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A Term Structure Interest Rate Model with the Exit Time from the Negative Interest Rate Policy

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Abstract

In the government bond markets in Japan and a number of European countries, negative interest rates have been observed in recent years. Incorporating a negative lower bound for interest rates into a term structure model makes it possible for the model to replicate yield curves that include negative rates. In this study, we propose a new term structure model with a stochastic lower bound where the short rate is defined as the sum of the quadratic form of the Gaussian process and a negative lower bound for interest rates. The lower bound is characterized by a Brownian bridge with the random interval pinned at zero at the starting time and the end time of a negative interest rate policy (NIRP). Under this setting, we derive a zero coupon bond price formula by imposing the no arbitrage condition. We calibrate our proposed model using Japanese yield curve data and estimate the implied posterior distribution of the time to exit from the NIRP.
Keywords: Yield curve, No arbitrage condition, Quadratic Gaussian term structure model, Brownian bridge, Negative interest rate policy.
JEL Classification E43, E52, G12

1 Introduction

In the government bond markets in Japan and a number of European countries, short- and medium-term interest rates have been negative for several years against the backdrop of quantitative easing (QE) and negative interest rate policy (NIRP) implemented by the Bank of Japan and the European Central Bank. Moreover, as Figure 1 indicates, even long-term interest rates in these countries have sometimes fallen below zero during these periods of QE and NIRP.

Previous studies of term structure models with negative interest rates have been motivated by two main goals. One is to construct a model to price interest rate derivatives in a negative interest rate environment. For example, Hagan et al. [11] extend the SABR model proposed by Hagan et al. [10] to incorporate a constant negative lower bound for the forward rate. This model is called the shifted SABR model. Additionally, Antonov et al. [3] propose the free boundary SABR model where forward rates can take negative values with no restrictions on their lower bounds.

The second purpose of a number of previous studies on term structure models with negative interest rates is to construct a model to extract information about market expectations on economic trends and future monetary policy developments. In this study, we focus on constructing a term structure model for this purpose.

In a standard affine Gaussian term structure model proposed by Duffie and Kan [7], interest rates can take negative values. This model ensures the tractability of parameter and

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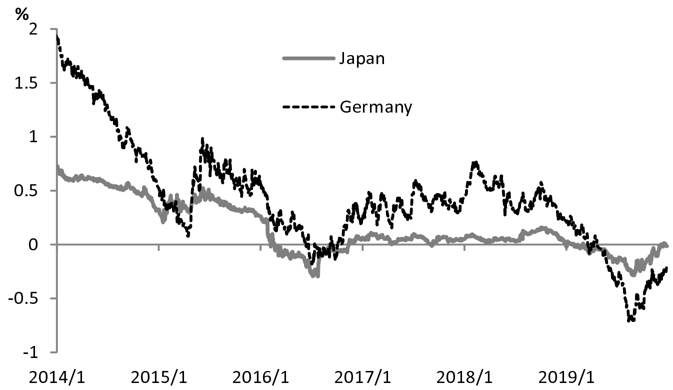


Figure 1: 10-Year Government Bond Yields in Japan and Germany

state variable estimation; hence, many studies (including Ang and Piazzesi [2] and Kim and Orphanides [14]) have used it to extract information about market expectations of economic trends and monetary policy developments. However, a standard affine Gaussian term structure model is more likely to show a higher probability of negative future interest rates than market participants expect because the model implied distribution of future interest rates follows a normal distribution. This shortcoming is an important cause of this type of model's low estimation accuracy in a low interest rate environment.

In the face of the low interest rate environment that has existed since the global financial crisis of 2007-2008, alternative term structure models have been studied to examine market participants' views on future economic trends and monetary policy developments. One of these models is the so-called shadow rate model. This model, proposed by Black [5] and analytically formulated by Gorovoi and Linetsky [9], defines the short rate as the larger of a state variable called the shadow rate and a constant threshold. The shadow rate model with a zero threshold exhibits its power in analyzing a low interest environment where the short-term interest rate is close to zero. To illustrate, Kim and Singleton [15] and Wu and Xia [21] set a threshold of zero in applying the shadow rate model to analyze the Japanese and U.S. government bond markets, respectively. Setting a threshold at a constant negative value in the shadow rate model produces a term structure model that generates negative interest rates. Lemke and Vladu [17] apply the shadow rate model with a negative threshold to Eurozone yield curve data.

Another alternative to a standard affine Gaussian term structure model is the quadratic Gaussian term structure model studied by Ahn et al. [1] and by Leippold and Wu [16]. This is a short rate model where the short rate is defined as a quadratic function of state variables and provides a term structure of interest rates with a lower bound. Nyholm and Vidova-Koleva [19] estimate this model using U.S. yield curve data. Kim and Singleton [15] provide an empirical comparison between this model and the shadow rate model using Japanese yield curve data. Although Nyholm and Vidova-Koleva [19] and Kim and Singleton [15] set a lower bound of interest rates at zero, incorporating a negative lower bound produces a term structure model that allows for negative interest rates.

Incorporating a constant negative lower bound for interest rates into the shadow rate model and the quadratic Gaussian term structure model could improve their ability to capture the yield curve shapes currently seen in Japan and in some European countries, compared with a standard affine Gaussian term structure model. However, a term structure model with a constant negative lower bound of interest rates does not update the lower bound based on a strengthening (or weakening) of QE or NIRP; in the markets in reality, strengthening (or tapering) these policies would make the negative lower bound deeper (less negative). As such, we cannot say that a term structure model with a constant negative lower bound for interest rates is sufficient to extract market information under a negative interest rate environment. A better approach would incorporate a stochastic lower bound of interest rates to extract the market information under a negative interest rate environment more accurately.

