then present an algorithm to derive a numerical solution to this system of nonlinear algebraic equations. We show that the approximate optimal portfolio is a nonlinear function of the state vector. This suggests that the timing aspect is important, as pointed out by Batbold *et al.* [5] in the optimal portfolio for the CRRA utility. Since we have access to the approximate analytical solution for the optimal portfolio, we can implement the above complex portfolio rebalancing to achieve market timing effects as long as we can precisely estimate the parameters and the latent factor process under the quadratic security market model. Finally, if the security market model is affine rather than quadratic, the affine model can be interpreted as a linear state-space model; thus, the model parameters and the state vector process can be estimated with high accuracy by the maximum likelihood method based on Kalman filtering. However, for the quadratic model, the state-space model is nonlinear. Therefore, some type of pseudo-maximum likelihood method based on nonlinear filtering is necessary. In a future study, we will estimate the quadratic model based on nonlinear filtering.

A Proofs

A.1 Proof of Proposition 1

First, optimal consumption control (4.16) is obtained as follows:

$$c_t^* = J_W^{-\frac{1}{\gamma}} = \left\{ \left(\frac{G}{W_t^*} \right)^{\gamma} \right\}^{-\frac{1}{\gamma}} = \frac{W_t^*}{G}. \tag{A.1}$$

Second, the derivatives of J are given by

$$WJ_W = (1-\gamma)J, \qquad J_X = \gamma J \frac{G_X}{G}, \qquad W^2 J_{WW} = -\gamma(1-\gamma)J,$$

$$WJ_{XW} = \gamma(1-\gamma)J\frac{G_X}{G}, \qquad J_{XX} = \gamma J \left\{ (\gamma - 1)\frac{G_X}{G}\frac{G_X'}{G} + \frac{G_{XX}}{G} \right\}.$$

Then, it is easy to see that $\mathcal{T}_t$ in eq. (4.8) is expressed as eq. (4.19), and that π_t in eq. (4.13) is rewritten as

$$\pi_t = J\left((\gamma-1)\bar{\Lambda}_t + \gamma(\gamma+\theta-1)\frac{G_X}{G}\right). \tag{A.2}$$

Therefore, by inserting eq. (4.19) into eq. (4.7), we obtain optimal investment control (4.17).

The first to third terms in PDE (4.14) are calculated from eqs. (A.2) as follows:

$$\begin{aligned}
&\frac{1}{2}\operatorname{tr}[J_{XX}] - \frac{\theta}{2(1-\gamma)J}J_X' J_X - \frac{\pi_t'\pi_t}{2W_t^2\left(J_{WW} - \frac{\theta J_W^2}{(1-\gamma)J}\right)}\\
=&\frac{\gamma}{2}J\left\{\operatorname{tr}\left[(\gamma-1)\frac{G_X}{G}\frac{G_X'}{G} + \frac{G_{XX}}{G}\right] + \frac{\gamma^2\theta}{2(\gamma-1)}\frac{G_X'}{G}\frac{G_X}{G}\right.\\
&\left. - \frac{1}{2(\gamma-1)(\gamma+\theta)}\left((\gamma-1)\bar{\Lambda}_t' + \gamma(\gamma+\theta-1)\frac{G_X'}{G}\right)\left((\gamma-1)\bar{\Lambda}_t' + \gamma(\gamma+\theta-1)\frac{G_X'}{G}\right)'\right\}\\
=&J\left\{\frac{\gamma}{2}\operatorname{tr}\left[\frac{G_{XX}}{G}\right] - \frac{\gamma-1}{2(\gamma+\theta)}\bar{\Lambda}_t'\bar{\Lambda}_t - \frac{\gamma(\gamma+\theta-1)}{\gamma+\theta}\bar{\Lambda}_t'\frac{G_X}{G}\right.\\
&\left. + \frac{\gamma}{2}\left(\gamma-1+\frac{\gamma\theta}{\gamma-1} - \frac{\gamma(\gamma+\theta-1)^2}{(\gamma-1)(\gamma+\theta)}\right)\frac{G_X'}{G}\frac{G_X}{G}\right\}\\
=&\gamma J\left\{\frac{1}{2}\operatorname{tr}\left[\frac{G_{XX}}{G}\right] - \frac{\gamma-1}{2\gamma(\gamma+\theta)}\bar{\Lambda}_t'\bar{\Lambda}_t - \frac{\gamma+\theta-1}{\gamma+\theta}\bar{\Lambda}_t'\frac{G_X}{G} + \frac{\theta}{2(\gamma-1)(\gamma+\theta)}\frac{G_X'}{G}\frac{G_X}{G}\right\}.
\end{aligned} \tag{A.3}$$

The sixth term in PDE (4.14) is calculated from eq. (A.1) as follows:

$$\frac{\gamma}{1-\gamma}J_W^{1-\frac{1}{\gamma}} = \gamma\frac{J}{W_t^*}\frac{W_t^*}{G} = \gamma\,\frac{J}{G}, \tag{A.4}$$

Substituting eqs. (A.3) and (A.4) into eq. (4.14) and dividing by γJ yields eq. (4.18).

A.2 Proof of Lemma 3

X_t is expressed as the solution to linear SDE (2.1) as follows:

$$X_t = Qe^{-tL}Q^{-1}X_0 + Q\int_0^t e^{-(t-s)L}Q^{-1}\,dB_s.$$

Thus, as $\lim_{t\to\infty} e^{-tL} = 0$, $\lim_{t\to\infty} \mathrm{E}[X_t] = 0$ holds.

Next, the following equation holds:

$$\begin{aligned}
X_t'AX_t = &\left\{Qe^{-tL}Q^{-1}X_0 + Q\int_0^t e^{-(t-s)L}Q^{-1}\Sigma\,dB_s\right\}'\\
&\qquad A\left\{Qe^{-tL}Q^{-1}X_0 + Q\int_0^t e^{-(t-s)L}Q^{-1}\,dB_s\right\}.
\end{aligned}$$

Because $\mathrm{E}[dB_s dB_t'] = \delta_{st} I_N ds$[8], the following equation holds:

$$
\begin{aligned}
\lim_{t\to\infty} \mathrm{E}[X_t' A X_t] &= \lim_{t\to\infty} \int_0^t \mathrm{tr}\left[(Q^{-1})' e^{-(t-s)L} Q' A\, Q e^{-(t-s)L} Q^{-1}\right] ds \\
&= \mathrm{tr}\left[(Q^{-1})' \lim_{t\to\infty} \int_0^t e^{-(t-s)L} Q' A\, Q e^{-(t-s)L} ds\, Q^{-1}\right] \\
&= \mathrm{tr}\left[(Q^{-1})' M Q^{-1}\right].
\end{aligned}
$$

Therefore, eq. (5.7) holds.

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[8] δ_{st} is the Kronecker's delta.

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