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On the Information Exchange between Risk-Averse Firms: The Mean and Variance Effects

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On the Information Exchange between Risk-Averse Firms: The Mean and Variance Effects *

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Abstract. There have been so many papers on the theory of oligopoly and information. In spite of a growing literature on this subject, however, we believe that there is nevertheless a conspicuously missing link in it. To our surprise, very few papers have ever attempted to investigate an important subject of "information exchange and risk aversion. " Although such a subject seems to demand a very tough computations and psychological pains, we strongly believe that someone must take up a challenge. The main purpose of this paper is to do our best for filling in such a missing gap, thus hoping to do a contribution to the important subject of oligopoly and information.

More specifically, this paper aims to discuss the value of additional information in Cournot duopoly when each firm faces its own cost uncertainty. If firms display risk aversion and thus maximize the expected utility of profits, the exchange of cost information between them affects the mean values of outputs as well as their variances. By effectively employing a constant absolute-risk-aversion model, we are able to show the *variance effect* may sometimes overpower the *mean effect*, whence information sharing may possibly make firms worse-off. As our daily experience shows, "going together" may sometimes be a better policy than "going alone."

* This paper is a completely revised version of Sakai-Yoshizumi (1991a). Sakai has exerted all his energy for revitalizing it in line with more recent developments of oligopoly theory under imperfect information. Sakai wishes to dedicate this new paper to the fond memory of Mr. Akihito Yoshizumi, who unfortunately retired from active academic work some time ago.

1 A Missing Link in Duopoly and Information

There have been so many papers on the theory of oligopoly and information. In spite of a growing literature on this subject, however, we believe that there is nevertheless a conspicuously missing link in it. To our surprise, very few papers have ever attempted to investigate an important subject of "information exchange and risk aversion. " Although such a subject seems to demand a very tough computations and psychological pains, we strongly believe that someone must take up a challenge. So, the main purpose of this paper is to do our best for filling in such a gap, thus contributing to the important subject of oligopoly and information. As the saying goes, there is a will, there is a way !

In his monumental essays, Arrow (1970) once remarked: [1)]

> From the time of Bernoulli on, it has been common to argue that (a) individuals tend to display aversion to the taking of risks, and (b) that risk aversion in turn is an explanation for many observed phenomena in the economic world, In this essay, I [Arrow] wish to discuss more specifically the measures of risk aversion and to show how, in conjunction with the expected-utility hypothesis, they can be used to derive quantitative rather than merely qualitative results in economic theory.
>
> (Arrow, 1970, the opening paragraph, p. 90)

Although around fifty long years have passed since then, it looks to us that Arrow's remark is still alive even today. To be honest, we are really in full agreement with his view. As was correctly pointed out by Arrow, from the old time of Bernoulli (1738) on, there should have been a two-way street between economic behavior and risk aversion. On one hand, observing people's economic behavior in their daily lives, they tend to display risk aversion. On the other, risk aversion explains very well many observed phenomena to the economic world. So far so good. If we turn to the issue of "oligopoly and information," however, it is quite unfortunate that such a two-way street has hardly been usable presumably because of some technical and computational difficulties. [2)]

In this paper, we are concerned with the value of additional information in a duopoly model in which risk-averse firms are confronted with private cost uncertainty. More specifically, we would like to investigate the question of whether and how much the exchange of information between firms is beneficial, or possibly harmful, to them when they display risk aversion.

One of the most fashionable topics in modern oligopoly theory is centered around the welfare implications of information sharing among firms. The line of research was already initiated in the 1970s, continued by the explosion of works in the 1980s and the 1990s. And, even in the new century, we have seen the continuation and further development of research on the theory of oligopoly and information. [3)]

A great variety of oligopoly models under risks have been studied so far. In some papers, firms are assumed to behave as Cournot competitors with output strategies , whereas in other papers, they are instead regarded as Bertrand competitors with price strategies. They may exist a common risk or else private (i.e., firm-specific) risks. Uncertainty may be about the demand side or the cost side. Besides, products may be homogeneous or differentiated.

While all the existing papers explore the problem of information sharing in oligopoly theory in a very extensive way, it is quite unfortunate that they all have one conspicuous defect in common. This is because they merely assume that firms behave as expected-profit maximizers, implying that firms are risk neutral players. The aim of this paper is to mend such a grave deficiency by making the assumption that firms display a certain degree of risk aversion.

To see how and to what degree the introduction of risk aversion into an oligopoly model influences the welfare results of information sharing, we consider here the most standard model of duopoly — Cournot duopoly with cost uncertainty. The basic idea behind our model should be simple and clear. There are two firms in an industry that produce homogeneous products. Each firm has information about its own cost, but not its rival's. At the starting point, suppose that both firms make a certain agreement concerning the exchange of cost information between them. Such an agreement may be made either by a binding contract or through a third independent agency such as a trade association. The question to ask here is how such an exchange agreement contributes to the welfare of participating firms. It is now well-known that within the framework aforementioned, information pooling between risk neutral firms leads to an increase in expected profits. So far so good: It agrees with our common sense indeed !

If, however, firms are *risk averse players* and thus maximize the expected utility of profits rather than the mere profits, the situation must change drastically. The exchange of cost information between risk-averse firms affects expected outputs in two distinct ways. Indeed, it increases both the mean and variances of outputs. Presumably, the *first mean effect* constitutes a plus factor for the expected utility of profits, whereas the *second variance effect* serves as a minus factor for the welfare of firms. It can naturally be conjectured that there would be the case in which the variance effect dominates the mean effect and thus information sharing makes firms worse off.

In their remarkable paper, Newbery and Stiglitz (1984) have demonstrated that free trade may be *Pareto inferior* to no trade whatever. Possibly, no information may be better than noisy information ! Our result is consistent with the Newberry-Stiglitz result if we regard the flow of information as one form of trade. In historical perspective, closing one's country to outsiders may sometimes be better than the complete opening of a country to the outside world. It really depends !

The remainder of this paper is organized as follows. In Section 2, we set up our analytical framework for the Cournot-type duopoly model in which each firm faces its own risk about the cost side. In Section 3, we focus on a simple yet interesting case, involving constant absolute-risk-aversion utility functions and normally distributed random variables. It is in this specific model that we are able to compute various equilibrium values under private and shared information. Section 4 is devoted to exploring the impact of information sharing on the welfare of producers. Computations may not be so easy. I do hope, however, that our efforts should really be rewarding. Section 5 provides some concluding remarks.

