# Analytic Approach
# for the Walrasian equilibrium Existence Theorem

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October 2020

**Abstract**

We will provide the Walrasian equilibrium exstence theorem by the methods of elementary analysis or differential geometry rather than topological methods.

JEL Classification: C62, D41, D51

# 1 Introduction

In this paper, we will investigate some kind of clasical existence theorem of Walrasian equilibrium for a perfecctly competitive complete market economy. In the field of general equilibrium theory, existence issue of Walrasian equilibria and research of their welfare properties have been considered as crucial and central problem. For the detaied developments of the area, see, for example, [2], [6], [8], [9].

Economic agents, consumers and producers, trade their privately owned commodities by using price or market mechanism embedded in the market economy in order to maximize their economic benefits. Then, price or market mechanism adjusts their behaviors, interactions between economic agents. In the Walrasian equilibrium prices, aggregate excess demand is equal to zero, and hence each economic agent maximizes their benefits under the feasible allocation. The adjustment by price system does not necessarily guarantee the realization of Walrasian equilibrium [4], however, the equilibrium concepts are understood to be important as cornerstones and benchmarks for the interpretations of economic activities. The existence issue of Walrasian equilibrium is concerned with the consistency of price or market mechanism. It makes the consistency condition in the economic analysis clear.

Topological methods in the existence issue of Walrasian equilibrium are main streams in the research areas of general equilibrium theory. On the other hand, analytical methods or methods using differntial geometry have been cultivated by many reasearchers [*1]. See, for example, [14]. Our approach to the existence issue is heavily dependent on one of the methods of analysis or differential geometry, which have been cultivated by, for example, [11], [12]. As clearly indicated by [16], the existence of Walrasian equilibrium is equivalent to Brouwer's fixed point Theorem. Therefore, the central problem is reduced to the some kinds of fixed point theorem, in paticular, Brouwer's fixed point Theorem. There are many methods of proof in (Brouwer's) fixed point theorem, those of which are deeply connected in geometrical natures. As stated in the above, our choice of method is one in differential geometry. The choices of various kind of methodlogies are crucial from the viewpoints of comparison, and those plays crucial roles in theoretical development. Our result seems to be transitinal in the sense that it is heavily analytical, and there is room to examin algebraic constructions using differential forms or homotopic considerations. Also, several parts of our assumptions are still strong, paticularly, Assumption 3. We would like to make remaining issues weakened, and to extend the result to imcomplete sequential economies, in the future research.

Our paper is constructed as the followings. In Section 2, assumptions of the economic primitives and main theorem are stated. In Section 3, series of propositions and lemmas are described, and thus, main result is completed in the proofs.

[*1] It is impossible to overemphasize the importance of the methods of differential topology in these areas. Differential topology have occupied the intermedeate and vast position among the topological methods and differential geometry. See, again, [8].

## 2 The Walrasian Model and Equilibrium Existence Theorem

In the perfectly competitive, complete markets of our Walrasian economy, $n$ kinds of perfectly divisible goods are traded, *i.e.* the commodity space is assumed to be $\mathbb{R}^n$. Then, our fundamental economic primitive for the perfectly competitive, complete markets is given by the $C^1$-excess demand function $x : R^n_{++} \longrightarrow \mathbb{R}^n$, where $\mathbb{R}^n_{++}$ is strictly positive orthant of $\mathbb{R}^n$, $\{p \in R^n : p^i > 0 \text{ for any } p = (p^1, \cdots, p^n)\}$. The domain $\mathbb{R}^n_{++}$ of $x$ is interpreted as the price space. We assume that excess demand function $x$ satisfies the following Assumption 1.

**Assumption 1.** [*2] *(i)* $x(tp) = x(p)$ *for any* $t \in \mathbb{R}_{++}$ *and* $p \in \mathbb{R}^n_{++}$. (*homogenity of degree* 0)

*(ii)* $\langle p|x(p)\rangle = 0$ *for any* $p \in \mathbb{R}^n_{++}$. [*3] (*Walras's law*)

*(iii)* $\lim_{p^k \to p} \|x(p^k)\| = +\infty$ *whenever* $\lim_{k\to+\infty} p^k = p \in \partial\mathbb{R}^n_{++}$*, where* $\partial\mathbb{R}^n_{++}$ *is the boundary of* $\mathbb{R}^n_{++}$. (*boundary condition, or, the desirability of commodities*)

