CRR DISCUSSION PAPER SERIES B

Discussion Paper No. B-17

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October 2019

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A Semi-analytical Solution to Consumption and International Asset Allocation Problem

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Abstract

We consider a finite continuous-time optimal consumption and international asset allocation problem for an agent with CRRA utility, assuming a quadratic factor international security market model in which, latent factors are constituted of global economy factors and currency specific factors. It is not generally straightforward to find an analytical solution to the partial differential equation (PDE, hereafter) for the agent's indirect utility function, since a non-homogeneous term appears in the PDE. We apply a method of Liu [11] and Batbold *et al.* [4] to the PDE, and derive a semi-analytical solution. In the optimal investment ratio based on the solution, the market price of currency specific risk, the disparities between domestic and foreign market prices of global economy risk, and the disparities between domestic and foreign market prices of currency specific risk appear.

1 Introduction

The importance of general household asset formation has been emphasized against the background of the public pension finance deterioration due to low growth and aged economy in most developed countries. International security investment in high growth countries, such as emerging countries, is essential for the general household in low growth country to effectively form the asset. Thus it is crucial for the government to lead the general household, whose investment knowledge tends to be insufficient, to enable effective

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international security investment. Considering that the general household has limited investment knowledge, we should promote an asset allocation to domestic and foreign government bonds and main indices including stock indices and REIT indices instead of active management.

The purpose of this paper is to derive a semi-analytical solution to an optimal consumption and international asset allocation problem assuming a highly general continuous-time international security market model, and to contribute to useful discussions on an exemplary international asset allocation for the general household.

Campbell and Viceira [6] considered an infinite continuous-time optimal consumption and investment problem under the assumption that an agent with CRRA utility invests in an instantaneously risk-free security and a zero-coupon bond with a constant time to maturity under the Vasicek one-factor term-structure model. A second-order partial differential equation (PDE, hereafter) for the value function is delivered from the Hamilton-Yacobi-Bellman (HJB, hereafter) equation, but it is not generally straightforward to find an analytical solution to the PDE, since a non-homogeneous term appears in the PDE. They derive an approximate analytical solution applying the log-linear approximation proposed by Campbell [5] to the non-homogeneous term.

On the other hand, Liu [11] examined a finite continuous-time horizon optimal consumption and investment problem under the assumption that an agent with CRRA utility invests in an instantaneously risk-free asset and risky securities under a highly general multi-factor security market model in which latent factors satisfy a diffusion process and both of the drift and diffusion functions are quadratic functions of the factors, and both of the market price of risk and the instantaneous interest rates are affine functions of the factors. He paid attention to a fact that a solution for the non-homogeneous PDE for the indirect utility function derived from the HJB equation is expressed as an integral of the solution for a homogeneous PDE ignoring the non-homogeneous term of the non-homogeneous PDE, and derived a system of ordinary differential equations (ODEs, hereafter) for unknown parameters constituting of the integrand.

Recently, Batbold, Kikuchi, and Kusuda [4] have considered a finite continuous-time optimal consumption and investment problem under the assumption that an agent with CRRA utility invests in an instantaneously risk-free asset, bonds, and indices under a highly general multi-factor security security market model in which latent factors satisfy a multi-dimensional version of diffusion Ornstein-Uhlenbeck process, and both of the market price of risk and the short-term interest rates are affine functions of the factors. They have expressed the indirect utility function as an integral of a solution for the above homogeneous PDE applying the method of Liu [11], and derived the system of ODEs for unknown parameters constituting of the integrand. They have solved the ODEs, and derived a semi-analytical solution which is a time-integrated analytic function.

In all of the above studies, one-country security market model is assumed. Surprisingly few studies has been made at continuous-time international security market model including both of stock markets and bond markets. Very recently, Kikuchi [9] has unified a quadratic international bond market model of Leippold and Wu [10] with a quadratic stock market model which is a generalized version of the affine one-country stock market model of Mamaysky [12], and proposed a quadratic international security market model.

We assume a stationary latent factor international security continuous-time model which eliminates a non-stationary factor in the Kikuchi's model and consider the same problem as Batbold *et al.* [4]. In the security market model, latent factors are constituted of global economy factors and currency specific factors. These factors satisfy the multi-dimensional version of the Ornstein-Uhlenbeck process. In each country, the market price of global economy risk and the market price of currency specific risk is an affine function of the international economy factors and of the currency specific factors, respectively, and the instantaneous interest rate, the dividend-rate, and the expected inflation-rate are quadratic functions of the international economy factors. Main results of this paper is summarized as follows.

We apply the method of Liu [11] and Batbold *et al.* [4] to our problem, and derive a semi-analytical solution. In the optimal investment ratio based on the solution, the market price of currency specific risk, the disparities between domestic and foreign market prices of global economy risk, and the disparities between domestic and foreign market prices of currency specific risk appear, while all of them do not appear in the optimal investment ratio for one-country security investment problem. It indicates that in international security investment, an investor should correctly estimate the global economy factors, the currency specific factors, the market price of disparities between domestic and foreign market prices of global economy risk, and the disparities between domestic and foreign market prices of currency specific risk.

The rest of this paper is organized as follows. In Section 2, we explain the stationary latent factor international security market model and the agent's optimal consumption and security investment problem. In Section 3, we derive a semi-analytical solution to the problem, and present an optimal consumption-wealth ration and investment ratio.

2 Stationary Quadratic International Security Market Model and Consumer's Problem

In this section, we first introduce the stationary quadratic international security market model, and present stochastic differential equations (SDEs, hereafter) which domestic and foreign security's return rate processes satisfy under no arbitrage condition.

2.1 Market Environment

We consider a frictionless international security market economy which consists of USA and N different currency areas with time span $[0, \infty)$. Agents' common subjective probability and information structure is modeled by a complete filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathrm{P})$ where $\mathbb{F} = (\mathcal{F}_t)_{t \in [0,\infty)}$ is the natural filtration generated by a $\bar{N}$-dimensional standard Brownian motion B_t. We indicate the expectation operator under P with E, and the conditional expectation operator with E_t, respectively.

In the US market, there are markets for the consumption commodity and securities at every date $t \in [0, \infty)$. The traded securities are nominal-risk-free security called the *money market account*, a continuum of zero-coupon bonds whose maturity dates are $(t, t + \bar{\tau}]$, each of which has a one US dollar payoff at its maturity date, J main indices (stock indices, REIT indices, *et al.*).

In the n-th currency area ($n \in \{1, \cdots, N\}$), there are security markets at every date $t \in [0, \infty)$. The traded securities are a continuum of zero-coupon bonds whose maturity dates are $(t, t + \hat{\tau}_n]$, each of which has a one currency payoff concerned at its maturity date, $\hat{J}_n$ main indices. There are foreign exchange markets between any two currency areas at $t \in [0, \infty)$.

At every date t, let P_t, P_t^T, and S_t^j denote the US dollar price of money market account, the zero-coupon bond with maturity date T, and the j-th index, respectively, in the US. Similarly, at every date, let $\hat{P}_{nt}^T$ and $\hat{S}_{nt}^j$ denote the n-th currency unit price of the zero-coupon bond with maturity date T, and the j-th index, respectively, in the n-th currency area.

2.2 Stationary Quadratic International Security Market Model

Very recently, Kikuchi [9] has presented a quadratic international security market model which unifies the international bond market model of Leippold and Wu [10] with a quadratic stock market model which is a generalized version of an affine stock market model presented in Mamaysky [12]. We assume the stationary quadratic international security market model in which we eliminate a non-stationary factor on stock prices in the Kikuchi's model.

Let $\mathbf{X}_t = (X_t', Y_t')' \in \mathbb{R}^{M+N}$ be some vector process with $X_t \in \mathbb{R}^M$ and $Y_t \in \mathbb{R}^N$, and let

$$B_t = \begin{pmatrix} B_t^X \\ B_t^Y \end{pmatrix}$$

with $B_t^X \in \mathbb{R}^M$ and $B_t^Y \in \mathbb{R}^N$.

Assumption 1. *The vector process $\mathbf{X}_t$ describes the state of the global economy, and the state vector processes X_t and Y_t are controlled by the following SDEs:*

$$dX_t = -K_X X_t \, dt + dB_t^X, \tag{2.1}$$

$$dY_t = -K_Y \, Y_t \, dt + dB_t^Y, \tag{2.2}$$

where K_X is an $M \times M$ constant matrix, K_Y is an $N \times N$ constant matrix, and each of these matrices is a positive lower triangular matrix.