2 A Stochastic Model of Duopoly under Private Cost Information

Let us deal with an ordinary Cournot duopoly game where each player treats an output as its strategic variable and is confronted with its own cost risk. We consider an industry in which there are two firms, namely firm 1 and firm 2, which produce homogeneous products. More specifically, let x_i be the output of firm i $(i = 1,2)$ and p its unit price.

Whenever we build an economic model, we like to obey the golden maxim: "Simple is best ! " Undoubtedly, linear equations are simple and beautiful, and so are exponential and logarithm functions. Besides, in a stochastic world, normal distribution functions, also known as Gaussian functions, are equally plain and cool !

Let the (inverse) demand function in the market be written as follows:

$$p = F(x_1 + x_2), \tag{1}$$

where F is a differentiable, decreasing function, so that $F' < 0$.

Now, let the cost function of firm i be denoted in the following way:

$$C_i = C_i \ (x_i ; k_i) \qquad (i = 1, 2) \qquad (2).$$

where the cost parameters k_1 and k_2 stand for random parameters. Let Φ (k_1, k_2) be the joint distribution function of k_1 and k_2. In particular, we assume that the form of the function Φ per se is a common knowledge for the two firms. If we utilize (1) and (2), we can express the profit of firm i in the following fashion:

$$\Pi_i = \Pi_i (x_1, x_2; \ k_i) = F(x_1 + x_2)\, x_i - C_i \ (x_i : k_i) \qquad (i = 1, 2) \qquad (3)$$

Here, we assume that each firm has a von-Neumann-Morgenstern utility function $U_i(\Pi_i)$. We especially suppose that U_i is an increasing and concave function, meaning that each firm displays risk aversion.

We are in a position to pay attention to the information structure we are dealing with. We are content to focus on the following two types:

① *Private information*, written as η^p , in which each firm knows its own cost, but not its rival's cost.

② *Shared information*, denoted by η^s , in which both firms share cost information with each other.

We would like to compare the two Cournot-Nash equilibriums; namely, the one under private information, η^p, and the other under shared information, η^s. On the one hand, a pair ($x_1^p(k_1)$, $x_2^p(k_2)$) of output strategies is said to be an equilibrium pair under η^p if for each k_1 and each k_2, the following pair of equations simultaneously hold :

$$x_1^p(k_1) = \arg\max E[U_1(\Pi_1(x_1, \ x_2^p(k_2)) \ | \ k_1] \qquad \text{for all } x_1 , \qquad (4)$$

$$x_2^p(k_2) = \arg\max E[U_2(\Pi_2(x_1^p(k_1), \ x_2) \ | \ k_2] \qquad \text{for all } x_2 . \qquad (5)$$

On the other hand, a pair ($x_1^p(k_1, k_2)$, $x_2^p(k_1, k_2)$) of output strategies is called an equilibrium pair under η^s if for each (k_1, k_2) , the following pair of equations simultaneously hold:

$$x_1^p(k_1, k_2) = \arg\max E \ [U_1(\Pi_1(x_1, x_2^p(k_1, k_2)) \,|\, (k_1, k_2)] \qquad \text{for all } x_1 , \qquad (6)$$

$$x_2{}^p(k_1, k_2) = \arg\max E\ [U_2(\Pi_2(x_1{}^p(k_1, k_2), x_2) \mid (k_1, k_2)] \quad \text{for all } x_2 .$$
(7)

It is assumed here that each firm wants to maximize expected utility of profits subject to information available to it. On the one hand, under η^p, the optimal output level of firm i is contingent on k_i (i = 1, 2). On the other hand, under η^s, it is contingent on both k_1 and k_2. Note that Cournot-Nash equilibrium represents a sort of "passive equilibrium" in the sense that once it is reached, no player has an "active incentive" to change its strategy unilaterally.

3 The Case of Constant Absolute-Risk-Aversion

3.1 A Cournot Duopoly Model with Simplifying Assumptions

The model we are going to work with is a simple Cournot duopoly model where each firm is subject to its own cost uncertainty. More specifically, let us make the following set of simplifying assumptions.

First of all, we suppose that the demand function each firm faces is linear:

$$F(x_1, x_2) = a - b(x_1 + x_2), \quad a, b > 0 , \tag{8}$$

where we assume without loss of generality that b is unity. Next, we also assume that the cost function of each firm is simple and linear:

$$C_i(x_i\ ; k_i) = k_i x_i . \qquad (i = 1, 2) \tag{9}$$

Concerning the cost uncertainty of the firms, we assume that the pair (k_1, k_2) of unit costs has a standard symmetric normal distribution of two variables with $E(k_i) = \mu$, $Var(k_i) = \sigma^2$, and $Cov(k_1, k_2) = \rho\sigma^2$ $(i = 1, 2)$. In other words, the relevant variance matrix should be of the following simple form:

$$\Sigma = \sigma^2 \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix} . \tag{10}$$

The aforementioned normal distribution may be depicted like a bell-shaped figure. As is seen in Fig. 1, the top of the bell is reached at $(k_1, k_2) = (\mu, \mu)$.

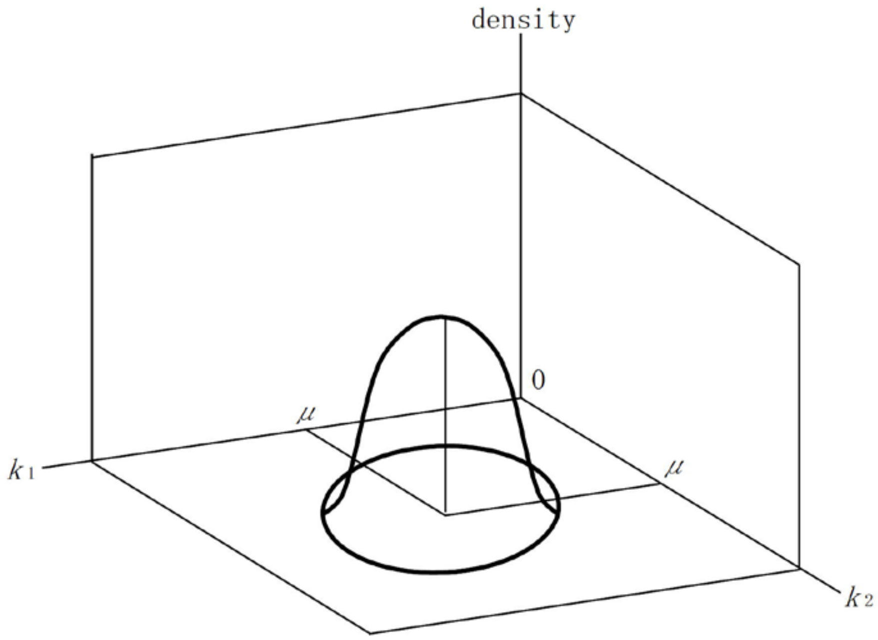

Fig. 1 The private cost risk is depicted like a bell-shaped figure

In what follows, we suppose that both firms have the same utility function, thereby occasionally dropping the subscript i. For the sake of simplicity, we assume that the utility function is exponential :

$$U_i(\Pi_i) = \beta - \gamma \exp[-R\,\Pi_i], \quad \beta, \gamma, R > 0, \tag{11}$$

in which R represents the coefficient of absolute risk aversion. When we put $\beta = \gamma = 1$ for combinience, (11) becomes a simpler equation:

$$U_i(\Pi_i) = 1 - \exp[-R\,\Pi_i], \quad R > 0, \tag{12}$$

whose figure is depicted in Fig. 2.