*(iv) There exists* $l \in \mathbb{R}^n$ *such that* $x(p) \gg l$. [*4] (*lower boundedness*)

Let $S^{n-1}$ be a unit sphere $\{p \in \mathbb{R}^n : \|p\| = 1\}$ of $\mathbb{R}^n$, and $S^{n-1}_{++}$ be a strictly positive orthant $\{p = (p^1, \cdots, p^n) \in S^{n-1} : p^i > 0 \text{ for any i}\}$ of $S^{n-1}$. By Assumption 1-(*i*), we can normalize price set $\mathbb{R}^n_{++}$ to $S^{n-1}_{++}$. Therefore, by Assumption 1-(*ii*), we can regard $x$ as a tangent vector field $x : S^{n-1}_{++} \longrightarrow \mathbb{R}^n$, which is defined as $p \in S^{n-1}_{++} \longmapsto x_p = x(p)$, where $x_p \in T_pS^{n-1}$ is a tangent vector at $p \in S^{n-1}$. [*5]

Thus, we can restate Assumption 1 as Assumption 2.

**Assumption 2.** *an excess demand for the Walrasian perfectly competitive economy is a* $C^1$*-tangent vector field* $x : S^{n-1}_{++} \longrightarrow \mathbb{R}^n$ *satisfying the following assumptions*

*(i)* $\langle p|x_p\rangle = 0$ *for any* $p \in S^{n-1}_{++}$. (*Walras's law*)

*(ii)* $\lim_{p^k \to p} \|x_{p^k}\| = +\infty$ *whenever* $\lim_{k\to+\infty} p^k = p \in \partial S^{n-1}_{++}$*, where* $\partial S^{n-1}_{++}$ *is the boundary of* $S^{n-1}_{++}$. (*boundary condition, or, the desirability of commodities*)

*(iii) There exists* $l \in \mathbb{R}^n$ *such that* $x_p \gg l$. (*lower boundedness*)

Note that $x_p$ is inward pointing near $\partial S^{n-1}_{++}$ by Assumption 2-(*ii*). Moreover, we add a technical assumption.

**Assumption 3.** *As function,* $x : S^{n-1}_{++} \longrightarrow \mathbb{R}^n$ *is uniformly continuous on* $S^{n-1}_{++}$.

Our main result is the following theorem.

**Theorem 1.** *There exists* $p \in S^{n-1}_{++}$ *such that* $x_p = 0$.

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[*2] Adding the another regular assumptions, it is wellknown that $C^2$-smooth preferences [3] on the common consumption set $\mathbb{R}^n_{++}$ in pure exchange economy generates the excess dmand function which satisfies Assumption 1. Production economy under the regular assumptions generates the the excess dmand function which satisfies Assumption 1, too [8].

[*3] $\langle\cdot|\cdot\rangle$ is a usual inner product on $\mathbb{R}^n \times \mathbb{R}^n$.

[*4] For $a = (a^1, \cdots, a^n), b = (b^1, \cdots, b^n) \in \mathbb{R}^n$, $a \gg b$ means that $a^i > b^i$ for any $i$.

[*5] $T_pS^{n-1}$ is a tangent vector space of $S^{n-1}$ at $p \in S^{n-1}_{++}$.

## 3 Series of Lemmas and Propositions

To lead a contradiction, suppose that $x_p \neq 0$ for any $p \in S^{n-1}_{++}$, and define a non-zero unit $C^1$-tangent vector field $X : S^{n-1}_{++} \longrightarrow \mathbb{R}^n$ on $S^{n-1}_{++}$as $p \in S^{n-1}_{++} \longmapsto X_p \equiv \dfrac{x_p}{\|p\|}$. By Assumption 2, $X$ satisfies the following facts.

**Fact 1.** *(i) $\langle p|X_p\rangle = 0$ for any $p \in S^{n-1}_{++}$.*
*(ii) $\|X_p\| = 1$ for any $p \in S^{n-1}_{++}$*

Note that $X_p$ is outward pointing at $\partial S^{n-1}_{++}$ by Fact 1-$(i)$. Let $S^{n-1}_{+}$ be a positive orthant $\{p = (p^1, \cdots, p^n) \in S^{n-1} : p^i \geqq 0$ for any i$\}$ of $S^{n-1}$.