The state vector processes X_t and Y_t follow multivariate Ornstein-Uhlenbeck processes with mean reversion. For identification purposes, the two processes are normalized to have zero long-run means and identity instantaneous variance.[1]

Each country's state price deflator is assumed to be orthogonally decomposed into a deflator related to the state process X_t and that related to Y_t.

Assumption 2. *The domestic and the n-th foreign state-price deflators π_t and $\hat{\pi}_t$ are expressed as*

$$\pi_t = \pi_t^X \, \pi_t^Y, \qquad \hat{\pi}_{nt} = \hat{\pi}_{nt}^X \, \hat{\pi}_{nt}^Y, \tag{2.3}$$

where π_t^X and $\hat{\pi}_{nt}^X$ are diffusion processes which depend on only B_t^X, and π_t^Y and $\hat{\pi}_{nt}^Y$ are expressed as

$$\frac{d\pi_t^Y}{\pi_t^Y} = -\Lambda_t^Y \, dB_t^Y, \qquad \frac{d\hat{\pi}_{nt}^Y}{\hat{\pi}_{nt}^Y} = -\hat{\Lambda}_{nt}^Y \, dB_t^Y. \tag{2.4}$$

Furthermore, any security price process is a diffusion process, and depends on only B_t^X.

[1] See Kikuchi[9] for a detailed discussion of the identification issue.

Then it is straightforward to see the following lemma.

Lemma 1. *Under Assumptions 1 and 2, the following 1 and 2 hold iff there is no arbitrage.*

1. π_t^X *and* $\hat{\pi}_{nt}^X$ *satisfy*

$$\frac{d\pi_t^X}{\pi_t^X} = -r_t\,dt - \Lambda_t^X\,dB_t^X, \qquad \frac{d\hat{\pi}_{nt}^X}{\hat{\pi}_{nt}^X} = -\hat{r}_{nt}\,dt - \hat{\Lambda}_{nt}^X\,dB_t^X, \quad (2.5)$$

where r_t *and* $\hat{r}_{nt}$ *is the domestic and the n-th foreign instantaneous interest rate, respectively, and* Λ_t^X *and* $\hat{\Lambda}_{nt}^X$ *is the market price of domestic risk and the market price of n-th foreign risk, respectively.*

2. *The process of exchange rate against the n-th foreign currency satisfies*

$$\frac{d\varepsilon_{nt}}{\varepsilon_{nt}} = \left(r_t - \hat{r}_{nt} + \begin{pmatrix} \Lambda_t^X - \hat{\Lambda}_{nt}^X \\ \Lambda_t^Y - \hat{\Lambda}_{nt}^Y \end{pmatrix}' \begin{pmatrix} \Lambda_t^X \\ \Lambda_t^Y \end{pmatrix} \right) dt + \begin{pmatrix} \Lambda_t^X - \hat{\Lambda}_{nt}^X \\ \Lambda_t^Y - \hat{\Lambda}_{nt}^Y \end{pmatrix}' dB_t. \quad (2.6)$$

Proof. See Appendix A.1. □

Remark 1. *Lemma 1 implies that* Y_t *describes the state of currency and that* X_t *describes the state of the global economy except the state of currency. We call* X_t *global economy factor and* Y_t *currency factor, hereafter.*

Remark 2. *Leippold and Wu [10] estimate their international bond market model using U.S. and Japanese LIBOR and swap rates and the exchange rate between the two economies. They conclude that independent currency factors are essential to capture the portion of the exchange rate movement that is independent of the term structure movement.*

Assumption 3. *1. The market prices of domestic risk and the n-th foreign risk are affine functions of the global economy factors* X_t.

$$\Lambda_t^X = \lambda_X + \Lambda_X X_t, \qquad \hat{\Lambda}_{nt}^X = \hat{\lambda}_X^n + \hat{\Lambda}_X^n X_t, \quad (2.7)$$

where $K_X + \Lambda_X$ *is regular.*

2. The market prices of domestic currency risk and n-th foreign currency risk are affine functions of the currency factors Y_t.

$$\Lambda_t^Y = \lambda_Y + \Lambda_Y Y_t, \qquad \hat{\Lambda}_{nt}^Y = \hat{\lambda}_Y^n + \hat{\Lambda}_Y^n Y_t. \quad (2.8)$$

3. The domestic and the n-th foreign instantaneous interest rates are quadratic functions of the global economy factors X_t.

$$r_t = \rho^0 + \rho' X_t + \frac{1}{2} X_t' R X_t, \qquad \hat{r}_{nt} = \hat{\rho}_n^0 + \hat{\rho}_n' X_t + \frac{1}{2} X_t' \hat{R}_n X_t, \tag{2.9}$$

where R and $\hat{R}_n$ are symmetric matrices.

4. The domestic and the n-th foreign dividend processes are quadratic functions of the global economy factors X_t.

$$\begin{aligned} D_t^j &= \left(d_j^0 + d_j' X_t + \frac{1}{2} X_t' \Delta_j X_t \right) \exp\left(b_j^0 t + b_j' X_t + \frac{1}{2} X_t' \Sigma_j X_t \right), \\ \hat{D}_{nt}^j &= \left(\hat{d}_{nj}^0 + \hat{d}_{nj}' X_t + \frac{1}{2} X_t' \hat{\Delta}_{nj} X_t \right) \exp\left(\hat{b}_{nj}^0 t + \hat{b}_{nj}' X_t + \frac{1}{2} X_t' \hat{\Sigma}_j X_t \right), \end{aligned}$$

where $\Delta_j, \Sigma_j, \hat{\Delta}_{nj}, \hat{\Sigma}_j$ are symmetric matrices.

5. The domestic price index satisfies

$$\frac{dp_t}{p_t} = i_t \, dt, \qquad p_0 = 1, \tag{2.10}$$

where i_t is the expected instantaneous inflation rate, and it is a quadratic function of X_t.

$$i_t = \iota^0 + \iota' X_t + \frac{1}{2} X_t' I_p X_t, \tag{2.11}$$

where I_p is a symmetric matrix.

2.3 Domestic and Foreign Return Rate Processes and Budget Constraint

Let I_n and $\tau = T - t$ denote $n \times n$ identity matrix and the time to maturity of the bond P_t^T, respectively. We use the following notation.

$$\Lambda_t = \begin{pmatrix} \Lambda_t^X \\ \Lambda_t^Y \end{pmatrix}, \qquad \hat{\Lambda}_{nt} = \begin{pmatrix} \hat{\Lambda}_{nt}^X \\ \hat{\Lambda}_{nt}^Y \end{pmatrix}.$$

Kikuchi [9] shows the following lemma.

Lemma 2. *Under Assumptions 1-3, the following hold:*

1. Arbitrage-free domestic security price processes satisfy the following:

(i) *The short-term bond:*

$$\frac{dP_t}{P_t} = r_t\, dt, \qquad\qquad P_0 = 1. \tag{2.12}$$

(ii) *The default-free bond with time to maturity* τ*:*

$$\frac{dP_t^T}{P_t^T} = \left(r_t + (b(\tau) + \Sigma(\tau)X_t)'\Lambda_t^X\right) dt + (b(\tau)+\Sigma(\tau)X_t)' dB_t^X, \; P_T^T = 1, \tag{2.13}$$

where

$$\frac{d\Sigma(\tau)}{d\tau} = \Sigma^2(\tau) - 2\Sigma(\tau)(K_X + \Lambda_X) - R, \quad \Sigma(0) = 0, \tag{2.14}$$

$$\frac{db(\tau)}{d\tau} = \left(\Sigma(\tau) - (K_X + \Lambda_X)'\right) b(\tau) - \Sigma(\tau)\lambda_X - \rho, \quad b(0) = 0, \tag{2.15}$$

(iii) *The* j*-th index:*

$$\frac{dS_t^j + D_t^j dt}{S_t^j} = \left(r_t + (b_j + \Sigma_j X_t)'\Lambda_t^X\right) dt + (b_j + \Sigma_j X_t)' dB_t^X, \tag{2.16}$$

where

$$\Sigma_j^2 - (K_X + \Lambda_X)'\Sigma_j + \frac{1}{2}(\Delta_j - R_j) = 0, \tag{2.17}$$

$$b_j = (K_X + \Lambda_X - \Sigma_j)'^{-1}(d_j - \rho - \Sigma_j \lambda_X). \tag{2.18}$$