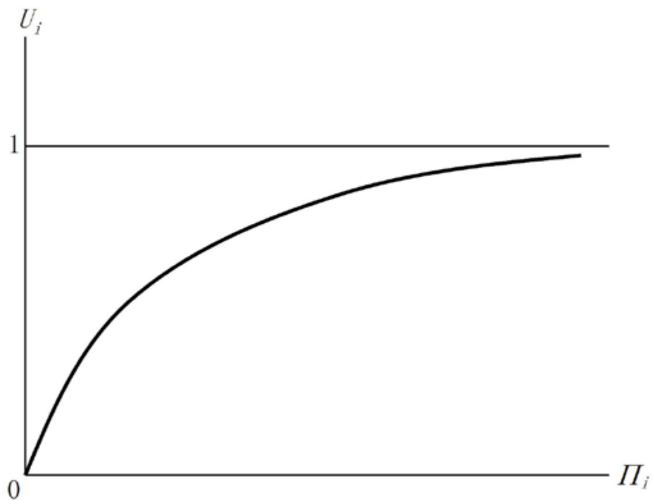

Fig. 2 The utility function, $U_i(\Pi_i)$, as an exponential function.

3.2 Equilibrium under Private Information

Let us begin our investigation with the case of private information, η^p. First of all, since b is assumed to be unity, we can write the profit of firm i as follows:

$$\begin{aligned}\Pi_i &= [a - b(x_i + x_j)]x_i - k_i x_i \\ &= x_i[a - k_i - x_i - x_j] \qquad (i, j = 1, 2;\quad i \neq j)\end{aligned} \tag{13}$$

In the light of (12), given the value of k_i and the rival's output strategy $x_j^p(k_j)$, firm i is supposed to choose its output x_i so as to maximize the following equation:

$$\begin{aligned}&E\ [U_i(\Pi_i(x_i, x_j^p(k_j); k_i) \mid k_i] \\ &= 1 - E\{\exp[-R\,\Pi_i(x_i, x_j^p(k_j); k_i)] \mid k_i\} \\ &= 1 - E\{\exp[-R x_i(a - k_i - x_i - x_j^p(k_j))] \mid k_i\}\ .\end{aligned} \tag{14}$$

Under those simplifying linearity assumptions aforementioned, we should expect to find the result that under η^p, the equilibrium output of firm i is also linear in k_i:

$$x_i^p(k_i) = -A\ (k_i - \mu) + B \qquad (i = 1, 2), \tag{15}$$

where A and B represent the quantities to be determined as below. Since both firm 1 and firm 2 are symmetrically treated in this paper, $x_1^p(k_1)$ and $x_2^p(k_2)$ should have the same functional form indicated by (15), so that we also obtain

$$x_j^p(k_j) = -A\ (k_j - \mu) + B \qquad (j = 1, 2), \tag{16}$$

. So, in the light of (16), (14) above becomes

$$E\ [U_i(\Pi_i(x_i, x_j^p(k_j)) \mid k_i]$$
$$= 1 - E\ 【\exp[-Rx_i(a - k_i - x_i + A(k_j - \mu) - B)] \mid k_i\ 】 \tag{17}$$

.

We note that the *conditional density of the quantity* $(k_j - \mu)$*, given* k_i, is normal with mean $\rho(k_i - \mu)$ and variance $\sigma^2(1-\rho^2)$. Consequently, the conditional density of the quantity $(a - k_i - x_i + A(k_j - \mu) - B)$, given k_i, is normal with mean $(a - k_i - x_i + A\rho(k_i - \mu) - B)$ and variance $A^2\sigma^2(1-\rho^2)$. [4)]

Besides, we find it very helpful to use the following mathematical lemma:

LEMMA (Expectation of Exponential Function over Normal Distribution)

Let a stochastic variable y be normally distributed with mean μ and variance σ^2, and γ be a constant. Then we obtain the following properties:

$$① \quad E\ \exp[\gamma y] = \exp[\gamma\mu + (1/2)\gamma^2\sigma^2]. \tag{18}$$

$$② \quad E\ \exp[-\gamma y^2] = [1/(1+2\gamma\sigma^2)^{1/2}]\exp[-\gamma\mu^2/(1+2\gamma\sigma^2)] \tag{19}$$

The proof of this lemma is rather straightforward, but perhaps will cause some psychological pain. So, we would like to omit it in this paper. [5)]

In what follows, we want to stress the applicability of the lemma in many ways. Indeed, if we make use of the two properties ① and ②, we may clearly transform (17) into the following equation:

$$E\ [U_i(\Pi_i(x_i, x_j^p(k_j)) \mid k_j]$$
$$= 1 - \exp[-Rx_i(a - k_i - x_i + A\rho(k_i - \mu) - B)$$

$$+ \quad (1/2)\, R^2 (x_i)^2 A^2 \sigma^2 (1 - \rho^2) \quad]$$

$$= \quad 1 - \exp [\, - R x_i^2 (a - k_i - (1 + (1/2)\, QA^2) + A\rho (k_i - \mu) - B)]\,, \qquad (20)$$

where Q is defined by $Q = R\ \sigma^2 (1 - \rho^2)$. Hence, the first-order condition for expected utility-maximization with respect to x_i is clearly provided in the following way.

$$a - k_i - (2 + QA^2)\, x_i^p (k_i) + A\ \rho (k_i - \mu) - B \quad = \quad 0\,, \qquad (21)$$

from which follows the equation:

$$x_i^p (k_i) = \ - [(1 - A\rho) / (2 + QA^2)] (k_i - \mu) + (a - \mu - B) / (2 + QA^2). \qquad (22)$$