At the end of NIRP[1], the lower bound of interest rates rises to zero. Given this, introducing a stochastic lower bound of interest rates into a model is desirable. In this study, we model a stochastic lower bound of interest rates using a Brownian bridge, which is a Brownian motion pinned at the origin at the starting time and the end time. We regard the starting time of Brownian bridge as the date when we observed negative interest rates for the first time. For the end date of Brownian bridge, we assume that it is the end date of QE or NIRP.

To our knowledge, there are few studies of term structure models that explicitly introduce an end date for QE or NIRP. However, Marumo et al. [18][2] assume that the short rate stays at zero until the end of the Bank of Japan's zero interest rate policy (ZIRP) and evolves based on the Vasicek model after the ZIRP ends. They model the exit time from the ZIRP with a random variable. As with Marumo et al. [18], we model the end date of QE or NIRP as a random variable accordingly. Since we model the lower bound of interest rates with a Brownian bridge, we make the end time of the Brownian bridge stochastic. Here, note that in general, the Brownian bridge has a deterministic end time; however, Bedini et al. [4] recently formulated a Brownian bridge with a random time interval. Thus, in our modeling of a stochastic lower bound for interest rates, we apply the Brownian bridge with a random interval using the formulation in Bedini et al. [4].

In this study, we construct a term structure model based on a short rate model. We define the short rate as the sum of the quadratic function of state variables and a stochastic lower bound provided by the Brownian bridge with a random interval as mentioned above. We then derive the zero coupon bond price formula under the no arbitrage condition. In addition, we calibrate our proposed model to Japanese government bond zero coupon rates to examine the fit, and extract market participants' expectations about when the Bank of Japan will end its unconventional monetary policy.

The rest of this paper is organized as follows. We provide the model setup in section 2. In section 3, we derive the zero coupon bond pricing formula in the case where the end date of the NIRP is deterministic. In section 4, we derive the zero coupon bond pricing formula in the case where the end date of the NIRP is random. In section 5, we calibrate our model to Japanese government bond zero coupon yield data. Section 6 concludes the paper.

2 Setup

We define a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\le t}, \mathbb{P})$ where the filtration $(\mathcal{F}_t)_{0\le t}$ satisfies the usual conditions of right-continuity and completeness and is the natural filtration generated by two stochastic processes X_t and y^τ_t as defined below. $\mathbb{P}$ denotes the physical measure. We assume that the market is complete and has no arbitrage opportunities, so that there exists the unique risk-neutral measure $\mathbb{Q}$. We focus only on a theory on $\mathbb{Q}$ in this study. $W^{\mathbb{Q}}_{t,x} \in \mathbb{R}^n$ and $W^{\mathbb{Q}}_{t,y} \in \mathbb{R}^1$ are independent standard Brownian motions under $\mathbb{Q}$.

[1]The Bank of Japan has conducted quantitative and qualitative easing with Yield Curve Control (YCC) since September 2016, while a negative interest rate of -0.1% has been applied to current accounts of financial institutions. For simplicity, in this paper, we collectively refer to monetary policies in countries where negative interests are observed as NIRP.

[2]Futami [8] extends Marumo et al. [18] to the setting of the multi-factor with a single regime shift.

The state variable X_t satisfies the following stochastic differential equation:

$$dX_t = K_X^{\mathbb{Q}}(\theta^{\mathbb{Q}} - X_t)dt + \Sigma_X dW_{t,x}^{\mathbb{Q}}. \tag{2.1}$$

The risk-free short rate r_t is assumed to be the sum of a quadratic function of X_t and y_t^τ:

$$r_t = X_t' \Psi X_t + y_t^\tau, \tag{2.2}$$

where X_t' represents the transposition of X_t and Ψ is assumed to be positive definite. Equation (2.2) implies that y_t^τ is the lower bound of r_t.

We model y_t^τ as the Brownian bridge process with $y_0^\tau = 0$, $y_\tau^\tau = 0$, and $y_t^\tau = 0$ for $t \geq \tau$ defined as

$$y_t^\tau = \sigma_y W_{t,y}^{\mathbb{Q}} - \frac{\sigma_y t}{\tau \vee t} W_{\tau \vee t, y}^{\mathbb{Q}}. \tag{2.3}$$

For the time being, we assume that τ is a strictly positive constant value. Equation (2.3) is equivalent to equation (2.4) in the stochastic differential equation form:

$$dy_t^\tau = \left\{ \begin{array}{ll} -\dfrac{y_t^\tau}{\tau - t} dt + \sigma_y dW_{t,y}^{\mathbb{Q}} & (t < \tau) \\ 0 & (\tau \leq t) \end{array} \right\}. \tag{2.4}$$

τ is interpreted as the exit time from the NIRP policy.

3 Bond pricing in the case where τ is deterministic

In this section, we derive a bond pricing formula in the case where τ is deterministic. We assume that τ is a strictly positive constant. Hereinafter, we denote a normal policy period, $\tau \leq t$ (post-NIRP period) by a superscript of letter "n" and an abnormal policy period, $t < \tau$ (NIRP period) by a superscript of letter "a".

3.1 Bond pricing in a normal policy period, the post-NIRP

In this subsection, we derive a zero coupon bond pricing formula in a normal policy period, $\tau \leq t$. This period corresponds to the post-NIRP period.