2. *Arbitrage-free* n*-th foreign security price processes in domestic currency term satisfy the following:*

(i) *The default-free bond with time to maturity* τ*:*

$$\frac{d(\hat{P}_{nt}^T \varepsilon_t)}{\hat{P}_{nt}^T \varepsilon_{nt}} = \left\{ r_t + \left(\begin{pmatrix} \hat{b}_n(\tau) + \hat{\Sigma}_n(\tau)X_t \\ 0 \end{pmatrix} + \begin{pmatrix} \Lambda_t^X - \hat{\Lambda}_{nt}^X \\ \Lambda_t^Y - \hat{\Lambda}_{nt}^Y \end{pmatrix} \right)' \begin{pmatrix} \Lambda_t^X \\ \Lambda_t^Y \end{pmatrix} \right\} dt + \left(\begin{pmatrix} \hat{b}_n(\tau) + \hat{\Sigma}_n(\tau)X_t \\ 0 \end{pmatrix} + \begin{pmatrix} \Lambda_t^X - \hat{\Lambda}_{nt}^X \\ \Lambda_t^Y - \hat{\Lambda}_{nt}^Y \end{pmatrix} \right)' dB_t, \tag{2.19}$$

where

$$\frac{d\hat{\Sigma}_n(\tau)}{d\tau} = \hat{\sigma}_n^2(\tau) - 2\hat{\Sigma}_n(\tau)(K_X + \hat{\Lambda}_X^n) - \hat{R}_n, \quad \hat{\Sigma}(0) = 0, \tag{2.20}$$

$$\frac{d\hat{b}_n(\tau)}{d\tau} = \left(\hat{\Sigma}_n(\tau) - (K_X + \hat{\Lambda}_X)'\right) \hat{b}_n(\tau) - \hat{\Sigma}_n(\tau)\hat{\lambda}_X - \hat{\rho}_n, \quad b(0) = 0. \tag{2.21}$$

(ii) The j-th index:

$$\frac{d(\hat{S}^j_{nt}\varepsilon_{nt}) + \hat{D}^j_{nt}\varepsilon_{nt}dt}{\hat{S}^j_{nt}\varepsilon_{nt}} = \left\{ r_t + \left(\begin{pmatrix} \hat{b}_{nj} + \hat{\Sigma}_{nj}X_t \\ 0 \end{pmatrix} + \begin{pmatrix} \Lambda^X_t - \hat{\Lambda}^X_{nt} \\ \Lambda^Y_t - \hat{\Lambda}^Y_{nt} \end{pmatrix} \right)' \begin{pmatrix} \Lambda^X_t \\ \Lambda^Y_t \end{pmatrix} \right\} dt$$
$$+ \left(\begin{pmatrix} \hat{b}_{nj} + \hat{\Sigma}_{nj}X_t \\ 0 \end{pmatrix} + \begin{pmatrix} \Lambda^X_t - \hat{\Lambda}^X_{nt} \\ \Lambda^Y_t - \hat{\Lambda}^Y_{nt} \end{pmatrix} \right)' dB_t, \quad (2.22)$$

where

$$\hat{\Sigma}^2_j - (K_X + \hat{\Lambda}_X)'\hat{\Sigma}_j + \frac{1}{2}(\hat{\Delta}_{nj} - \hat{R}_n) = 0, \quad (2.23)$$

$$\hat{b}_{nj} = (K_X + \hat{\Lambda}^n_X - \hat{\Sigma}_{nj})'^{-1}(\hat{d}_{nj} - \hat{\rho}_n - \hat{\Sigma}_{nj}\hat{\lambda}^n_X). \quad (2.24)$$

Proof. See Appendix A.2. □

Remark 3. *Λ^Y_t does not appear in domestic price processes (i.e., eqs. (2.13) and (2.16)), but appears in the exchange rate process (i.e., eqs. (2.6), (2.19), and (2.22)) as a market price of risk. Thus we call Λ^Y_t the market price of currency risk as in Leippold and Wu [10]. Similarly, we call $\hat{\Lambda}^Y_{nt}$ the market price of n-th foreign currency risk.*

Remark 4. *In the exchange rate process (i.e., eq. (2.6)), the disparity between domestic and foreign prices of global economy risk (i.e, $\Lambda^X_t - \hat{\Lambda}^X_{nt}$) and the disparity between domestic and foreign prices of currency risk (i.e, $\Lambda^Y_t - \hat{\Lambda}^Y_{nt}$) are volatilities in the exchange rate. As a result, the exchange rate's expected return rate depends not only on the disparity between domestic and foreign instantaneous interest rate but also on the disparities between domestic and foreign prices of these market risks. Similarly, volatilities in the n-th foreign security prices in domestic currency term (i.e., eqs. (2.19) and the disparity between domestic and foreign prices of global economy risk and the disparity between domestic and foreign prices of currency risk, and these prices of market risks also appear in the expected return rates in these securities.*

Let Φ^j_t and $\hat{\Phi}^j_{nt}$ denote the portfolio share in the j-th domestic index and in the j-th foreign index, respectively. Regarding the default-free bond, let $\varphi_t(\tau)$ and $\hat{\varphi}_{nt}(\tau)$ denote the portfolio share density in the domestic bond with τ-time to maturity and in the foreign bond with τ-time to maturity[2].

[2] We suppose that the functional spaces of portfolio share densities in domestic and foreign bonds include distributions.

Define Ψ_t by

$$\Psi_t = \begin{pmatrix} \Psi_t^X \\ 0 \end{pmatrix} + \begin{pmatrix} \hat{\Psi}_t^X \\ \hat{\Psi}_t^Y \end{pmatrix}, \tag{2.25}$$

where

$$\Psi_t^X = \int_0^{\bar{\tau}} \varphi_t(\tau) b(\tau)\, d\tau + \sum_{j=1}^{J} \Phi_t^j b_j,$$

$$\hat{\Psi}_t^X = \sum_{n=1}^{N} \int_0^{\hat{\tau}_n} \hat{\varphi}_{nt}(\tau) \left(\hat{b}_n(\tau) + (\Lambda_t^X - \hat{\Lambda}_{nt}^X)\right) d\tau + \sum_{n=1}^{N} \sum_{j=1}^{\hat{J}_n} \hat{\Phi}_{nt}^j \left(\hat{b}_{nj} + (\Lambda_t^X - \hat{\Lambda}_{nt}^X)\right),$$

$$\hat{\Psi}_t^Y = \sum_{n=1}^{N} \int_0^{\bar{\tau}_n} \hat{\varphi}_{nt}(\tau)\, d\tau\, (\Lambda_t^Y - \hat{\Lambda}_{nt}^Y) + \sum_{n=1}^{N} \sum_{j=1}^{\hat{J}_n} \hat{\Phi}_{nt}^j (\Lambda_t^Y - \hat{\Lambda}_{nt}^Y).$$

We call Ψ_t the investment control, hereafter.

Let W_t denote the real wealth process. Then the agent's budget-constraint is expressed as in the following lemma.

Lemma 3. *Under Assumptions 1 and 3, given an investment control Ψ_t and the consumption control c_t, the budget-constraint satisfies*

$$dW_t = \left\{ W_t \left(\bar{r}_t + \Psi_t' \Lambda_t \right) - c_t \right\} dt + W_t \Psi_t' \, dB_t, \tag{2.26}$$

where $\bar{r}_t = r_t - i_t$.

Proof. See Appendix A.3. □

The budget constraint (2.26) shows that the real weal process is determined by the control $u_t = (c_t, \Psi_t)$.

2.4 Consumption and Asset Allocation Problem

Assumption 4. *The agent maximizes following CRRA utility under the budget-constraint (2.26).*

$$U(c) = \mathrm{E}\left[\int_0^T \alpha\, e^{-\beta t} \frac{c_t^{1-\gamma}}{1-\gamma} dt + (1-\alpha)\, e^{-\beta T} \frac{W_T^{1-\gamma}}{1-\gamma} \right]. \tag{2.27}$$

Let $\mathbf{X}_t = (W_t, X_t', Y_t')'$. We call a control $u_t = (c_t, \Psi_t)$ satisfying the budget-constraint (2.26) with the initial state $\mathbf{X}_0 = (W_0, X_0', Y_0')'$ the admissible control and denote $\mathcal{B}(\mathbf{X}_0)$ the set of admissible controls.