Now, we are ready to compare the two equations, (15) and (22). Since they should unquestionably be identical equations, we obtain the following results:

$$A \ = \ (1 - A\rho) / (2 + QA^2)\,, \qquad (23)$$

$$B \ = \ (a - \mu - B) / (2 + QA^2)\,. \qquad (24)$$

If we rearrange (22) and (23), then we may easily derive the following equations:

$$R\ \sigma^2 (1 - \rho^2) A^3 \ + \ (2 + \rho) A \ - 1 \ = \ 0\,, \qquad (25)$$

$$B \ = \ (a - \mu) / [\, 3 + R\sigma^2 (1 - \rho^2) A^2] \quad . \qquad (26)$$

It is noted that Eq. (25) represents a *cubic equation with respect to* A except for the perfect correlation case $\rho = \pm 1$, where the equation becomes just linear.. Although the A^2- term is not present in (25), the equation per se is nevertheless fairly complicated and very hard to solve for A. In order to inquire into the properties of such cubic equation more deeply, let us newly introduce the function $g(A)$ as follows:

$$g(A) \ = \ R\ \sigma^2 (1 - \rho^2) A^3 \ + \ (2 + \rho) A \ - 1 \quad . \qquad (27)$$

Then, differentiating (26) with respect to A, we find the following derivative:

$$g'(A) = 3R\sigma^2(1-\rho^2)A^2 + (2+\rho), \tag{28}$$

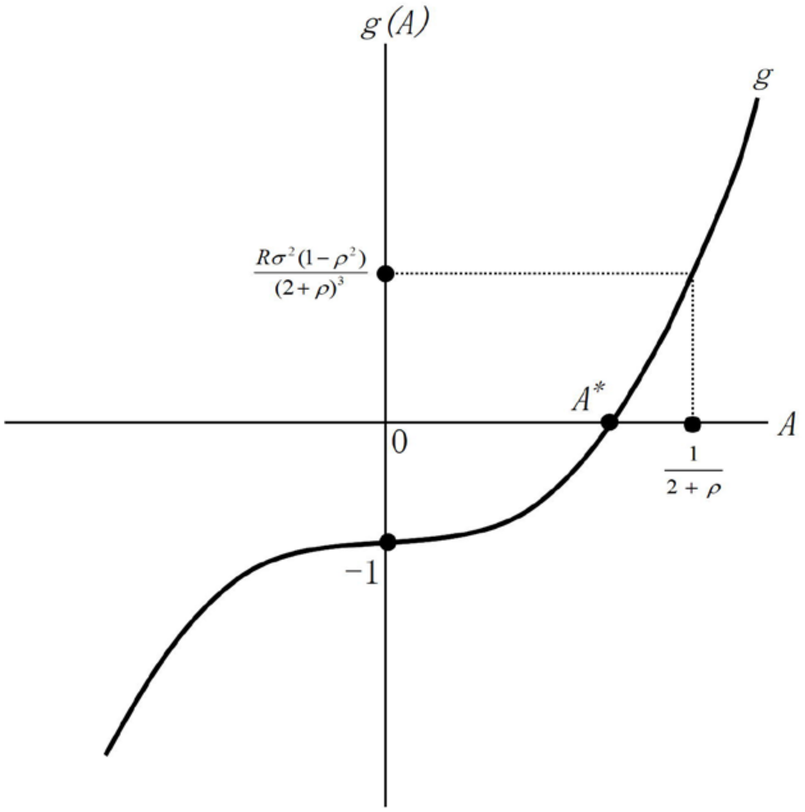

Fig. 3 The graphical representation of $g(A)$

which is always positive. Consequently, $g(A)$ is an increasing function. If we differentiate (28) once more, we immediately obtain the following second derivative:

$$g''(A) = 6R\sigma^2(1-\rho^2)A \quad , \tag{29}$$

implying that $g''(A) \gtreqless 0$ if and only if $A \gtreqless 0$.

As is easily shown, $g(0) = -1$ and $g(1/(2+\rho)) > 0$. Since $g(A)$ is increasing by (28), it follows from Fig. 3 that letting g(A) = 0 gives only one real root, A^*, whose value must be lie between zero and $1/(2+\rho)$. Besides, by virtue of (29), $g(A)$ is concave (or convex) if A is negative (or positive). So, the point (0, − 1) gives us the only one reflection point of the cubic curve g(A).

Very long time ago, Gerolamo Cardano (1501-1576) , a spirited mathematician from Italy, successfully attempted to find a general formula to solve cubic equations. Today, modern mathematicians are a bit smarter than Cardano, thus cleverly offering us a better formula than his old formula. Specifically, we are now ready to use the following improved formula for cubic equations : [6)]

THEOREM (The Cubic Formula of Cardano, with Drawbacks Corrected)

Let us consider the cubic equation $ax^3 + bx^2 + c = 0$. Then we have the solution as follows :

$$x = 【p + (p^2 + q^3)^{1/2}】^{1/3} + 【p - (p^2 + q^3)^{1/2}】^{1/3} - (b/3a), \quad (30)$$

where

$$p = (-b^2/27a^3) + (bc/6a^2) - (d/2a), \quad (31)$$

$$q = (c/3a) - (b^2/9a^2). \quad (32)$$

The exact proof of the Cardano Formula is so complicated that it is wisely omitted here, letting other mathematical papers handle it. [6)]

Now, let us dare to apply the Cardano Formula (30) to the cubic function $g(A) = 0$. Then after some calculations, we will find the following solution:

$$A = 【1/(2Q)^{1/3}】【[1 + (1+L)^{1/2}]^{1/3} + [1 - (1+L)^{1/2}]^{1/3}】, \quad (33)$$

where

$$Q = R\sigma^2(1 - \rho^2) \quad \text{and} \quad L = 4(2 + \rho^3)/27Q. \quad (34)$$

If we insert (33) and (34) into (26), we can calculate the value of B in the following way:

$$B = (a - \mu)/(3 + QA^2). \; . \quad (35)$$

In the light of (15), we can compute the expected value E and variance V of $x_i{}^p(k_i)$ as follows:

$$E\ x_i{}^p = B = (a - \mu)/[3 + QA^2], \quad (36)$$

$$V\ x_i{}^p = E\ (x_i{}^p - E x_i{}^p)^2 = \sigma^2 A^2 \; . \quad (37)$$

We are ready to compute the expected utility of equilibrium profits, $EU_i^{\,p}$.
In the light of Eq. (16) above, we find it convenient to start with the following *conditional expectation over* k_i :

$$E_j\,[U_i(\Pi_i(x_i, x_j^{\,p}(k_j))\mid k_i]$$
$$=1-E_j\ \exp\ 【-Rx_i^{\,p}(k_i)\ [a-k_i-x_i^{\,p}(k_i)+A(k_j-\mu)-B)]\mid k_i\ 】.\quad (38)$$