**Proposition 1.** *There exists a non-zero unit $C^0$-tangent vector field $\widetilde{X} : S^{n-1}_{+} \longrightarrow \mathbb{R}^n$ on $S^{n-1}_{+}$ such that $\widetilde{X}_p = X_p$ for any $p \in S^{n}_{++}$.*

*Proof.* Fix $\bar{p} \in \partial S^{n-1}_{+}$. Then, there exists a sequence $\{q^k\}$ such that $q^k \in S^{n-1}_{++}$ and $\lim_{k\to+\infty} q^k = \bar{p}$, and a convergent subsequence of $\{X_{q^k}\}$, putting the limit as $X_{\bar{p}} \in S^{n-1}_{+}$. $X_{\bar{p}}$ satisfies that $\langle \bar{p}|X_{\bar{p}}\rangle = 0$, $i.e, X_{\bar{p}} \in T_{\bar{p}}\partial S^{n-1}_{+}$ and $\|X_{\bar{p}}\| = 1$ by Fact 1-$(ii)$. Vector field $X$ is uniformly continuous on $S^{n-1}_{++}$ by Assumption 3. Hence, for any $\epsilon > 0$, there exists $\delta > 0$ such that

$\|p^k - \bar{p}\| < \delta$ implies $\|X_{p^k} - X_{\bar{p}}\| \leqq \|X_{p^k} - X_{q^l}\| + \|X_{q^l} - X_{\bar{p}}\| < \epsilon$. (If necessary, take $l$ large enough.)

Thus, unique tnagent vector $X_{\bar{p}} \in T_{\bar{p}}\partial S^{n-1}_{++}$ can be taken, and we can define a non-zero unit $C^0$-tangent vector field $\widetilde{X} : S^{n-1}_{+} \longrightarrow \mathbb{R}^n$ on $S^{n-1}_{+}$ as

$$p \in S^{n-1}_{+} \longmapsto \widetilde{X}_p = \begin{cases} X_p & : p \in S^{n+1}_{++} \\ X_{\bar{p}} & : p = \bar{p} \in \partial S^{n+1}_{+}. \end{cases}$$

□

Applying the Hopf's extension argument [1], [5], [11], there exists an open set $U$ such that $S^{n-1}_{+} \subset U$ and $C^1$-vector field $\widetilde{X}^{ext} : U \longrightarrow \mathbb{R}^n$ such that $\widetilde{X}^{ext}_p = \widetilde{X}_p$ for any $p \in S^{n-1}_{+}$. Abusing the notation, we will merely denote $\widetilde{X}^{ext}$ as $\widetilde{X}$ in the following argument.

**Lemma 1.** *There exist a Lipschitz constant $c > 0$ such that $\|\widetilde{X}_p - \widetilde{X}_q\| \leqq c\|p - q\|$ for any $p, q \in S^{n-1}_{+}$.*

*Proof.* Let $A$ be a cube, say, $A = \prod_{i=1}^{j} [a_i, b_i], a_i, b_i \in \mathbb{R}, a_i < b_i, 1 \leqq j \leqq n$. We can identify $\widetilde{X}$ with $C^1$-function $\widetilde{x} : U \longrightarrow \mathbb{R}^n$ since $\widetilde{X}$ is $C^1$-tangent vector field on $U$. Applying mean value theorem into any coordinate function $\widetilde{x}^i : U \longrightarrow \mathbb{R}$,

$$\|\widetilde{X}_p - \widetilde{X}_q\| = \|\widetilde{x}(p) - \widetilde{x}(q)\| \leqq \sum_{i=1}^{n} |\widetilde{x}^i(p) - \widetilde{x}^i(q)| = \sum_{i=1}^{n} |\left\langle \partial_{\xi^i}\widetilde{x}^i|p - q\right\rangle| \leqq \sum_{i=1}^{n} \|\partial_{\xi^i}\widetilde{x}^i\|\|p - q\|,$$
$$\xi^i = \theta_i p + (1 - \theta_i)q, \theta_i \in [0, 1].$$

We can take $c^i = \max_{\xi\in A} \|\partial_\xi \widetilde{x}^i\|$ since $A$ is compact and $\widetilde{x}$ is $C^1$. Then, $\|\widetilde{X}_p - \widetilde{X}_q\| \leqq \sum_{i=1}^{n} c^i \|p-q\|$. Putting $c = \sum_{i=1}^{n} c^i$, $\|\widetilde{X}_p - \widetilde{X}_q\| \leqq c\|p-q\|$ uniformly.