An infinitesimal generator of X_t for $\tau \leq t$ is provided as

$$\mathscr{D}_t^n = (K_X^{\mathbb{Q}}(\theta^{\mathbb{Q}} - X_t))' \frac{\partial}{\partial X_t} + \frac{1}{2} \mathrm{Tr} \left(\Sigma_X \Sigma_X' \frac{\partial^2}{\partial X_t \partial X_t'} \right). \tag{3.1}$$

Applying the Feynman-Kac theorem to the zero coupon bond price $P_{t,u}^n$ with maturity date $T = t + u$ leads to the following partial differential equation (PDE):

$$\left[\frac{\partial}{\partial t} + \mathscr{D}_t^n \right] P_{t,u}^n = r_t P_{t,u}^n, \quad P_{t,0}^n = 1. \tag{3.2}$$

We guess the solution form of equation (3.2) as follows:

$$P_{t,u}^n = \exp(X_t' A_u^n X_t + (b_u^n)' X_t + c_u^n). \tag{3.3}$$

Substituting equation (3.3) into equation (3.2), we obtain the following system of ordinary differential equations (ODEs) for A_u^n, b_u^n, and c_u^n.

$$\begin{aligned} &- \dot{A}_u^n - 2K_X^{\mathbb{Q}\prime} A_u^n + 2A_u^n \Sigma_X \Sigma_X' A_u^n - \Psi = 0, \\ &- (\dot{b}_u^n)' + 2(K_X^{\mathbb{Q}} \theta^{\mathbb{Q}})' A_u^n - b_u^{n\prime} K_X^{\mathbb{Q}} + 2b_u^{n\prime} \Sigma_X \Sigma_X' A_u^n = 0, \\ &- \dot{c}_u^n + (K_X^{\mathbb{Q}} \theta^{\mathbb{Q}})' b_u^n + \mathrm{Trace} \left(\Sigma_X \Sigma_X' \left(A_u^n + \frac{1}{2} b_u^n (b_u^n)' \right) \right) = 0, \end{aligned} \tag{3.4}$$

where the boundary conditions are $A_0^n = 0$, $b_0^n = 0$, and $c_0^n = 0$ and $\dot{A}_u^n$, $\dot{b}_u^n$, and $\dot{c}_u^n$ represent the derivatives of A_u^n, b_u^n, and c_u^n with respect to the variable u.

3.2 Bond pricing under a negative interest rate policy

In this subsection, we derive a zero coupon bond pricing formula in the case where $t < \tau$. This corresponds to the period when a central bank is conducting a negative interest rate policy.

First, we suppose that the bond's maturity date T is before the end date of the NIRP τ. In this case, we denote the zero coupon bond price by $P_{t,u,w}^{a,1}$ where $u = T - t$ and $w = \tau - T$. The price $P_{t,u,w}^{a,1}$ is provided as follows:

$$\begin{aligned} P_{t,u,w}^{a,1} &= E\left[\exp\left(-\int_t^T r_s ds\right) |\mathcal{F}_t\right] = E\left[\exp\left(-\int_t^T (X_s'\Psi X_s + y_s^\tau)ds\right) |\mathcal{F}_t\right] \\ &= E\left[\exp\left(-\int_t^T X_s'\Psi X_s ds\right) |\mathcal{F}_t\right] E\left[\exp\left(-\int_t^T y_s^\tau ds\right) |\mathcal{F}_t\right] \\ &= P_{t,u}^n E\left[\exp\left(-\int_t^T y_s^\tau ds\right) |\mathcal{F}_t\right], \end{aligned} \tag{3.5}$$

where $E[\ ]$ is the expectation operator under $\mathbb{Q}$. By the assumption of independence between X_t and y_t^τ, the third equality in equation (3.5) holds true.

When we set

$$P_{t,u,w}^y = E\left[\exp\left(-\int_t^T y_s^\tau ds\right) |\mathcal{F}_t\right], \tag{3.6}$$

calculating the zero coupon bond price in equation (3.5) reduces to the calculation of $P_{t,u,w}^y$.

An infinitesimal generator of y_t^τ for $t < \tau$ is provided as

$$\mathscr{D}_t^a = -\frac{y_t^\tau}{\tau - t}\frac{\partial}{\partial y_t^\tau} + \frac{1}{2}\sigma_y^2\frac{\partial^2}{\partial y_t^2} = -\frac{y_t^\tau}{u + w}\frac{\partial}{\partial y_t^\tau} + \frac{1}{2}\sigma_y^2\frac{\partial^2}{\partial y_t^2}. \tag{3.7}$$

Applying the Feynman-Kac theorem to $P_{t,u,w}^y$ in equation (3.6), we obtain the following PDE:

$$\left[\frac{\partial}{\partial t} + \mathscr{D}_t^a\right] P_{t,u,w}^y = y_t^\tau P_{t,u,w}^y, \quad P_{t,0,w}^y = 1. \tag{3.8}$$

We guess the solution of equation (3.8) to be of the following form:

$$P_{t,u,w}^y = \exp(d_{u,w}^{a,1} y_t^\tau + f_{u,w}^{a,1}). \tag{3.9}$$

Substituting equation (3.9) into equation (3.8), we obtain the following ODEs.

$$\begin{aligned} \dot{d}_{u,w}^{a,1} + \frac{d_{u,w}^{a,1}}{u + w} + 1 &= 0, \\ \dot{f}_{u,w}^{a,1} &= \frac{1}{2}\sigma_y^2 (d_{u,w}^{a,1})^2, \end{aligned} \tag{3.10}$$

where the boundary conditions are $d_{0,w}^{a,1} = 0$ and $f_{0,w}^{a,1} = 0$, and $\dot{d}_{u,w}^{a,1}$ and $\dot{f}_{u,w}^{a,1}$ represent the derivatives of $d_{u,w}^{a,1}$ and $f_{u,w}^{a,1}$ with respect to the variable u, respectively. The first equation in equation (3.10) is known as d'Alembert's equation and its solution is as follows:

$$d_{u,w}^{a,1} = -\frac{u(u + 2w)}{2(u + w)}. \tag{3.11}$$

Equation (3.11) and the second equation in equation (3.10) lead to the following solution of $f^{a,1}_{u,w}$:

$$\begin{aligned} f^{a,1}_{u,w} &= \int_0^u \frac{1}{2}\sigma_y^2 (d^{a,1}_{v,w})^2 dv = \frac{1}{2}\sigma_y^2 \int_0^u \frac{v^2(v+2w)^2}{4(v+w)^2} dv \\ &= \frac{\sigma_y^2}{24}\left((u+w)^3 - 6w^2u + 2w^3 - \frac{3w^4}{u+w}\right). \end{aligned} \tag{3.12}$$