As mentioned above, the *conditional density of the quantity* $(k_j-\mu)$*, given* k_i, is normal with mean $\rho(k_i-\mu)$ and variance $\sigma^2(1-\rho^2)$. So, if we effectively employ the mathematical lemma aforementioned, then we can obtain the following derivations :

$$E_j\ [U_i(\Pi_i(x_i, x_j^{\,p}(k_j))\mid k_i]$$
$$=1-\ \exp\ 【-Rx_i^{\,p}(k_i)(a-k_i-x_i^{\,p}(k_i)+A\rho(k_j-\mu)-B)$$
$$+(1/2)R^2\ (x_i^{\,p}(k_i))^2\ A^2\sigma^2(1-\rho^2)\ 】.$$
$$=1-\ \exp\ 【-Rx_i^{\,p}(k_i)[(a-k_i-(1+(1/2)QA^2)x_i^{\,p}(k_i)$$
$$+A\rho(k_j-\mu)-B]\ 】\quad (39)$$

In the light of (21) , we note the following relation:

$$a-k_i-(1+(1/2)QA^2)x_i^{\,p}(k_i)+A\ \rho(k_j-\mu)-B$$
$$=\ (1+(1/2)QA^2)x_i^{\,p}(k_i)\ .\quad (40)$$

If we take account of (40), (39) can be transformed to the following:

$$E_j\ [U_i(\Pi_i(x_i, x_j^{\,p}(k_j))\mid k_i]$$

$$=1-\ \exp\ 【-R\ (1+(1/2)QA^2)\ (x_i^{\,p}(k_i))^2\ 】,\quad (41)$$

Now, taking the further *expectation over* k_i of the conditional expectation (41) and later employing (18), we obtain the following equation:

$$EU_i^{\,p}=E_i\ 【\ E_j[U_i(\Pi_i(x_i, x_j^{\,p}(k_j))\mid k_i]\ 】$$

$$=\ 1-(1/F^{1/2})\exp[-G],\quad (42)$$

where F and G represent the quantities defined by the following :

$$F = 1 + 2R[1 + (1/2)QA^2]\, Vx_i^p \quad , \tag{43}$$

$$G = [1 + (1/2)QA^2](Ex_i^p)^2 \;/\; 【 1 + 2R[1 + (1/2)QA^2]\, Vx_i^p 】. \tag{44}$$

3-3 Equilibrium under Shared Information

Let us turn to the case of shared information, η^s, in which each firm can know both values of k_1 and k_2. In this case, for a given k_i, firm i chooses its output x_i so as to maximize the following utility of profit :

$$U_i(\Pi_i(x_i, x_j^s(k_1, k_2); k_i) = 1 - \exp[-Rx_i(a - k_i - x_i - x_j^s(k_1, k_2))】. \tag{45}$$

The first-order condition gives us the following equation:

$$a - k_i - 2x_i^s(k_1, k_2) - x_j^s(k_1, k_2) = 0. \quad (i, j = 1,2\,;\, i \neq j) \tag{46}$$

If we solve a set of equations for x_1^s and x_2^s, then we obtain the following:

$$x_i^s(k_1, k_2) = [a - 2k_i + k_j] / 3 \qquad (\mathrm{i, j} = 1,2\,;\, i \neq j)\,, \tag{47}$$

from which we can easily compute the following:

$$Ex_i^s \equiv E\ [x_i^s(k_1, k_2)] = (a - \mu)/3 \quad , \tag{48}$$

$$Vx_i^s \equiv E\ [(x_i^s(k_1, k_2) - Ex_i^s]^2 = (5 - 4\rho)\ \sigma^2/\ 9. \tag{49}$$

Using (45) and (46), we immediately find

$$U_i(\Pi_i(x_i^s(k_1, k_2), x_j^s(k_1, k_2); k_i) = 1 - \exp[-R\ (x_i^s(k_1, k_2))^2】. \tag{50}$$

Note that by virtue of (47), $x_i^s(k_1, k_2)$ is a normally distributed random variable whose mean and variance are respectively given by (48) and (49). Hence, taking the expectation of (41) *over both k_1 and k_2*, and later using (18), we find

$$\begin{aligned} EU_i^s &= 1 - E\ \exp[-R\ (x_i^s(k_1, k_2))^2】 \\ &= 1 - (1/\boldsymbol{H}^{1/2})\exp[-K], \end{aligned} \tag{51}$$

where H and K stand for the quantities given by the following:

$$H = 1 + 2R\,Vx_i^s \qquad (52)$$

$$K = R(Ex_i^s)^2 / [1 + 2R\,Vx_i^s]\,. \qquad (53)$$

4 The Impact of Information Exchange: The Mean and Variance Effects

The problem we now wish to ask is how and to what extent the information exchange between firms affects the their output levels *in terms of expected values and variances.* While the exchange tends to increase the expected outputs of firms, it is also likely to increase their variances. The first effect may be called the *mean effect*, and the second the *variation effect*. Which one of the two effects is a dominating actor on the stage of information exchange should be of utmost importance.

4.1 The Impact of Information Sharing on Outputs

We are in a position to establish the following proposition of great importance:

PROPOSITION 1. Under the set of linearity-normality assumptions aforementioned, we obtain the following results:

① (the mean effect) $\boldsymbol{Ex_i^s \geqq Ex_i^p}$, **(54)**
where the equality holds if and only if $\rho = \pm 1$.

② (the variance effect) $\boldsymbol{Vx_i^s \geqq Vx_i^p}$, **(55)**
where the equality holds if and only if $\rho = \pm 1$.

To prove ① of this proposition, let us write again here the mean and variance of Ex_i^p and Ex_i^s (see Eqs. (36) and (48)) :

$$Ex_i^p = (a - \mu) / [3 + QA^2]\,, \quad Ex_i^s = (a - \mu) / 3\,;\ Q = R\,\sigma^2 (1 - \rho^2)$$

which clearly shows that Ex_i^s is al least as great as Ex_i^p, and that they are just equal if and only if $\rho^2 = 1$. This proves ①.