Let $A_\alpha$ be an open cube such that $S^{n-1}_{++} \subset \bigcup_\alpha A_\alpha \subset U$. We can assume $\{A_\alpha\}$ is a fimite collection since $S^{n-1}_{+}$ is compact.

$(i)$ If $p, q \in A_\alpha$ for some $\alpha$, there exists $c^\alpha$ such that $\|\widetilde{X}_p - \widetilde{X}_q\| \leqq c^\alpha \|p-q\|$ uniformly.

$(ii)$ If $(p,q) \in (A\times A) \setminus \bigcup_\alpha (A_\alpha \times A_\alpha)$, $0 < \|p-q\|$ uniformly. Since $(A\times A) \setminus \bigcup_\alpha (A_\alpha \times A_\alpha)$ is compact, a function $F : (A_\alpha \times A_\alpha) \setminus \bigcup_\alpha (A_\alpha \times A_\alpha) \longrightarrow \mathbb{R}$, defined by $(p,q) \longmapsto F(p,q) = \|\widetilde{X}_p - \widetilde{X}_q\|$ is uniformly continuous. Therefore, $\|\widetilde{X}_p - \widetilde{X}_q\| \leqq \|p-q\|$ uniformly on $(A\times A) \setminus \bigcup_\alpha (A_\alpha \times A_\alpha)$.

Putting $c = \max\{1, (c^\alpha)_\alpha\}$, $\|\widetilde{X}_p - \widetilde{X}_q\| \leqq c\|p-q\|$ uniformly.

□

For any $t \in \mathbb{R}$, define a function $f_t : A \longrightarrow \mathbb{R}^n$ as $p \in A \longmapsto f_t(p) = p + t\widetilde{X}_p$.

**Lemma 2.** *$f_t$ is injective for $|t|$ small enough.*

*Proof.* Choose $t$ so that $|t| < c^{-1}$, and let $f_t(p) = f_t(q)$ to indicate $p = q$. Suppose $p \neq q$. Then, since $p - q = t(\widetilde{X}_q - \widetilde{X}_p)$,

$$\|p-q\| = |t|\|\widetilde{X}_p - \widetilde{X}_q\| \leqq c|t|\|p-q\| < \|p-q\|, \text{ which leads to a contradiction.}$$

□

**Proposition 2.** $\mathrm{vol}\left(f_t(S^{n-1}_{+})\right)$ *is a polynomial function of $t$.*

*Proof.* Differentiating $f_t$ at $p \in S^{n-1}_{+}$,

$$\partial_p f_t = I_n + t\left[\frac{\partial \widetilde{x}^i}{\partial p^j}\right],$$

where $I_n$ is the identity matrix of n-th order, and, $\left[\frac{\partial \widetilde{x}^i}{\partial p^j}\right]$ is the Jacobian matrix of $\widetilde{x}$.

Therefore, we can put $\det \partial_p f_t = 1 + \sum_{i=1}^{n} \pi_i(p) t^i$, which is a polynomial function of $t$. Note that any $\pi_i(p)$ is continuous in $U$ by the definition of the determinant. Since $S^{n-1}_{+}$ is compact, any $\pi_i(p)$ is uniformly bounded. Hence, $\det \partial_p f_t > 0$ uniformly over $S^{n-1}_{+}$ for $|t|$ small enough. By the change of variable formula [*6],

$$\begin{aligned}\mathrm{vol}\left(f_t(S^{n-1}_{+})\right) &= \int_{S^{n-1}_{+}} |\det \partial_p f_t| dp = \int_{S^{n-1}_{+}} \det \partial_p f_t dp = \int_{S^{n-1}_{+}} (1 + \sum_{i=1}^{n} \pi_i(p) t^i) dp \\ &= \int_{S^{n-1}_{+}} 1 dp + \sum_{i=1}^{n} \left(\int_{S^{n-1}_{+}} \pi_i(p) dp\right) t^i.\end{aligned}$$

□

[*6] See, for example, [13].