Then the indirect utility function is defined by

$$J(t, \mathbf{X}_t^u) = \mathrm{E}_t \left[\int_t^T \alpha\, e^{-\beta t} \frac{c_t^{1-\gamma}}{1-\gamma} dt + (1-\alpha)\, e^{-\beta T} \frac{W_T^{1-\gamma}}{1-\gamma} \right], \qquad \forall t \in [0, T]. \tag{2.28}$$

The agent's consumption and portfolio choice problem and the value function is defined by

$$V(\mathbf{X}_0) = \sup_{u \in \mathcal{B}(\mathbf{X}_0)} J(0, \mathbf{X}_0). \tag{2.29}$$

3 Semi-analytical Solution and Optimal Control

In this section, we first derive the PDE for an unknown function constituting of the indirect utility function from the HJB equation. Then we derive the semi-analytical solution to the PDE and present the optimal consumption and asset allocation.

3.1 PDE for the Indirect Utility Function

The HJB equation is expressed as

$$\sup_{u \in \mathcal{B}(\mathbf{X}_0)} \left\{ J_t(t, \mathbf{X}^u) + \mu_t' J_{\mathbf{X}}(t, \mathbf{X}^u) + \frac{1}{2} \operatorname{tr}\left[J_{\mathbf{XX}}(t, \mathbf{X}^u) \right] + \alpha e^{-\beta t} \frac{c^{1-\gamma}}{1-\gamma} \right\} = 0 \tag{3.1}$$

$$\text{s.t.} \quad J(T, \mathbf{X}_T^u) = (1-\alpha)\, e^{-\beta T} \frac{W_T^{1-\gamma}}{1-\gamma},$$

where

$$\mu_t = \begin{pmatrix} \mu_W \\ \mu_X \\ \mu_Y \end{pmatrix} = \begin{pmatrix} W_t(\bar{r}_t + \Psi_t' \Lambda_t) - c_t \\ -K_X X_t \\ -K_Y Y_t \end{pmatrix}, \qquad \Sigma_t = \begin{pmatrix} W_t(\Psi_t^X)' & W_t(\Psi_t^Y)' \\ I_M & 0 \\ 0 & I_N \end{pmatrix}.$$

Then the optimal control $u^* = (c^*, \Psi^*)$ satisfies the following:

$$c_t^* = \alpha^{\frac{1}{\gamma}} e^{-\frac{\beta}{\gamma} t} J_W^{-\frac{1}{\gamma}}, \tag{3.2}$$

$$\Psi_t^* = \frac{\psi_t}{W_t^{*2} J_{WW}}, \tag{3.3}$$

where

$$\psi_t = -W_t^* \left\{ J_W \begin{pmatrix} \Lambda_t^X \\ \Lambda_t^Y \end{pmatrix} + \begin{pmatrix} J_{XW} \\ J_{YW} \end{pmatrix} \right\}. \tag{3.4}$$

The consumption related terms are computed as

$$-c_t^* J_W + \alpha e^{-\beta t}\frac{c_t^{*1-\gamma}}{1-\gamma} = \frac{c_t^*}{1-\gamma}\Big\{(\gamma-1)J_W + \alpha e^{-\beta t} c_t^{*-\gamma}\Big\} = \frac{\gamma}{1-\gamma} c_t^* J_W, \quad (3.5)$$

Regarding the portfolio related terms, we should note

$$W_t^* J_W (\Psi_t^*)'\Lambda_t + W_t^*(\Psi_t^{X*})' J_{XW} + W_t^*(\Psi_t^{Y*})' J_{YW} = -W_t^{*2} J_{WW} (\Psi_t^*)'\Psi_t^*. \quad (3.6)$$

Then we obtain

$$\begin{aligned} & W_t^* J_W (\Psi_t^*)'\Lambda_t \\ & + \frac{1}{2}\,\mathrm{tr}\left[\begin{pmatrix} W_t^*(\Psi_t^{X*})' & W_t^*(\Psi_t^{Y*})' \\ I_M & 0 \\ 0 & I_N \end{pmatrix}\begin{pmatrix} W_t^*(\Psi_t^{X*})' & W_t^*(\Psi_t^{Y*})' \\ I_M & 0 \\ 0 & I_N \end{pmatrix}'\begin{pmatrix} J_{WW} & J_{WX} & J_{WY} \\ J_{XW} & J_{XX} & J_{XY} \\ J_{YW} & J_{YX} & J_{YY} \end{pmatrix}\right] \\ & = W_t^* J_W (\Psi_t^*)'\Lambda_t \\ & + \frac{1}{2}\,\mathrm{tr}\left[\begin{pmatrix} W_t^{*2}\Big((\Psi_t^{X*})'\Psi_t^{X*} + (\Psi_t^{Y*})'\Psi_t^{Y*}\Big) & W_t^*(\Psi_t^{X*})' & W_t^*(\Psi_t^{Y*})' \\ W_t^*\Psi_t^{X*} & I_M & 0 \\ W_t^*\Psi_t^{Y*} & 0 & I_N \end{pmatrix}\begin{pmatrix} J_{WW} & J_{WX} & J_{WY} \\ J_{XW} & J_{XX} & J_{XY} \\ J_{YW} & J_{YX} & J_{YY} \end{pmatrix}\right] \\ & = \frac{1}{2}\,\mathrm{tr}\left[J_{XX} + J_{YY}\right] - \frac{\psi_t'\psi_t}{2W_t^{*2} J_{WW}}. \quad (3.7) \end{aligned}$$

Substituting the optimal control (3.2) and (3.3) into the HJB eq. (3.1), and using eqs. (3.5) and (3.7) yield the following PDE for J.

$$\begin{aligned} & J_t + \frac{1}{2}\,\mathrm{tr}\left[J_{XX} + J_{YY}\right] - \frac{\psi_t'\psi_t}{2W_t^{*2} J_{WW}} \\ & \quad + \bar{r}_t W_t^* J_W + (-K_X X_t)' J_X + (-K_Y Y_t)' J_Y + \frac{\gamma}{1-\gamma} c_t^* J_W = 0. \quad (3.8) \end{aligned}$$

From the above PDE, we guess that indirect utility function takes the form

$$J(t, \mathbf{X}_t) = e^{-\beta t}\frac{W_t^{1-\gamma}}{1-\gamma}\big(G(t, X_t, Y_t)\big)^{\gamma}. \quad (3.9)$$

where $G(t, X_t, Y_t)$ is function of (t, X_t, Y_t).

Then the sufficient condition for optimization in the left-hand side of the HJB equation is confirmed since the following Hessian $\mathbf{H}$ is negative definite for any control $(c, \Psi) \in \mathbb{R}_+ \times \mathbb{R}^N$.

$$\mathbf{H} = \begin{pmatrix} -\alpha\gamma e^{-\beta t} c^{-\gamma-1} & 0 & \cdots & 0 \\ 0 & -\gamma e^{-\beta t} W_t^{1-\gamma} G^{\gamma} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & -\gamma e^{-\beta t} W_t^{1-\gamma} G^{\gamma} \end{pmatrix}. \quad (3.10)$$

Putting eqs. (3.2), (3.3) and partial derivatives of J into the PDE (3.8), we obtain the following proposition.