Next, we calculate the price of a zero coupon bond with maturity date that comes at or after the end date of the NIRP; in other words, $t < \tau \leq T$. In this case, we denote the zero coupon bond price by $P^{a,2}_{t,u,w}$ where $u = T - t$ and $w = \tau - T$. Then, $P^{a,2}_{t,u,w}$ is provided as follows:

$$\begin{aligned} P^{a,2}_{t,u,w} &= E\left[\exp\left(-\int_t^T r_s ds\right)|\mathcal{F}_t\right] = E\left[\exp\left(-\int_t^T (X_s'\Psi X_s + y_s^\tau)ds\right)|\mathcal{F}_t\right] \\ &= E\left[\exp\left(-\int_t^T X_s'\Psi X_s ds\right)|\mathcal{F}_t\right] E\left[\exp\left(-\int_t^T y_s^\tau ds\right)|\mathcal{F}_t\right] \\ &= E\left[\exp\left(-\int_t^T X_s'\Psi X_s ds\right)|\mathcal{F}_t\right] E\left[\exp\left(-\int_t^\tau y_s^\tau ds\right)|\mathcal{F}_t\right] \\ &= P^n_{t,u} P^y_{t,u+w,0}. \end{aligned} \tag{3.13}$$

Here, it should be noted that $P^y_{t,u,w} = P^y_{t,u+w,0}$ when $w \leq 0$.

$P^y_{t,u+w,0}$ in equation (3.13) is calculated from equation (3.9), (3.11), and (3.12) as follows:

$$\begin{aligned} P^y_{t,u+w,0} &= \exp(d^{a,1}_{u+w,0} y_t^\tau + f^{a,1}_{u+w,0}) \\ &= \exp\left(-\frac{u+w}{2} y_t^\tau + \frac{\sigma_y^2}{24}(u+w)^3\right). \end{aligned} \tag{3.14}$$

4 Bond pricing in the case where τ is random

In this section, we derive a zero coupon bond pricing formula in the case where the end date of the NIRP τ is random. We define the risk-free short rate r_t as $r_t = X_t'\Psi X_t + y_t$ instead of using equation (2.2). By this definition, y_t becomes the lower bound of interest rates. In this section, we model the lower bound of interest rates y_t as the Brownian bridge with a random time interval τ as shown in Bedini et al. [4].

Let $\tau : \Omega \to (0, +\infty)$ be a strictly positive random variable whose distribution function is denoted by $F(t) = \mathbb{Q}(\tau \leq t)$. We assume that τ is independent of $W^{\mathbb{Q}}_{t,x}$ and $W^{\mathbb{Q}}_{t,y}$. When we denote $(C, \mathbf{C})$ the space of continuous real-valued functions on $\mathbb{R}_+$ endowed with the σ-algebra generated by the canonical process, we define a Brownian bridge with a random time interval τ as the map from $(\Omega, \mathcal{F})$ to $(C, \mathbf{C})$ as follows:

Definition 1. The process $y_t(\omega)$ given by

$$y_t(\omega) = y_t^{\tau(\omega)}(\omega),$$

is the Brownian bridge with a random interval τ, where y_t^r is the Brownian bridge with a deterministic time interval r as defined in equation (2.3).

Bedini et al. [4] prove that y_t given in Definition 1 is measurable. They also prove the following corollary:

Corollary 1. Let $\sigma(\tau)$ denote the σ-algebra generated by τ and $\mathcal{B}(A)$ denote the Borel set of A.

If $h : ((0,+\infty) \times C, \mathcal{B}((0,+\infty)) \otimes \mathbf{C}) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ is a measurable function such that $E[|h(\tau, y)|] < +\infty$, then $E[h(\tau, y)|\sigma(\tau)](\omega) = E[h(r, y^r)]|_{r=\tau(\omega)}$, $\mathbb{Q}$-a.s.

Corollary 1 plays a role in deriving the zero coupon bond pricing formula that will be shown later.

Let $f(x)$ be the prior density function of τ under $\mathbb{Q}$. We define $G(t, y_t)$ as

$$G(t, y_t) = \int_t^\infty \varphi_t(v, y_t) f(v) dv, \tag{4.1}$$

where $\varphi_t(r, y)$ represents the density of y_t^r provided in equation (2.3) and is calculated as follows:

$$\varphi_t(r, y) = \sqrt{\frac{r}{2\pi t(r-t)\sigma_y^2}} \exp\left(-\frac{y^2 r}{2t(r-t)\sigma_y^2}\right). \tag{4.2}$$

Note that $\dfrac{\varphi_t(v, y_t) f(v)}{G(t)}$ can be interpreted as the posterior density of τ conditioned on y_t while $f(v)$ is its prior density.

Let $\mathcal{F}_t^y$ denote the natural completed filtration generated by y_t; that is, $\mathcal{F}_t^y = \sigma(y_s; 0 \leq s \leq t) \vee \mathcal{N}$. In this section, we derive the formula for the coupon bond price $P_{t,T-t}$ with maturity date T at time t. We obtain the following proposition for $P_{t,T-t}$ during an NIRP period.