To prove ②, let us rewrite the mean and variance of Vx_i^p and Vx_i^s (see Eqs. (37) and (49)) :

$$Vx_i^p = \sigma^2 A^2, \quad Vx_i^s = \sigma^2[(5-4\rho)/9]; \quad 0 < A^2 < 1/(2+\rho^2)^2$$

We can show that Vx_i^s is at least as great as Vx_i^p , and that they are just equal if and only if $\rho^2 = 1$. This proves ② .

At this point, a graphical explanation of Proposition 1 would be very instructive. To this end, let us consider the relevant reaction functions under η^p and η^s. In Fig. 4, the horizontal axis measures the expected output of firm i, and the vertical axis the expected output of firm j. On the one hand, it is not a difficult job to find that under η^p, a pair of *derived reaction functions in terms of* Ex_1 *and* Ex_2 are provided in the following way: [7]

$$a - \mu - [2 + R\sigma^2(1-\rho^2)A^2]\, Ex_i^p - Ex_j^p = 0. \quad (i, j = 1,2\,;\, i \neq j) \tag{56}$$

On the other hand, in a similar fashion, a pair of derived reaction functions under η^s are given follows: [8]

$$a - \mu - 2\, E_i E_j x_i^p - E_i E_j x_j^p = 0. \quad (i, j = 1,2\,;\, i \neq j) \quad . \tag{57}$$

or more simply,

$$a - \mu - 2\, Ex_i^p - Ex_j^p = 0. \quad (i, j = 1,2\,;\, i \neq j) \quad . \tag{58}$$

In Fig. 4, Points Q^p and Q^s respectively represent Cournot-Nash equilibrium under η^p and η^s. Unless k_1 and k_2 are perfectly (positively or negatively) correlated, Point Q^s lies northeast of Q^p. Clearly, this implies that Ex_i^s is greater than Ex_i^p .

Let us take a more careful look again at Proposition 2. Interestingly, the exchange of cost information between risk averse firms affects outputs in two distinctive ways. First, it leads to an increase in the expected value of each output. This may be called the *mean effect*. Second, it results in a rise in the variance of each output as well. This is because information sharing makes the production activities of firms more responsive to changes in cost conditions: it can be named the *variation effect.* Which plays a dominant part, the mean effect or the variance effect ? In short, when we consider the impact of private information between risk-averse players, this must be a very critical question to ask.

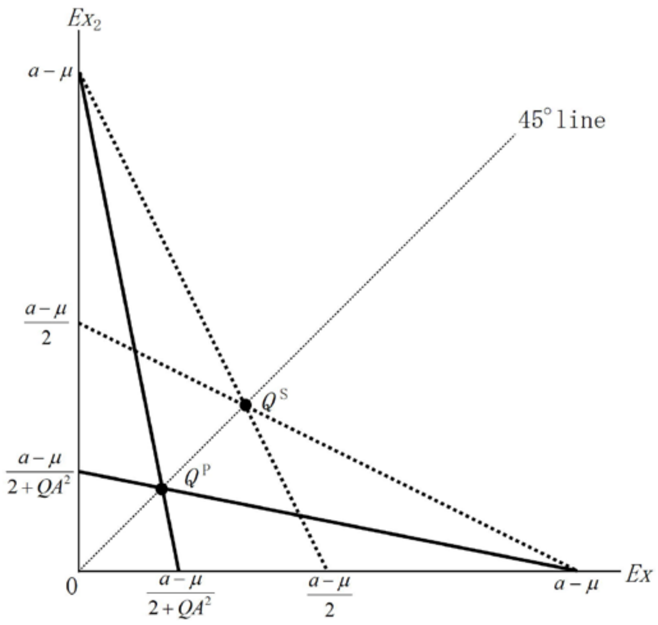

Fig. 4 The Impact of Information Sharing on Outputs: $Ex_i^{s} \geqq Ex_i^{p}$

4.2 The Impact on the Welfare of Firms

We are now interested in the welfare aspect of information exchange. To this end, we are ready to establish the following welfare proposition of utmost importance:

PROPOSITION 2. Under the set of linearity-normality assumptions aforementioned, we have the following results:

$$EU_i^{s} \gtreqless EU_i^{p} \quad \text{according to whether} \quad (1/2RT)\log(H/F) \gtreqless (a-\mu)^2, \qquad (59)$$

where

$$T = [2+QA^2] / \{2[3+QA^2]^2 [1+R\sigma^2A^2(2+QA^2)]\} - 1/[9+2(5-4\rho)R\sigma^2]. \qquad (60)$$

To prove this proposition, we note by virtue of (42) and (51) the following equivalence relations:

$$E U_i^{s} \gtreqless E U_i^{p} \quad \Leftrightarrow 1 - (1/H^{1/2}) \exp(-K) \gtreqless 1 - (1/F^{1/2}) \exp(-G)$$
$$\Leftrightarrow \quad (H/F)^{1/2} \quad \gtreqless \quad \exp(G - K)$$
$$\Leftrightarrow \quad (1/2) \quad \log(H/F) \quad \gtreqless \quad G - K \qquad (61).$$

Now we have to prove that ① H is greater than F, and that ② T, defined by (60), is positive. To this end, if we substitute Eqs. (10), (11), (14) and (15) into the definitions of G and K, we find the following:

$$G - K = R T \ (\mathrm{a} - \mu)^2 \quad . \qquad (62)$$

①: Let $y = R\sigma^2 A^2 > 0$. Then from (24), we immediately have

$$(1 - \rho^2) y + 2 + \rho = 1/A, \qquad (63)$$

so that

$$R\sigma^2 = y/A^2 = [(1 - \rho^2) y + 2 + \rho]^2 \quad y. \qquad (64) .$$

Substituting these equations into (60) above, we obtain

.

$$T = I \ / \ J, \qquad (65)$$

where

$$I = 【9 + 2(5-4\rho)[(1-\rho^2)y + 2 + \rho]^2 \ \mathrm{y}】【2 + (1-\rho^2)y】$$
$$- \ 2 \ 【3 + (1-\rho^2)y】^2 \ 【1 + \mathrm{y}(2 + (1-\rho^2)y)】$$
$$= \quad (1-\rho^2)y \ 【8(1-\rho^2)^2(1-\rho)y^3 + 4(1-\rho)^2(1+\rho)(11+4\rho)y^2$$
$$+ 2(1-\rho)(38 + 30\rho + 4\rho^2)y + (41 + 16\rho)】, \qquad (66)$$

$$J = 2【3 + (1-\rho^2)y】^2 【1 + [2 + (1-\rho^2)y]y】【9 + 2(5-4\rho)[(1-\rho^2)y + 2 + \rho]^2 \ \mathrm{y}】. \quad (67)$$

Cleary, we see that both I and J are positive., so that T should be positive.