Proposition 1. *Under Assumptions 1-4, the indirect utility function, the optimal consumption, and the optimal investment for the problem (2.29) satisfy eqs. (3.9), (3.11), and (3.12), respectively. The function $G(t, X_t, Y_t)$ constituting of the indirect utility function is a solution to the PDE (3.13).*

$$c_t^* = \alpha^{\frac{1}{\gamma}} \frac{W_t^*}{G}, \tag{3.11}$$

$$\Psi_t^* = \frac{1}{\gamma} \begin{pmatrix} \Lambda_t^X \\ \Lambda_t^Y \end{pmatrix} + \begin{pmatrix} \frac{G_X}{G} \\ \frac{G_Y}{G} \end{pmatrix}, \tag{3.12}$$

$$\frac{\partial}{\partial t} G(t, X_t, Y_t) + \mathcal{L} G(t, X_t, Y_t) + \alpha^{\frac{1}{\gamma}} = 0,$$
$$G(T, X_T, Y_T) = (1-\alpha)^{\frac{1}{\gamma}}, \tag{3.13}$$

where $\mathcal{L}$ is a linear differential operator defined by

$$\begin{aligned} \mathcal{L}G = & \frac{1}{2} \operatorname{tr}\left[G_{XX} + G_{YY}\right] \\ & + \left(-K_X X - \frac{\gamma-1}{\gamma}(\lambda_X + \Lambda_X X)\right)' G_X + \left(-K_Y Y - \frac{\gamma-1}{\gamma}(\lambda_Y + \Lambda_Y Y)\right)' G_Y \\ & - \left\{ \frac{\gamma-1}{2\gamma^2}\Big((\lambda_X + \Lambda_X X)'(\lambda_X + \Lambda_X X) + (\lambda_Y + \Lambda_Y Y)'(\lambda_Y + \Lambda_Y Y)\Big) \right. \\ & \left. + \frac{\gamma-1}{\gamma}\left(\rho^0 - \iota^0 + (\rho - \iota)' X + \frac{1}{2} X'(R - I_p) X\right) + \frac{\beta}{\gamma} \right\} G. \end{aligned} \tag{3.14}$$

Proof. See Appendix A.4. □

3.2 A Semi-analytical Solution

A non-homogeneous term $\alpha^{\frac{1}{\gamma}}$ appears in the PDE (3.13), and it makes difficult to derive an analytical solution. Liu [11] presents a method to derive a semi-analytical solution exploiting an analytical solution to a homogeneous PDE which abandons the non-homogeneous term. Following his method, we examine the homogeneous PDE (3.15).

$$\frac{\partial}{\partial \tau} g(\tau, X, Y) = \mathcal{L} g(\tau, X, Y), \qquad g(0, X, Y) = 1. \tag{3.15}$$

An analytical solution to the (3.15) is expressed as

$$g(\tau, Z) = \exp\left(a_0(\tau) + a'(\tau) Z + \frac{1}{2} Z' A(\tau) Z\right), \tag{3.16}$$

where

$$Z = \begin{pmatrix} X \\ Y \end{pmatrix}, \qquad a(\tau) = \begin{pmatrix} a_X(\tau) \\ a_Y(\tau) \end{pmatrix}, \qquad A(\tau) = \begin{pmatrix} A_X(\tau) & A_{XY}(\tau) \\ A'_{XY}(\tau) & A_Y(\tau) \end{pmatrix}, \tag{3.17}$$

and $A_X(\tau)$, $A_Y(\tau)$ is a symmetric matrix.

Then we can confirm that a semi-analytical solution for the PDE (3.13) is expressed as

$$G(t, Z) = \alpha^{\frac{1}{\gamma}} \int_0^{T-t} g(s, Z)\, ds + (1-\alpha)^{\frac{1}{\gamma}} g(T-t, Z). \tag{3.18}$$

We use the following notation.

$$\lambda = \begin{pmatrix} \lambda_X \\ \lambda_Y \end{pmatrix}, \quad \Lambda = \begin{pmatrix} \Lambda_X & 0 \\ 0 & \Lambda_Y \end{pmatrix}, \quad L = \begin{pmatrix} K_X + \frac{\gamma-1}{\gamma}\Lambda_X & 0 \\ 0 & K_Y + \frac{\gamma-1}{\gamma}\Lambda_Y \end{pmatrix}.$$

Substituting g and its derivatives into the PDE (3.16) and paying attention to $Z'L'AZ = Z'AL'Z$, we get

$$\begin{aligned} \frac{da_0}{d\tau} + Z'\frac{da}{d\tau} + \frac{1}{2}Z'\frac{dA}{d\tau}Z &= \frac{1}{2}(a'a + \mathrm{tr}[A]) + Z'Aa + \frac{1}{2}Z'AZ \\ &\quad - \frac{\gamma-1}{\gamma}\lambda' a - \frac{\gamma-1}{\gamma}Z'A\lambda - Z'L'a - \frac{1}{2}Z'L'AZ - \frac{1}{2}Z'ALZ \\ &\quad - \frac{\gamma-1}{2\gamma^2}\lambda'\lambda - \frac{\gamma-1}{\gamma^2}Z'\Lambda'\lambda - \frac{\gamma-1}{2\gamma^2}Z'\Lambda'\Lambda Z \\ &\quad - \frac{\gamma-1}{\gamma}(\rho^0 - \iota^0) - \frac{\gamma-1}{\gamma}Z'\begin{pmatrix} \rho - \iota \\ 0 \end{pmatrix} - \frac{\gamma-1}{2\gamma}Z'\begin{pmatrix} R - I_p & 0 \\ 0 & 0 \end{pmatrix}Z - \frac{\beta}{\gamma}. \end{aligned} \tag{3.19}$$

Since the above equations are identical equations on Z, the following system of ordinary differential equations (ODEs) for (a_0, a, A) is derived.

$$\frac{da_0}{d\tau} = \frac{1}{2}(a'\,a + \mathrm{tr}[A]) - \frac{\gamma-1}{\gamma}\lambda' a - \left(\frac{\gamma-1}{2\gamma^2}\lambda'\lambda + \frac{\gamma-1}{\gamma}(\rho^0 - \iota^0) + \frac{\beta}{\gamma}\right), \quad a_0(0) = 0, \tag{3.20}$$

$$\frac{da}{d\tau} = (A - L)a - \frac{\gamma-1}{\gamma}A\lambda - \frac{\gamma-1}{\gamma^2}\Lambda'\lambda - \frac{\gamma-1}{\gamma}\begin{pmatrix} \rho - \iota \\ 0 \end{pmatrix}, \qquad a(0) = 0, \tag{3.21}$$

$$\frac{dA}{d\tau} = A^2 - L'A - AL - \frac{\gamma-1}{\gamma^2}\Lambda'\Lambda - \frac{\gamma-1}{2\gamma}\begin{pmatrix} R - I_p & 0 \\ 0 & 0 \end{pmatrix}, \qquad A(0) = 0. \tag{3.22}$$

We also use the following notation.

$$a^*(t,Z_t) = \frac{\alpha^{\frac{1}{\gamma}}\int_0^\tau g(s,Z_t)a(s)\,ds + (1-\alpha)^{\frac{1}{\gamma}}g(\tau,Z_t)a(\tau)}{\alpha^{\frac{1}{\gamma}}\int_0^\tau g(s,Z_t)\,ds + (1-\alpha)^{\frac{1}{\gamma}}g(\tau,Z_t)},$$

$$A^*(t,Z_t) = \frac{\alpha^{\frac{1}{\gamma}}\int_0^\tau g(s,Z_t)A(s)\,ds + (1-\alpha)^{\frac{1}{\gamma}}g(\tau,Z_t)A(\tau)}{\alpha^{\frac{1}{\gamma}}\int_0^\tau g(s,Z_t)\,ds + (1-\alpha)^{\frac{1}{\gamma}}g(\tau,Z_t)}.$$

Then we have Proposition 2.