Proposition 1. The following equation holds $\mathbb{Q}$-a.s.:

$$\begin{aligned} 1_{\{t<\tau\}} P_{t,T-t} &= P_{t,T-t}^n E\left[\exp\left(-\int_t^T y_s ds\right) 1_{\{t<\tau\}} | \mathcal{F}_t^y\right] \\ &= \frac{1_{\{t<\tau\}}}{G(t, y_t)} \left(\int_T^{+\infty} P_{t,T-t,v-T}^{a,1} \varphi_t(v, y_t) f(v) dv + \int_t^T P_{t,T-t,v-T}^{a,2} \varphi_t(v, y_t) f(v) dv\right). \end{aligned}$$

Proof. The first equality holds true due to the independence between X_t and y_t. The second term of the right hand side of the first equality is calculated $\mathbb{Q}$-a.s. as follows:

$$\begin{aligned} E\left[\exp\left(-\int_t^T y_s ds\right) 1_{\{t<\tau\}} | \mathcal{F}_t^y\right] &= E\left[\exp\left(-\int_t^T y_s ds\right) | y_t\right] 1_{\{t<\tau\}} \\ &= E\left[E\left[\exp\left(-\int_t^T y_s ds\right) | \sigma(\tau) \vee \sigma(y_t)\right] | y_t\right] 1_{\{t<\tau\}} \\ &= E\left[E\left[\exp\left(-\int_t^T y_s^r ds\right) | y_t^r\right]_{r=\tau} | y_t\right] 1_{\{t<\tau\}} \\ &= E\left[P_{t,T-t,\tau-T}^y(y_t) | y_t\right] 1_{\{t<\tau\}}. \end{aligned} \tag{4.3}$$

The third equality in the above equation holds true due to Corollary 1. As shown in Bedini et al. [4], the right hand side of the final equality in the above equation is provided as follows:

$$E\left[P_{t,T-t,\tau-T}^y(y_t) | y_t\right] 1_{\{t<\tau\}} = \frac{1}{G(t, y_t)} \left(\int_T^{+\infty} P_{t,T-t,v-T}^y \varphi_t(v, y_t) f(v) dv + \int_t^T P_{t,T-t,v-T}^y \varphi_t(v, y_t) f(v) dv\right) 1_{\{t<\tau\}}.$$

Therefore, equations (3.5), (3.13), and (4.3) lead to the conclusion of this proposition. □

We obtain the following pricing formula for $P_{t,T-t}$ by Proposition 1.

Proposition 2. The following equation holds $\mathbb{Q}$-a.s.:

$$P_{t,T-t} = P^n_{t,T-t}1_{\{\tau\leq t\}} + \frac{1_{\{t<\tau\}}}{G(t,y_t)}\left(\int_T^{+\infty} P^{a,1}_{t,T-t,v-T}\varphi_t(v,y_t)f(v)dv + \int_t^T P^{a,2}_{t,T-t,v-T}\varphi_t(v,y_t)f(v)dv\right).$$

If we have state variables X_t and y_t and all of the parameters of the model, we can calculate $P_{t,T-t}$ for any T by integrating numerically the integrands of the right hand side of the equation in Proposition 2.

5 Calibration

In this section, we calibrate our proposed model to Japanese government bond yield data. After describing the calibration procedure, we present some results including a fit to the market data and the implied posterior distributions of the time to exit from the NIRP.

5.1 Calibration Procedure

For the calibration process, we use market data for zero coupon yields of Japanese government bond with maturities of 6 months, 1, 2, 3, 5, 7, 10, and 20 years. These yields are estimated based on the B-spline regression in Steeley [20], and in Kikuchi and Shintani [13], using Japanese government bond prices from the Japan Securities Dealers Association.

Prior to calibrating the model to the market data, we first determine the model parameters $K^{\mathbb{Q}}$, $\theta^{\mathbb{Q}}$, Σ_X, and Ψ in equations (2.1) and (2.2) using yield curve time series data before negative interest rates were observed in the market. By assuming that $K^{\mathbb{Q}}$, $\theta^{\mathbb{Q}}$, Σ_X, and Ψ are time-invariant even after the first observation of negative interest rates, we use these same estimates for the calibration.

In determining these parameters, we define the short rate as $r_t = X_t'\Psi X_t$ with Ψ being positive definite, so that the zero coupon bond prices are provided as in equations (3.3) and (3.4).

Suppose that X_t is a three-dimensional latent state variable. We regard our model as the state space model to estimate the model parameters including $K^{\mathbb{Q}}$, $\theta^{\mathbb{Q}}$, Σ_X, Ψ, and X_t. To estimate X_t, we rely on the filtering method; therefore, we need the dynamics of X_t under the physical measure $\mathbb{P}$ as well as under the risk-neutral measure $\mathbb{Q}$. This process is assumed to be as follows:

$$dX_t = K^{\mathbb{P}}_X(\theta^{\mathbb{P}} - X_t)dt + \Sigma_X dW^{\mathbb{P}}_{t,x}, \tag{5.1}$$

where $W^{\mathbb{P}}_{t,x}$ is a standard Brownian motion under $\mathbb{P}$. Setting X_t's dynamics as shown in equation (5.1) implies that we assume the essentially affine market price of risk as first introduced in Duffee [6]. Errors in the observation equation are assumed to be normally distributed with a zero mean vector and a diagonal covariance matrix Σ_η and to be independent of other random variables. It should be noted that we can apply the invariant transformation in Ahn et al. [1] and Leippold and Wu [16] to our model before performing this estimation. By applying the invariant transformation, we allow $K^{\mathbb{P}}_X$ to be the lower triangular matrix , $\theta^{\mathbb{P}}$ the zero vector, and Σ_X the identity matrix. In addition to this setting, we assume that $K^{\mathbb{Q}}_X$ is the lower triangular matrix as with $K^{\mathbb{P}}_X$. From the above, we estimate the lower triangular matrices $K^{\mathbb{Q}}_X$ and $K^{\mathbb{P}}_X$, $\theta^{\mathbb{Q}}$, the positive definite matrix Ψ, and the diagonal observation error covariance matrix Σ_η.