② Adopting a method similar to ①, we also find that

$$H - F = 【2(5-4\rho)/9】【(1-\rho^2)y + 2】^2 y \ - 【2 + (1-\rho^2)y】y$$
$$= 【\mathrm{y}(1-\rho^2)/9】【2(5-4\rho)[(1-\rho^2)\mathrm{y}^2 + (31-12\rho-16\rho^2)\mathrm{y} + 2(11+4\rho)】, \qquad (68)$$

which must be positive. Hence, we should find H greater than F. Q.E.D.

In what follows, we would like to consider economic implications of the Second Proposition in many possible ways. As the proposition shows, the impact of information sharing on the welfare of risk-averse producers may go in either direction, depending upon many factors to be listed below:

(i) the expected values of net demand intercept, $a - \mu$,
(ii) the degree of risk aversion, R,
(iii) the degree of risk per se. σ^2,
(iv) the value of correlation, ρ.

Clearly, there are so many combinations of those four factors conceivable. In some situations, firms may gain from exchanging their cost information with each other. In other situations, however, they may not gain from such exchange. To find more definite answers, we must have the courage to go several steps further,

First of all, we would like to examine how welfare gain or loss is related to Factor (i) above. As we can see from the First Proposition above, there exist two different channels through which information exchange affects the welfare of risk averse firms; namely, the mean effect and the variance effect. If the factor $(a - \mu)$ is sufficiently small, the mean effect presumably dominates the variance effect, so that information sharing makes both firms better off. When the factor $(a - \mu)$ is sufficiently large, however, the situation must change drastically. Risk aversion now plays such a critical role in determining the welfare impact of information sharing between firms is *harmful* rather than beneficial to them.

Fig. 5 indicates the significance of the Second Proposition in a more visible fashion. The horizontal axis measures the quantity $(a - \mu)$, and the vertical axis the quantity $(EU^s - EU^p)$. For convenience, we assume that R is ten and σ^2 is unity. Let us suppose that ρ takes on five different values; namely, $\rho = -0.95, -0.7, 0, 0.7, 0.95$. Then, as is seen in Fig. 5, we have five U-shaped curves corresponding to those values of ρ. When any one of those curves lies above the horizontal line OH, we are in the normal situation in which information exchange is beneficial to firms. If, however, a U-shaped curve happens to lie below the line OH, we enter the anomalous world in which non-sharing is even better than sharing. As our experience teaches us, "going alone" is sometimes better than "going together" !

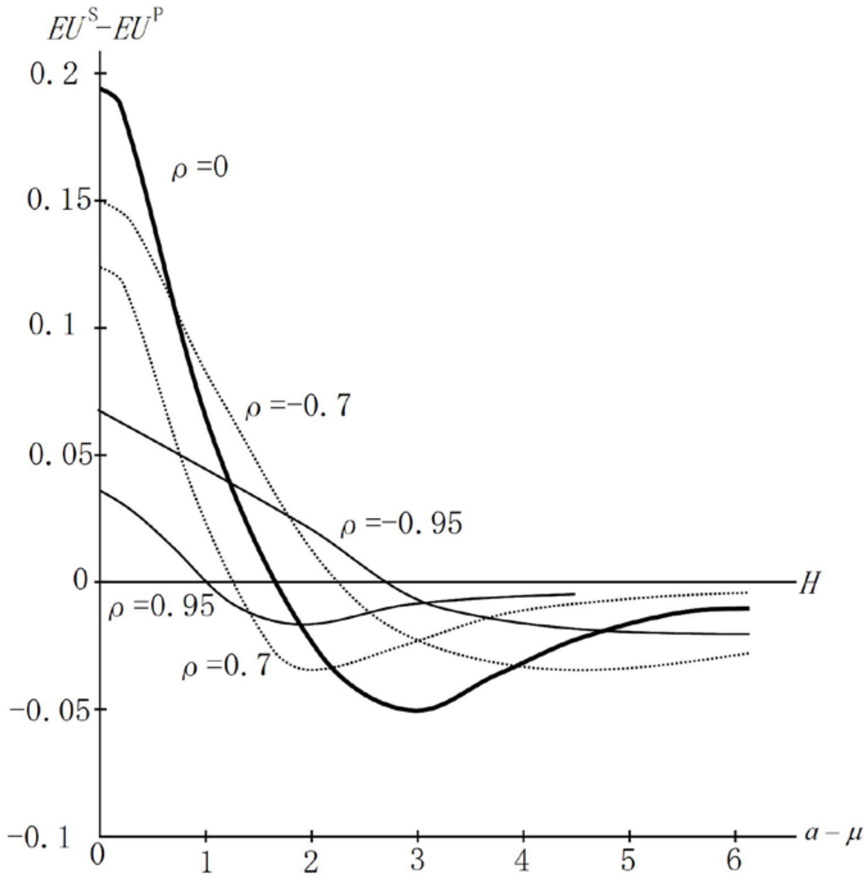

Fig. 5 The Impact of Information Sharing on the Expected Utility of Profits: Changes in $(a-\mu)$

Fig. 6 gives us another look at the problem of how the welfare gain or loss from information is sensitive to Factor (iv); namely, the value of correlation, ρ. Note that as before, R is ten and σ^2 is unity, and that $(a-\mu)$ is now fixed at two. When ρ is negative, the two firms are stochastically in a complementary relation, so that the exchange of information is likely to contribute positively to the welfare of firms. By contrast, in case ρ is positive, the conflict of interests between rival firms arises so seriously that information pooling may be rather harmful to them. In short, between friends, "going together" is always helpful. Between rivals, however, "going alone" may sometimes be a better policy. Human relation are really complicated indeed !