Proposition 2. *Under Assumptions 1-4, an optimal control for the problem (2.29) satisfies*

$$c_t^* = \frac{\alpha^{\frac{1}{\gamma}}W_t^*}{\alpha^{\frac{1}{\gamma}}\int_0^\tau g(s,Z_t)\,ds + (1-\alpha)^{\frac{1}{\gamma}}g(\tau,Z_t)}, \tag{3.23}$$

$$\begin{aligned}\Psi_t^* &= \frac{1}{\gamma}\big(\lambda+\Lambda Z_t\big) + a^*(t,Z_t) + A^*(t,Z_t)Z_t \\ &= \frac{1}{\gamma}\begin{pmatrix}\lambda_X+\Lambda_X X_t\\ \lambda_Y+\Lambda_Y Y_t\end{pmatrix}' + \begin{pmatrix}a_X^*(t,Z_t)+A_X^*(t,Z_t)X_t+A_{XY}^*(t,Z_t)Y_t\\ a_Y^*(t,Z_t)+A_{XY}^*(t,Z_t)'X_t+A_Y^*(t,Z_t)Y_t\end{pmatrix},\end{aligned} \tag{3.24}$$

where (a_0, a, A) *is given by (3.25)-(3.27)and* A *is*

$$a_0(\tau) = \int_0^\tau \left\{\frac{1}{2}(a(s)'a(s)+\mathrm{tr}[A(s)]) - \frac{\gamma-1}{\gamma}\lambda' a(s) - \left(\frac{\gamma-1}{2\gamma^2}\lambda'\lambda + \frac{\gamma-1}{\gamma}(\rho^0-\iota^0)+\frac{\beta}{\gamma}\right)\right\}ds, \tag{3.25}$$

$$\begin{aligned}a(\tau) &= \exp\left(\int_0^\tau (A(s)-L)ds\right)\\ &\times \int_0^\tau \left(-\frac{\gamma-1}{\gamma}A(s)\lambda - \frac{\gamma-1}{\gamma^2}\Lambda'\lambda - \frac{\gamma-1}{\gamma}\begin{pmatrix}\rho-\iota\\0\end{pmatrix}\right)e^{-\int_0^s(A(s)-L)dt}ds,\end{aligned} \tag{3.26}$$

$$A(\tau) = C_2(\tau)C_1^{-1}(\tau), \tag{3.27}$$

where

$$\begin{pmatrix}C_1(\tau)\\C_2(\tau)\end{pmatrix} = \exp\left(\tau\begin{pmatrix}L & -I_{\bar N}\\ Q & -L'\end{pmatrix}\right)\begin{pmatrix}I_{\bar N}\\ 0_{\bar N}\end{pmatrix}, \tag{3.28}$$

where

$$Q = -\frac{\gamma-1}{\gamma^2}\Lambda'\Lambda - \frac{\gamma-1}{2\gamma}\begin{pmatrix}R-I_p & 0\\ 0 & 0\end{pmatrix}. \tag{3.29}$$

Proof. See Appendix A.5. □

3.3 Example for Optimal Investment

Let J and $\hat{J}_n$ denote the number of domestic bonds (or bond groups) and of the n-th foreign bonds (or bond groups), respectively. Suppose $\bar{N} = I + J + \sum_{n=1}^{N}(\hat{I}_n + \hat{J}_n)$. Then we can uniquely determine the optimal investment strategy.

Let Φ_t^P and Φ_t^S denote the investment-wealth ratio of the domestic bonds and of the domestic indices, respectively. Let B^P and B^S denote the volatilities of the domestic bonds and of the domestic indices, respectively. Similarly, let Φ_{nt}^P and Φ_{nt}^S denote the investment ratio of the n-th foreign bonds and of the n-th foreign indices, respectively. Let B_n^P and B_n^S denote the volatilities of the n-th foreign bonds and of the n-th foreign indices, respectively.

Let Φ_t and B denote the investment ratio vector process and the volatility matrix of the investment ratio vector process defined by

$$\Phi_t = \begin{pmatrix} \Phi_t^P \\ \Phi_t^S \\ \hat{\Phi}_{1t}^P \\ \hat{\Phi}_{1t}^S \\ \vdots \\ \hat{\Phi}_{Nt}^P \\ \hat{\Phi}_{Nt}^S \end{pmatrix}, \qquad B = \begin{pmatrix} B^P \\ B^S \\ \hat{B}_1^P \\ \hat{B}_1^S \\ \vdots \\ \hat{B}_N^P \\ \hat{B}_N^S \end{pmatrix}. \tag{3.30}$$

We use the following notation.

$$\Delta\Lambda_t^X = \begin{pmatrix} 0_{M\times(I+J)} & \Delta\Lambda_{1t}^X & \Delta\Lambda_{2t}^X & \cdots & \Delta\Lambda_{Nt}^X \end{pmatrix}, \tag{3.31}$$

$$\Delta\Lambda_t^Y = \begin{pmatrix} 0_{N\times(I+J)} & \Delta\Lambda_{1t}^Y & \Delta\Lambda_{2t}^Y & \cdots & \Delta\Lambda_{Nt}^Y \end{pmatrix}, \tag{3.32}$$

where $\Delta\Lambda_{nt}^X$ is an $M \times (\hat{I}_n + \hat{J}_n)$matrix, and $\Delta\Lambda_{nt}^Y$ is an $N \times (\hat{I}_n + \hat{J}_n)$ matrix, which are given by

$$\Delta\Lambda_{nt}^X = \begin{pmatrix} \Lambda_t^X - \hat{\Lambda}_{nt}^X & \Lambda_t^X - \hat{\Lambda}_{nt}^X & \cdots & \Lambda_t^X - \hat{\Lambda}_{nt}^X \end{pmatrix}, \tag{3.33}$$

$$\Delta\Lambda_{nt}^Y = \begin{pmatrix} \Lambda_t^Y - \hat{\Lambda}_{nt}^Y & \Lambda_t^Y - \hat{\Lambda}_{nt}^Y & \cdots & \Lambda_t^Y - \hat{\Lambda}_{nt}^Y \end{pmatrix}. \tag{3.34}$$

Suppose that each country's bond index is incorporated into the portfolio, *i.e.*, $I = \hat{I}_1 = \cdots = \hat{I}_N = 1$. Let $\psi_t(\tau)$ and $\hat{\psi}_{nt}(\tau)$ denote the domestic incorporation ratio and the n-th foreign incorporation ratio, respectively. Note that

$$\int_0^{\bar{\tau}} \psi_t(\tau)d\tau = \int_0^{\bar{\tau}} \hat{\psi}_{nt}(\tau)d\tau = 1, \quad \forall n \in \{1, \cdots, N\}.$$

We use the following notation.

$$\Phi_t^S = \begin{pmatrix} \Phi_t^1 \\ \Phi_t^2 \\ \vdots \\ \Phi_t^J \end{pmatrix}, \qquad B^P = \int_0^{\bar{\tau}} \psi_t(\tau) b(\tau)' d\tau, \qquad B^S = \begin{pmatrix} b_1' \\ b_2' \\ \vdots \\ b_J' \end{pmatrix},$$

$$\hat{\Phi}_{nt}^S = \begin{pmatrix} \hat{\Phi}_{nt}^1 \\ \hat{\Phi}_{nt}^2 \\ \vdots \\ \hat{\Phi}_{nt}^{\hat{J}_n} \end{pmatrix}, \quad \hat{B}_n^P = \int_0^{\hat{\tau}_n} \hat{\psi}_{nt}(\tau) \hat{b}_n(\tau)' d\tau, \quad \hat{B}_n^S = \begin{pmatrix} \hat{b}_{n1}' \\ \hat{b}_{n2}' \\ \vdots \\ \hat{b}_{\hat{J}_n}' \end{pmatrix},$$

for all $n \in \{1, \cdots, N\}$.

Then it follows from eqs. (2.25)(3.24) that the investment vector Φ_t is calculated as

$$\begin{aligned} \Phi_t = \frac{1}{\gamma} \begin{pmatrix} B' + \Delta\Lambda_t^X \\ \Delta\Lambda_t^Y \end{pmatrix}^{-1} \begin{pmatrix} \lambda_X + \Lambda_X X_t \\ \lambda_Y + \Lambda_Y Y_t \end{pmatrix}' \\ + \begin{pmatrix} B' + \Delta\Lambda_t^X \\ \Delta\Lambda_t^Y \end{pmatrix}^{-1} \begin{pmatrix} a_X^*(t, Z_t) + A_X^*(t, Z_t) X_t + A_{XY}^*(t, Z_t) Y_t \\ a_Y^*(t, Z_t) + A_{XY}^*(t, Z_t)' X_t + A_Y^*(t, Z_t) Y_t \end{pmatrix}'. \end{aligned} \tag{3.35}$$

Remark 5. *In order to compare international investment with domestic investment, we consider a case of domestic investment, i.e.,* $\bar{N} = I + J$ *and*

$$\Phi_t = \begin{pmatrix} \Phi_t^P \\ \Phi_t^S \end{pmatrix}, \qquad B = \begin{pmatrix} B^P \\ B^S \end{pmatrix}. \tag{3.36}$$

Then the optimal investment Φ_t *is given by*

$$\tilde{\Phi}_t = \frac{1}{\gamma} B'^{-1} \big(\lambda_X + \Lambda_X X_t\big) + B'^{-1} \big(a_X + A_X X_t\big). \tag{3.37}$$