Since the observation equation of our state space model is nonlinear, we perform the estimation based on the unscented Kalman filter proposed by Julier and Uhlmann [12] and the quasi-maximum likelihood method. The data frequency is monthly and the observation period is from May 2009 to August 2015.

Once we obtain estimates of $K^{\mathbb{Q}}$, $K_X^{\mathbb{P}}$, $\theta^{\mathbb{Q}}$, Ψ, and Σ_η as described above, we can calibrate to the market data. In our calibration, we do not use estimates of $K_X^{\mathbb{P}}$ and Σ_η from the above parameters. As mentioned above, we assume that $K^{\mathbb{Q}}$, $\theta^{\mathbb{Q}}$, the identity matrix Σ_X, and Ψ are time-invariant.

We calibrate our model to Japanese government bond zero coupon yield curve data on October 30, 2015, February 29, 2016, and December 30, 2016. The maturities along the yield curves we use for calibration consist of 6 month, 1, 2, 3, 5, 7, 10, and 20 year. The model implied zero coupon yield with maturity date T at time t is provided as

$$-\frac{1}{T-t}\log P_{t,T-t}. \tag{5.2}$$

In our calibration, parameters to be optimized are σ_y, X_t, and y_t as well as the parameters that determine the prior distribution for the time τ to exit from the NIRP. In terms of the distribution of the time to exit from the NIRP, we assume that it follows the Gamma distribution with the shape parameter α and the scale parameter β; hence, its prior density is written as follows:

$$f(x;\alpha,\beta) = \frac{\beta^\alpha x^{\alpha-1}e^{-\beta x}}{\Gamma(\alpha)}. \tag{5.3}$$

Let $Yield_t$ be the observation vector with the zero coupon yield curve estimated from observed market prices and $\widetilde{Yield_t}$ be the vector of the model implied zero coupon yield curve. Here, t denotes the monthly observation date with $t = 0$ corresponding to September 30, 2015.

The objective function for the calibration is measured by the L_2-norm; thus, calibration relies on the nonlinear least squares as follows:

$$\min_{\alpha,\beta,\sigma_y,X_t,y_t} \left|\left| Yield_t - \widetilde{Yield_t} \right|\right|, \tag{5.4}$$

where $||A||$ represents the L_2-norm of the vector A.

If we obtain all parameters and state variables to minimize the objective function in equation (5.4), we can compute the posterior distribution of τ,

$$\frac{\varphi_t(v,y_t)f(v;\alpha,\beta)}{G(t)},$$

where $G(t)$ is provided in equation (4.1) and $\varphi_t(v,y_t)$ is provided in equation (4.2).

5.2 Calibration Results

5.2.1 Estimation Result using Pre-NIRP Period Data

As mentioned above, we assume that the lower triangular matrix $K_X^{\mathbb{Q}}$, $\theta^{\mathbb{Q}}$, the identity matrix Σ_X, and the positive definite matrix Ψ are invariant over time.

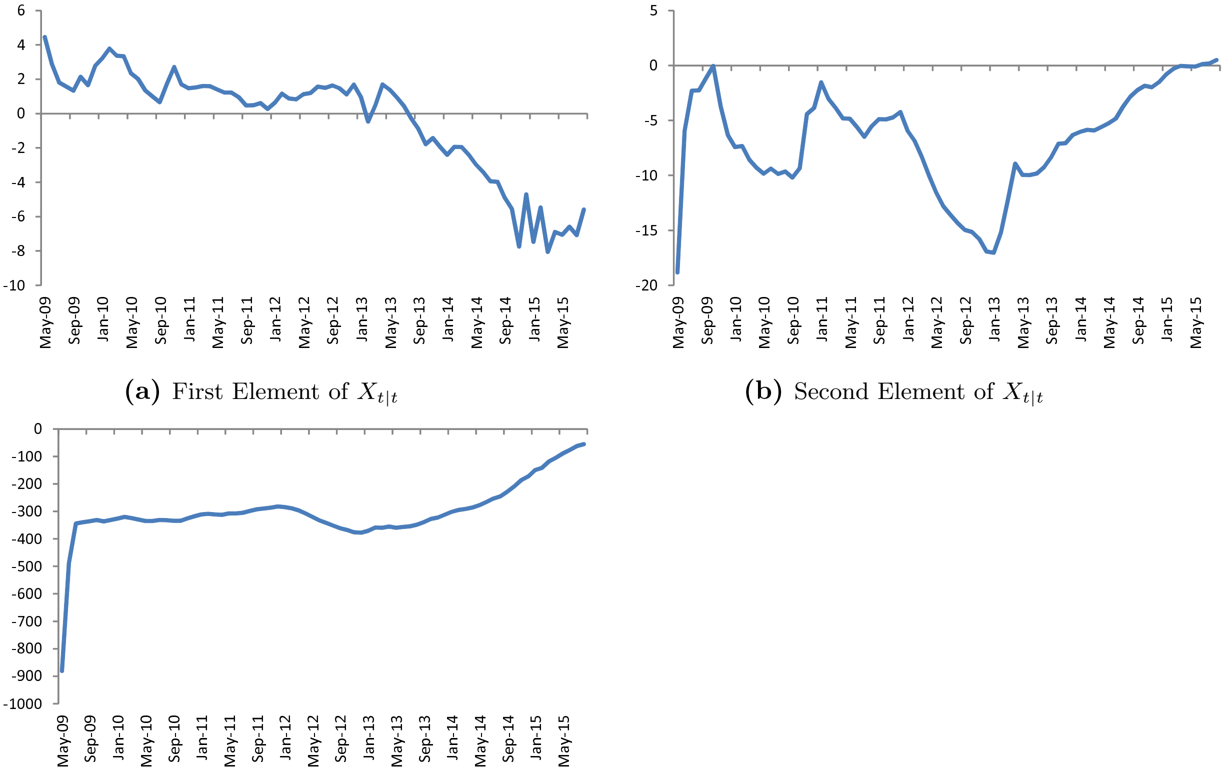
The estimates of $K_X^{\mathbb{Q}}$, $K_X^{\mathbb{P}}$, Ψ, and $\theta^{\mathbb{Q}}$ are provided in Table 1. Figure 2 reports a time series of the filtered values denoted by $X_{t|t}$.