Let $A \subset \mathbb{R}^n$ be a compact region whose indicator function is integrable, $\mathcal{N}A$ be a open set containing $A$, and $Y : \mathcal{N}A \longrightarrow \mathbb{R}^n$ be a $C^1$-vector field.

**Lemma 3.** *There exist a Lipschitz constant $c > 0$ such that $\|Y_p - Y_q\| \leqq c\|p - q\|$ for any $p, q \in A$.*

*Proof.* The proof can be done as similarly as the proof of Lemma 1.

□

Take $\alpha > 0$, and let

$$S^{n-1}_{\alpha,+} = \{p \in \mathbb{R}^n_+ : \|p\| = \alpha\}, S^{n-1}_{\alpha,t,+} = \left\{p \in \mathbb{R}^n_+ : \|p\| = \alpha\sqrt{1 + t^2}\right\}.$$

Define a $C^1$-function $f_{\alpha,t} : S^{n-1}_{\alpha,+} \longrightarrow \mathbb{R}^n$ as $p \in S^{n-1}_{\alpha,+} \longmapsto f_{\alpha,t}(p) = p + t\|p\|\widetilde{X}_{\frac{p}{\|p\|}}$.

**Proposition 3.** *$f_{\alpha,t} : S^{n-1}_{\alpha,+} \longrightarrow \mathbb{R}^n$ is a bijective mapping onto $S^{n-1}_{\alpha,t,+}$ for $|t|$ small enough.*

*Proof.* Applying Lemma 3, we can take $c > 0$ such that $\|\widetilde{X}_{\frac{q}{\|q\|}} - \widetilde{X}_{\frac{p}{\|p\|}}\| \leqq c\|p - q\|$ for any $p, q \in S^{n-1}_{\alpha,+}$. Then, choose $t$ so that $|t| < c^{-1}\alpha^{-1}$, and let $f_{\alpha,t}(p) = f_{\alpha,t}(q)$ to indicate $p = q$. Suppose $p \neq q$. Then, since $p - q = t \cdot \alpha(\widetilde{X}_{\frac{q}{\|q\|}} - \widetilde{X}_{\frac{p}{\|p\|}})$,

$$\|p - q\| = \alpha|t|\|\widetilde{X}_{\frac{q}{\|q\|}} - \widetilde{X}_{\frac{p}{\|p\|}}\| \leqq c\alpha|t|\|p - q\| < \|p - q\|, \text{ which leads to a contradiction.}$$

Hence, $f_{\alpha,t}$ is injective.

Let $c_A > 0$ be the Lipschitz constant of a vector field $\|p\|\widetilde{X}_{\frac{p}{\|p\|}}$ for $A = \left\{p \in \mathbb{R}^n_+ : \dfrac{\alpha}{2} \leqq \|p\| \leqq \dfrac{3\alpha}{2}\right\}$ in Lemma 3, let $t \in \mathbb{R}$ be such that $|t| < c_A^{-1}$ and $|t| < \dfrac{1}{3}$, and fix $P_0 \in S^{n-1}_{\alpha,+}$. Defin a function $g_t : A \longrightarrow \mathbb{R}^n$ as $p \in A \longmapsto g_t(p) = P_0 - t\|p\|\widetilde{X}_{\frac{p}{\|p\|}}$. Then,

$$\frac{\alpha}{2} \leqq |\|P_0\| - |t|\|p\|| \leqq \|g_t(p)\| \leqq \|P_0\| + |t|\|p\| \leqq \frac{3\alpha}{2}$$

Therefore, $g_t(p) \in A$. For any $p, q \in A$,

$$\|g_t(p) - g_t(q)\| = |t| \left\| \|p\|\widetilde{X}_{\frac{p}{\|p\|}} - \|q\|\widetilde{X}_{\frac{q}{\|q\|}} \right\| \leqq |t|c_A\|p - q\|, \text{ and } 0 < |t|c_A < 1.$$

Consequently, function $g_t$ has a fixed point $p^* \in A$ by Banach's shrinking mapping theorem since $A$ is a compact, hence, a Banach space. Therefore,

$$p^* = g_t(p^*), i.e. P_0 = p^* + t\|p^*\|\widetilde{X}_{\frac{p^*}{\|p^*\|}} = f_{\alpha,t}(p^*).$$