Comparing eq. (3.35) with eq. (3.37), in the optimal international investment ratio, the market price $\lambda_Y + \Lambda_Y Y_t$ *of currency risk, the disparities* $\Delta\Lambda_t^X$ *between domestic and foreign market prices of global economy risk, and the disparities* $\Delta\Lambda_t^Y$ *between domestic and foreign market prices of currency risk appear, while all of them do not appear in the optimal domestic investment ratio. This indicates that in international investment, investor should customarily estimate the market price of currency risk, the disparities between domestic and foreign market prices of global economy risk, and the disparities between domestic and foreign market prices of currency risk as well as the global economy factor, the currency factor, and the market price of the global economy risk.*

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A Proofs

A.1 Proof of Lemma 1

Suppose that π_t^X is given by

$$\frac{d\pi_t^X}{\pi_t^X} = \mu_t^X dt + (\sigma_t^X)' dB_t^X. \tag{A.1}$$

Since any domestic security price process $\tilde{S}_t$ does not depend on the currency factor by Assumption 1, the arbitrage-free price process satisfies

$$\frac{dS_t}{S_t} = (r_t + \sigma_t' \Lambda_t^X) dt + \sigma_t' \, dB_t^X. \tag{A.2}$$

Thus by Assumptions 1 and 2, the product of the state-price deflator and the security price $\tilde{S}_t$ satisfies

$$\begin{aligned} \frac{d(\pi_t S_t)}{\pi_t S_t} &= \frac{d\pi_t}{\pi_t} + \frac{dS_t}{S_t} + \left(\frac{d\pi_t}{\pi_t}\right)\left(\frac{dS_t}{S_t}\right) \\ &= \frac{d\pi_t^X}{\pi_t^X} + \frac{d\pi_t^Y}{\pi_t^Y} + \frac{dS_t}{S_t} + \left(\frac{d\pi_t^X}{\pi_t^X}\right)\left(\frac{dS_t}{S_t}\right) \\ &= \left(\mu_t^X + r_t + \sigma_t'(\sigma_t^X + \Lambda_t^X)\right) dt + (\sigma_t^X + \sigma_t) dB_t^X - \Lambda_t^Y \, dB_t^Y. \end{aligned}$$

By definition of state-price deflator, the product of the state-price deflator and the security price is an exponential martingale, which implies

$$\mu_t^X + r_t + \sigma_t'(\sigma_t^X + \Lambda_t^X) = 0. \tag{A.3}$$

Hence, we obtain eq. (2.5).

Secondly, we prove eq. (2.6). Note the following holds by definition of state-price deflator.

$$\hat{\pi}_{nt} = \pi_{nt} \varepsilon_t^n. \tag{A.4}$$

Thus putting eq. (2.3) into eq. (A.4) and taking logarithm of both sides of the equation yield

$$\log \varepsilon_t^n = \log \hat{\pi}_{nt}^X + \log \hat{\pi}_{nt}^Y - \log \pi_t^X - \log \pi_t^Y. \tag{A.5}$$

Differentiating the above equation and substituting eqs. (2.5) and (2.4), we obtain eq. (2.6).

A.2 Proof of Lemma 2

Following Kikuchi citeKi, we show the proof. It follows from Girsanov's theorem that the process $\tilde{B}_t^X$ defined by

$$\tilde{B}_t^X = B_t^X + \int_0^t \Lambda_s^X \, ds, \tag{A.6}$$

is a standard Brownian motion under the risk-neutral measure. Then the SDE for X_t under the the risk-neutral measure is rewritten as

$$\begin{aligned} dX_t &= \left(-K_X X_t - \Lambda_t^X\right) dt + d\tilde{B}_t^X \\ &= \{-\lambda_X - (K_X + \Lambda_X)X_t\} \, dt + d\tilde{B}_t^X. \end{aligned}$$

Regard the default-free bond P_t^T as a derivative written on the instantaneous interest rate r_t. Since r_t is a quadratic function of X_t, P_t^T is expressed as an analytic function $f(X_t, t)$, *i.e.*,

$$P_t^T = f(X_t, t). \tag{A.7}$$

It follows from arbitrage-free condition that f is a solution to the PDE:

$$f_t + \{-\lambda_X - (K_X + \Lambda_X)X_t\}' f_X + \frac{1}{2}\mathrm{tr}[f_{XX}] - \left(\rho^0 + \rho' X_t + \frac{1}{2}X_t' R X_t\right) f = 0,$$
$$f(X_T, T) = 1. \tag{A.8}$$

Then f is expressed as

$$f(X_t, t) = e^{b^0(\tau) + b(\tau)' X_t + \frac{1}{2}X_t' \Sigma(\tau) X_t}, \qquad (b^0(0), b(0), \Sigma(0)) = (0, 0, 0), \tag{A.9}$$

where $b^0(\tau), b(\tau), B(\tau)$ are analytic functions of $\tau = T - t$and $\Sigma(\tau)$ is a symmetric matrix. Differentiating (A.9) and putting the result into eq. (A.8), we have

$$\begin{aligned} -\frac{db^0(\tau)}{d\tau} - X_t' \frac{db(\tau)}{d\tau} - \frac{1}{2}X_t' \frac{d\Sigma(\tau)}{d\tau} X_t + \{-\lambda_X - (K_X + \Lambda_X)X_t\}'(b(\tau) + \Sigma(\tau)X_t) \\ + \frac{1}{2}\left(b(\tau)' b(\tau) + \mathrm{tr}[\Sigma(\tau)]\right) + X_t' \Sigma(\tau) b(\tau) + \frac{1}{2}X_t' \Sigma^2(\tau) X_t \\ - \left(\rho^0 + \rho' X_t + \frac{1}{2}X_t' R X_t\right) = 0. \end{aligned} \tag{A.10}$$

Since the above eq. is an identical equation on X_t, eq. (2.15) is obtained. Finally, differentiating eq. (A.9), we get eq. (2.13).

On the j-th index, Kikuchi [9] shows that S_t^j is given by

$$S_t^j = \exp\left(b_j^0 t + b_j' X_t + \frac{1}{2} X_t' \Sigma_j X_t\right). \tag{A.11}$$

Hence, the dividend rate process is

$$\frac{D_t^j}{S_t^j} = d_j^0 + d_j' X_t + \frac{1}{2} X_t' \Sigma_j X_t. \tag{A.12}$$

Then the following identical equation on X_t is obtained from eqs. (A.11) and (A.12) and arbitrage-free condition that

$$\begin{aligned} b_j^0 + \{-\lambda_X - (K_X + \Lambda_X) X_t\}'(b_j + \Sigma_j X_t) + \frac{1}{2}\big(b_j' b_j + \mathrm{tr}[\Sigma_j]\big) + X_t' \Sigma_j b_j + \frac{1}{2} X_t' \Sigma_j^2 X_t \\ + \left(d_j^0 + d_j' X_t + \frac{1}{2} X_t' \Delta_j X_t\right) - \left(\rho^0 + \rho' X_t + \frac{1}{2} X_t' R X_t\right) = 0. \end{aligned} \tag{A.13}$$

Thus we have eq. (2.18).

On the n-th foreign country's default-free bond, the following equation holds from arbitrage-free condition,

$$\frac{d\hat{P}_{nt}^T}{\hat{P}_{nt}^T} = \left(\hat{r}_t + (\hat{b}_n(\tau) + \hat{\Sigma}_n(\tau) X_t)' \hat{\Lambda}_t^X\right) dt + (\hat{b}_n(\tau) + \hat{\Sigma}_n(\tau) X_t)' dB_t^X, \tag{A.14}$$

Then we have eq. (2.19). In the similar way, we obtain eq. (2.22).