Table 2 reports the standard deviations of the measurement errors in yields, as provided in diagonal elements of Σ_η. Although the standard deviation of measurement errors for the 20-year maturity yield is 8.239 bps, the largest among errors for all maturity yields, this is within the allowable range.

$$K_X^{\mathbb{Q}} = \begin{pmatrix} 0.0080 & 0 & 0 \\ -0.0110 & 0.0380 & 0 \\ 2.6729 & 2.3056 & 0.0063 \end{pmatrix}, \quad K_X^{\mathbb{P}} = \begin{pmatrix} 0.0475 & 0 & 0 \\ 0.0339 & 0.0383 & 0 \\ 1.2859 & -0.6746 & 0.0240 \end{pmatrix},$$

$$10^8 \times \Psi = \begin{pmatrix} 0.0513 & -0.3917 & -0.0357 \\ -0.3917 & 11.8089 & -0.0380 \\ -0.0357 & -0.0380 & 0.0684 \end{pmatrix}, \quad \theta_X^{\mathbb{Q}} = \begin{pmatrix} -5.7197 \\ -1.3423 \\ 1.3934 \end{pmatrix}.$$

Table 1: Estimates of $K_X^{\mathbb{Q}}$, $K_X^{\mathbb{P}}$, $\theta_X^{\mathbb{Q}}$, and Ψ



(a) First Element of $X_{t|t}$

(b) Second Element of $X_{t|t}$

(c) Third Element of $X_{t|t}$

Figure 2: Time Series of Filtered Values $X_{t|t}$

6 month	1 year	2 year	3 year	5 year	7 year	10 year	20 year
0.03172	0.01676	0.01591	0.01934	0.02695	0.02295	0.04680	0.08239

Table 2: Standard Deviations of Measurement Errors in Yields (indicated in percent)

5.2.2 Calibration to market data during the NIRP period

First, we provide results of the calibration using market data on October 30, 2015. Table 3 shows the optimal parameters obtained as a result of the calibration.

α	β	σ_y	$X_{t,1}$	$X_{t,2}$	$X_{t,3}$	y_t
3.69265	22.08477	0.00165	–7.72593	–23.45741	12.67852	–0.00009

Table 3: Calibration Parameters on October 30, 2015

Parameter y_t in Table 3 is shown as a monthly rate; when annualized, $y_t = -0.110\%$. In other words, the lower bound of interest rates on October 30, 2015 is estimated as -0.110%.

Table 4 shows the resulting fitting errors. "Diff." in Table 4 represents the absolute values of the measurement errors.

	6m	1y	2y	3y	5y	7y	10y	20y
Obs.(%)	–0.02597	–0.00979	0.01240	0.02101	0.02897	0.09841	0.31602	1.18575
Model(%)	–0.02578	–0.01037	0.01347	0.01993	0.02967	0.09800	0.31613	1.18575
Diff.(bps)	0.03154	0.08362	0.13946	0.13774	0.09240	0.05648	0.01533	0.00087

Table 4: Comparison between Market Observations and Model Values on October 30, 2015

According to Table 4, measurement errors of 2 and 3 year yields are larger than other yields. However, these are within an allowable range.

Second, we provide results of the calibration using market data on February 29, 2016. Table 5 shows the optimal parameters obtained as a result of the calibration.

α	β	σ_y	$X_{t,1}$	$X_{t,2}$	$X_{t,3}$	y_t
3.35122	41.61736	0.00093	–9.85032	–8.04864	–174.68880	–0.00019

Table 5: Calibration Parameters on February 29, 2016

Parameter y_t in Table 5 is shown as a monthly rate; when annualized, $y_t = -0.225\%$. Thus, the lower bound of interest rates on February 29, 2016 is estimated as -0.225%. Compared with the lower bound of interest rates calibrated on October 30, 2015, this value is more negative. It is assumed that this was caused by the introduction of Quantitative and Qualitative Monetary Easing with a Negative Interest Rate (QQE with NIRP) by the Bank of Japan in January 2016.

Table 6 shows the resulting fitting errors. Table 6 shows a good fit as with the calibration result based on data from October 30, 2015.

	6m	1y	2y	3y	5y	7y	10y	20y
Obs.(%)	–0.20732	–0.21742	–0.23376	–0.24501	–0.24574	–0.20229	–0.05421	0.66325
Model(%)	–0.20742	–0.21780	–0.23382	–0.24504	–0.24552	–0.20256	–0.05400	0.66298
Diff.(bps)	0.01027	0.03820	0.00544	0.00283	0.02212	0.02725	0.02078	0.02647

Table 6: Comparison between Market Observations and Model Values on February 29, 2016

Third, we provide results of the calibration using market data from December 30, 2016. Table 7 shows the optimal parameters obtained as a result of this calibration.