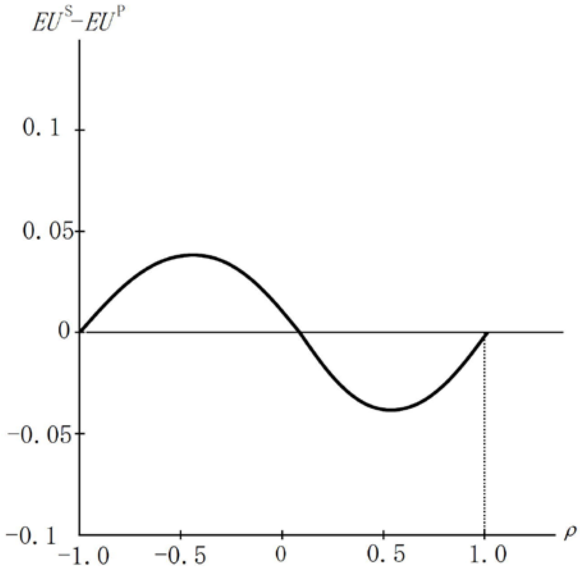

Fig. 6 The Impact on the Welfare of Firms: Changes in ρ

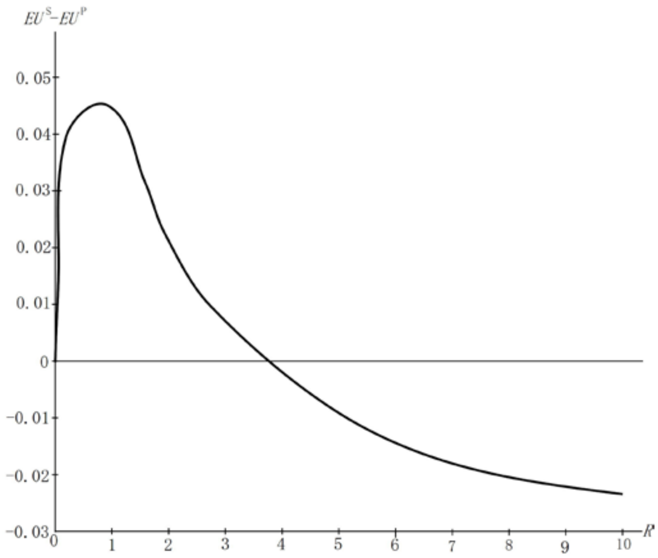

Fig. 7 The Relationship between the Degree of Risk Aversion

and the Value of Information

Finally, we can see the relationship between the degree of risk aversion and the value of information. Note that the horizontal axis now measures the value of R. Just for the sake of convenience, we assume that $\sigma^2 = 0.8$, $\rho = 0$, and $a - \mu = 1.9$. Then, we can draw a unique figure like a playground slide in Fig. 7.

When the value of R is sufficiently small (in fact, between one and four in the present case), EU^s exceeds EU^p. When the value of R becomes sufficiently large, however, the non-normal situation under which information exchange is rather harmful may emerge. This clearly demonstrates that an increasing degree of risk aversion has a negative impact on the information exchange between firms. As the saying goes, a wise man keeps clear of danger ! [7] .

5 Concluding Results

In this paper, we have been intensively concerned with the important issue of risk aversion and duopoly. The fundamental question to ask is simple like this:

" Is the information exchange between the two firms always beneficial to them ?
Or possibly, is it rather hurtful to them ? "

In our daily life, we tend to be guided, and perhaps perplexed, by the following two opposing kinds of proverbs. As one kind of proverbs teaches us the following:

" Knowledge is power,"
"Two heads are better than one."

There is another kind of proverbs, however, that says exactly the opposite :

" Ignorance is bliss."
"He toucheth pitch shall be defiled."

What we have shown in this paper is that both kinds of proverbs are not contradictory as they seem, and that their validity depends on the real circumstances we are confronting with. More specifically, we have devoted all our energy into an investigation of how and to what extent the presence of risk aversion affects the outputs and welfare of producers. On the one hand. information pooling tends to increase the means and variances of output simultaneously. On the other hand, information sharing may sometimes be harmful rather than beneficial to firms. Therefore, the welfare results when the firms display risk aversion are different from those of many existing papers, in which the firms engaging in information sharing are usually supposed to be risk neutral,

Admittedly, our duopoly model with private cost exchange is a very simple one, involving constant absolute-risk-aversion utility functions, normally distributed random variables, together with linear demand and cost functions. It is true that weakening some of those specific assumptions could make our model more general. On theory, we have no objection against this generalization whatever. It would make our task of deriving equilibrium values under alternative information structures, however, which already requires fairly troublesome computations, an even more formidable from a pragmatic point of view. Besides, we believe that even if we work with a more general framework than we have done in this paper, the possibility that the value of acquiring additional information may be negative still remain. We must bear in mind that both good and bad coins are in circulation. and that bad coins may sometimes drive out good ones.

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Footnotes

1) The year of 1970, when Arrow's Essays were first published, can be regarded as a very "memorial year" in the sense that it well-represents the dawn of a new era named "Risk and Uncertainty." Together with Akerlof, Spence and Stiglits, Arrow was a good representative of the "A-S Age," which was so called by collecting the initials of those four pioneers. In personal perspective, the Essays have given Sakai a great shock until today. For details, see Sakai (1982).

2) Bernoulli (1738) was first published in Latin as a mathematical paper at St. Petersburg, the capital of the Russian Empire, and more than two hundred years later translated in English. In this epoch-making and long standing paper, he boldly introduced the Law of Decreasing Marginal Utilities, which was an outstanding achievement in the history of economic thought, being far ahead of the times of the Marginal Revolution in the 1980s. This clearly tells as that Bernoulli was also historically first scholar who introduced the concept of Risk Aversion in the Theory of Decision Making under Risk.

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3) For those long years from the 1970s to the present, there have been a vast volume of papers on oligopoly and information. See Basar and Ho (1974), Ponssard (1979a, 1979b), Clark (1983), Vives (1984, 1999, 2002, 2008), Okada (1984), Sakai (1985, 1987, 1993), Shapiro (1986), Gal-Or (1985), Demange and Laroque (1995), Raith (1996), Jin (1998), and many others. For a summary of those and related articles, see Sakai (1990, 1991).

4) For the properties of the *conditional probability of the multivariate normal distribution*, see Mood and Graybill (1963). More generally, let the two-dimensional random variable (x, y) have the bivariate normal distribution with mean (μ_x, μ_y) and variance $(\sigma_x{}^2, \sigma_y{}^2)$. Then the *conditional density of y, given x*, is normal with mean $\mu_x + (\rho\,\sigma_y/\sigma_x)(x - \mu_x)$ and variance $\sigma_y{}^2(1-\rho^2)$.

5) The detailed proof of this lemma is given in Sakai (2015), p. 177.

6) For Cardano and the solution of the cubic, see Dorsey, B., Downie, K.-L., Huber, M (2020). Also see Wikipedia Web (2020). As far as I know, the cubic formula a la Cardano was first applied to economics in Sakai and Yoshizumi (1991b).

7) Related topics relating to our research in this paper, are informational non-efficient markets and Pareto inferior trades. See Grossman and Stiglitz (1980), and Newbery and Stiglitz (1984).