On the other hand,

$$\|P_0\|^2 = \|f_{\alpha,t}(p^*)\|^2 = \|p^*\|^2 + t^2\|p^*\|^2 = (1 + t^2)\|p^*\|^2$$
$$\|p^*\| = \frac{\alpha}{\sqrt{1 + t^2}}.$$

Hence, we can write

$$\sqrt{1 + t^2}P_0 = f_{\alpha,t}\left(\sqrt{1 + t^2}\; p^*\right).$$

This implies that $f_{\alpha,t} : S^{n-1}_{\alpha,+} \longrightarrow \mathbb{R}^n$ is surjective onto $S^{n-1}_{\alpha,t,+}$.

□

Let $\alpha = 1$. Then,

$$S^{n-1}_{\alpha,+} = S^{n-1}_{+}, \text{ and } S^{n-1}_{\alpha,t,+} = S^{n-1}_{t,+}, \text{ where } S^{n-1}_{t,+} = \left\{ p \in \mathbb{R}^n_+ : \|p\| = \alpha\sqrt{1+t^2} \right\}.$$

Note that a $C^1$-function $f_t : S^{n-1}_+ \longrightarrow \mathbb{R}^n$ as $p \in S^{n-1}_+ \longmapsto f_t(p) = p + t\|p\|\widetilde{X}_{\frac{p}{\|p\|}}$ is bijective onto $S^{n-1}_{t,+}$. Let

$$B^n_{\alpha,+} = \{p \in \mathbb{R}^n_+ : \|p\| \leqq \alpha\}, B^{n-1}_{\alpha,t,+} = \left\{ p \in \mathbb{R}^n_+ : \|p\| \leqq \alpha\sqrt{1+t^2} \right\}.$$

Fix $a, b \in \mathbb{R}$ so that $0 < a < b$, and put a compact region $A^b_a$ as $B^n_{b,t} \setminus \mathrm{int}B^n_{a,t} = \{p \in \mathbb{R}^n_+ : a \leqq \|p\| \leqq b\}$, , and defin a vector field $\widetilde{X^b_a} : A^b_a \longrightarrow \mathbb{R}^n$ as $p \in A^b_a \longmapsto \widetilde{X}_p = \|p\| X_{\frac{p}{\|p\|}}$, which is a $C^1$- vetor field.

*Proof of Theorem 1.* First, let $n$ be an odd number. Define a function $f^b_{t,a} : A^b_a \longrightarrow B^n_{b,+} \setminus \mathrm{int}B^n_{a,+}$ as $p \in A^b_a \longmapsto p + t\|p\|\widetilde{X}_{\frac{p}{\|p\|}}$. Let $|t|$ be small enough so that Proposition 3 holds. Then, $f^b_{t,a} : A^b_a \longrightarrow B^n_{b,+} \setminus \mathrm{int}B^n_{a,+}$ is a bijective $C^1$-mapping onto $B^n_{b,+} \setminus \mathrm{int}B^n_{a,+}$. Since the volume of $n$-dim ball is proportional to the $n$-th power of the radius,

$$\mathrm{vol}\left(f_t(A)\right) = \left(\sqrt{1+t^2}\right)^n \cdot \mathrm{vol}A$$

must hold. But, the right hand side of the equation is not a polynomial function since n is an odd number. This is a contradiction to Proposition 2.

Next, let $n$ be an even number, and suppose $x_p \neq 0$ for any $p \in S^{n-1}_{++}$ , which satisfies Assumption 2 and 3. Then, applying Proposition 1 to $x_p$, we can get an extended $C^1$-vector field $\widetilde{X} : S^{n-1}_+ \longrightarrow \mathbb{R}^n$, which satisfies Fact 1, Proposition 1, and unifrom continuity. Define a $C^1$-vector field $\widehat{X} : S^n_+ \longrightarrow \mathbb{R}^{n+1}$ as $(p, q) \in S^n_+ \longmapsto (\widetilde{X}_{\frac{p}{\|p\|}}, 0)$. Since $\widehat{X}$ is uniformly continuous and outward pointing on $\partial S^n_{++}$, we can apply the argument in this chapter, which leads to a contradiction since $n+1$ is an odd number. Thus, proofs are completed.

□

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