α	β	σ_y	$X_{t,1}$	$X_{t,2}$	$X_{t,3}$	y_t
34.83639	9.19370	0.00035	–3.52948	–14.89407	474.62277	–0.00048

Table 7: Calibration Parameters on December 30, 2016

Parameter y_t in Table 7 is a monthly rate; when annualized, $y_t = -0.580\%$. Thus, the lower bound of interest rates on December 30, 2016 as -0.580%. In September 2016, the Bank of Japan changed its monetary policy and introduced QQE with Yield Curve Control (YCC), a new policy that targets both short-term and long-term interest rates. Although the Bank of Japan changed its policy target by introducing YCC, the negative interest rate charge on a portion of bank excess reserves was unchanged under the new policy. For this reason, it is considered that introducing YCC lowered the lower bound for interest rates.

Table 8 shows the resulting fitting errors.

	6m	1y	2y	3y	5y	7y	10y	20y
Obs.(%)	−0.30545	−0.26249	−0.19424	−0.14938	−0.10341	−0.05415	0.05323	0.63287
Model(%)	−0.30588	−0.26164	−0.19422	−0.15084	−0.10120	−0.05572	0.05363	0.63285
Diff.(bps)	0.04279	0.08539	0.00236	0.14637	0.22085	0.15774	0.04005	0.00166

Table 8: Comparison between Market Observations and Model Values on December 30, 2016

Table 8 also shows a good fit, as with the result of calibrations on other dates.

It can be concluded that our proposed model shows the goodness of fit with market yield curve data.

5.2.3 Implied posterior distribution of the time to the exit from the NIRP

Market participants are interested in knowing when the central bank will end the NIRP. Since we obtained estimates of X_t, y_t, and all parameters on October 30, 2015, February 29, 2016, and December 30, 2016 as shown above, we can compute the implied posterior density of the time to exit from the NIRP on each of the three dates. The posterior density is provided as

$$\frac{\varphi_t(v, y_t) f(v; \alpha, \beta)}{G(t)},$$

where v denotes the time to exit from the NIRP and $G(t)$, $\varphi_t(v, y_t)$, and $f(v; \alpha, \beta)$ are given in equations (4.1), (4.2), and (5.3), respectively.

Figure 3 shows the implied posterior distributions of the time to exit from the NIRP on the three dates.

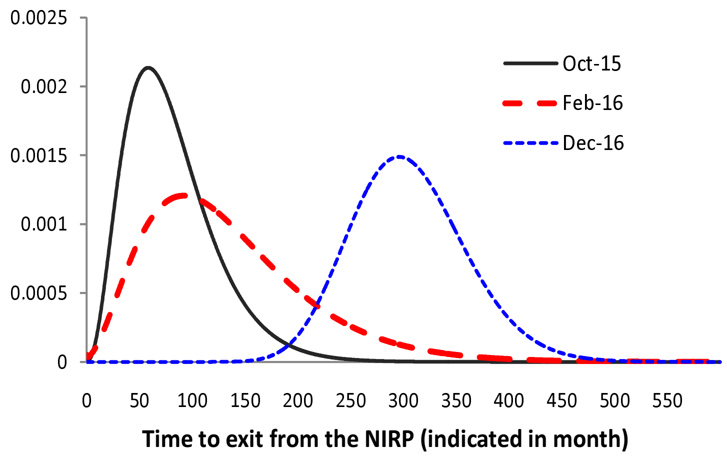


Figure 3: Implied Posterior Distributions of Time to Exit from the NIRP

We calculate the expected value of the time to exit from the NIRP based on the implied distribution shown in Figure 3. On October 30, 2015, the expected value is 80.4 months. On February 29, 2016, after QQE with a Negative Interest Rate was introduced by the Bank of Japan, the expected value increased to 133.2 months compared with the value on October 30, 2015. The expected value on December 30, 2016 is 305.1 months, much larger than the values on the prior dates. Since these distributions are based on the risk-neutral measure $\mathbb{Q}$, not on $\mathbb{P}$, it should be noted that the implied distributions do not always mirror market participants' expectations regarding monetary policy developments. However, it is likely that introducing YCC led market participants to expect negative interest rate environment for much longer.

6 Conclusion

In this study, we proposed a new term structure model that makes it possible to accurately replicate yield curves that include negative interest rates. We modeled the short rate as the sum of the quadratic function of Gaussian state variables and a stochastic lower bound of interest rates following a Brownian bridge pinned at zero at the start and end points. Our Brownian bridge representing a lower bound for interest rates is characterized as having a random time interval; thus, the interval can be regarded as the duration of the NIRP period. We formulated this Brownian bridge with a random time interval following Bedini et al. [4]; furthermore, we provided a zero coupon bond price formula under the no arbitrage condition. Additionally, we calibrated our proposed model using Japanese zero coupon yield data taken from the Bank of Japan's NIRP period. We showed that the calibration produces the goodness of fit of market data. We also calculated the implied posterior distribution of the time to exit from the NIRP using the parameters and state variables obtained through the calibration. This allows us to understand how market participants' views on the central bank's future monetary policy change.

It should be noted that in this study we focused on the $\mathbb{Q}$ distribution of the time to exit from the NIRP and we have not calculated its $\mathbb{P}$ distribution. To do that, we would need a formulation of the Brownian bridge with a random interval that represents a stochastic lower bound of interest rates under not only $\mathbb{Q}$ but also $\mathbb{P}$. In future work, we will incorporate a stochastic lower bound under $\mathbb{Q}$ and $\mathbb{P}$ simultaneously to extract market participants' subjective views on future monetary policy developments.

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