A.3 Proof of Lemma 3

Let $(\vartheta, (\vartheta(\tau)), (\vartheta^j), (\hat{\vartheta}_n(\tau)), (\hat{\vartheta}_n^j))$ denote the portfolio. The nominal value of wealth is given by

$$\tilde{W}_t = \vartheta_t P_t + \int_0^{\bar{\tau}} \vartheta_t(\tau) P_t(\tau) d\tau + \sum_{j=1}^{J} \vartheta_t^j S_t^j + \sum_{n=1}^{N} \int_0^{\hat{\tau}_n} \hat{\vartheta}_{nt}(\tau) P_{nt}(\tau) d\tau + \sum_{n=1}^{N} \sum_{j=1}^{\hat{J}_n} \hat{\vartheta}_{nt}^j S_{nt}^j. \tag{A.15}$$

Then given c_t, the self-financing portfolio $(\vartheta, (\vartheta(\tau)), (\vartheta^j), (\hat{\vartheta}_n(\tau)), (\hat{\vartheta}_n^j))$

satisfies

$$\begin{aligned}
\frac{d\tilde{W}_t}{\tilde{W}_t} &= \frac{1}{\tilde{W}_t}\Bigg\{\vartheta_t dP_t + \int_0^{\bar{\tau}} \vartheta_t(\tau)dP_t(\tau)d\tau + \sum_{j=1}^{J}\vartheta_t^j\Big(dS_t^j + D_t^j dt\Big) \\
&\quad + \sum_{n=1}^{N}\int_0^{\hat{\tau}_n} \hat{\vartheta}_{nt}(\tau)dP_{nt}(\tau)d\tau + \sum_{n=1}^{N}\sum_{j=1}^{\hat{J}_n}\vartheta_{nt}^{*j}\Big(dS_{nt}^j + D_{nt}^j dt\Big) - \frac{p_t}{\tilde{W}_t}c_t dt\Bigg\} \\
&= \frac{\vartheta_t P_t}{\tilde{W}_t}\frac{dP_t}{P_t} + \int_0^{\bar{\tau}} \frac{\vartheta_t(\tau)P_t(\tau)}{\tilde{W}_t}\frac{dP_t(\tau)}{P_t(\tau)}d\tau + \sum_{j=1}^{J}\frac{\vartheta_t^j S_t^j}{\tilde{W}_t}\frac{dS_t^j + D_t^j dt}{S_t^j} \\
&\quad + \sum_{n=1}^{N}\int_0^{\hat{\tau}_n} \frac{\hat{\vartheta}_{nt}(\tau)P_{nt}(\tau)}{\tilde{W}_t}\frac{dP_{nt}(\tau)}{P_{nt}(\tau)}d\tau + \sum_{n=1}^{N}\sum_{j=1}^{\hat{J}_n}\frac{\vartheta_{nt}^j S_{nt}^j}{\tilde{W}_t}\frac{dS_{nt}^j + D_{nt}^j dt}{S_{nt}^j} - \frac{c_t}{W_t}dt\Bigg\} \\
&= \left(1 - \int_0^{\bar{\tau}} \varphi_t(\tau)d\tau - \sum_{j=1}^{J}\Phi_t^j - \sum_{n=1}^{N}\int_0^{\hat{\tau}_n} \hat{\varphi}_{nt}(\tau)d\tau - \sum_{n=1}^{N}\sum_{j=1}^{\hat{J}_n}\hat{\Phi}_{nt}^j\right)\frac{dP_t}{P_t} \\
&\quad + \int_0^{\bar{\tau}} \varphi_t(\tau)\frac{dP_t(\tau)}{P_t(\tau)}d\tau + \sum_{j=1}^{J}\Phi_t^j\frac{dS_t^j + D_t^j dt}{S_t^j} \\
&\quad + \sum_{n=1}^{N}\int_0^{\hat{\tau}_n} \hat{\varphi}_{nt}(\tau)\frac{dP_{nt}(\tau)}{P_{nt}(\tau)}d\tau + \sum_{n=1}^{N}\sum_{j=1}^{\hat{J}_n}\hat{\Phi}_{nt}^j\frac{dS_{nt}^j + D_t^j dt}{S_{nt}^j} - \frac{c_t}{W_t}dt.
\end{aligned}$$

Thus the SDE for W_t is derived as

$$\begin{aligned}
\frac{dW_t}{W_t} &= \frac{d\tilde{W}_t}{\tilde{W}_t} - i_t dt \\
&= \left(1 - \int_0^{\bar{\tau}} \varphi_t(\tau)d\tau - \sum_{j=1}^{J}\Phi_t^j - \sum_{n=1}^{N}\int_0^{\hat{\tau}_n} \hat{\varphi}_{nt}(\tau)d\tau - \sum_{n=1}^{N}\sum_{j=1}^{\hat{J}_n}\hat{\Phi}_{nt}^j\right)\frac{dP_t}{P_t}
\end{aligned}$$

$$+\int_0^{\bar{\tau}} \varphi_t(\tau)\frac{dP_t(\tau)}{P_t(\tau)}d\tau + \sum_{j=1}^{J}\Phi_t^j\frac{dS_t^j}{S_t^j} + \sum_{n=1}^{N}\int_0^{\hat{\tau}_n} \hat{\varphi}_{nt}(\tau)\frac{d\hat{P}_{nt}(\tau)}{\hat{P}_{nt}(\tau)}d\tau + \sum_{n=1}^{N}\sum_{j=1}^{\hat{J}_n}\hat{\Phi}_{nt}^j\frac{d\hat{S}_{nt}^j}{\hat{S}_{nt}^j} - \frac{c_t}{W_t}dt.$$

Substituting eqs. (2.12), (2.13), (2.16), (2.19), and (2.22) into the above eq. and organizing the result yield eq. (2.26).

A.4 Proof of Proposition 1

Firstly, the optimal consumption control is calculated as

$$c_t^* = \alpha^{\frac{1}{\gamma}} e^{-\frac{\beta}{\gamma}t} J_W^{-\frac{1}{\gamma}} = \alpha^{\frac{1}{\gamma}} e^{-\frac{\beta}{\gamma}t} \left\{ e^{-\beta t} (W_t^*)^{-\gamma} G^{\gamma} \right\}^{-\frac{1}{\gamma}} = \alpha^{\frac{1}{\gamma}} \frac{W_t^*}{G},$$

and thus eq. (3.11) is obtained.

Secondly, derivatives of J are given by

$$J_t = -\beta J, \qquad W_t J_W = (1-\gamma) J, \qquad J_X = \gamma\, J \frac{G_X}{G}, \qquad J_Y = \gamma\, J \frac{G_Y}{G},$$

$$W_t^2 J_{WW} = -\gamma(1-\gamma) J, \qquad W_t J_{XW} = \gamma(1-\gamma) J \frac{G_X}{G}, \qquad W_t J_{YW} = \gamma(1-\gamma) J \frac{G_Y}{G},$$

$$J_{XX} = \gamma\, J \left\{ (\gamma-1) \frac{G_X}{G} \frac{G_X'}{G} + \frac{G_{XX}}{G} \right\}, \qquad J_{YY} = \gamma\, J \left\{ (\gamma-1) \frac{G_Y}{G} \frac{G_Y'}{G} + \frac{G_{YY}}{G} \right\}.$$

Then the nominator and the denominator of right-hand side of eq. (3.12) are rewritten as

$$\psi_t = J \left((\gamma-1) \begin{pmatrix} \Lambda_t^X \\ \Lambda_t^Y \end{pmatrix} + \gamma(\gamma-1) \begin{pmatrix} \frac{G_X}{G} \\ \frac{G_Y}{G} \end{pmatrix} \right), \tag{A.16}$$

$$W_t^2 J_{WW} = \gamma(\gamma-1) J. \tag{A.17}$$

Thus putting eqs. (A.16) and (A.17) into eq. (3.12), we have eq. (3.12). The second and third terms in eq. (3.8) are calculated from eqs. (A.16) and (A.17) as

$$\begin{aligned}
&\frac{1}{2} \operatorname{tr}\left[X J_{XX} + J_{YY} \right] - \frac{\psi_t' \psi_t}{2 W_t^2 J_{WW}} \\
&= \frac{\gamma}{2} J \operatorname{tr} \left[\left\{ (\gamma-1) \frac{G_X}{G} \frac{G_X'}{G} + \frac{G_{XX}}{G} \right\} + \left\{ (\gamma-1) \frac{G_Y}{G} \frac{G_Y'}{G} + \frac{G_{YY}}{G} \right\} \right] \\
&- \frac{1}{2\gamma(\gamma-1)} J \left((\gamma-1) \begin{pmatrix} \Lambda_t^X \\ \Lambda_t^Y \end{pmatrix} + \gamma(\gamma-1) \begin{pmatrix} \frac{G_X}{G} \\ \frac{G_Y}{G} \end{pmatrix} \right)' \left((\gamma-1) \begin{pmatrix} \Lambda_t^X \\ \Lambda_t^Y \end{pmatrix} + \gamma(\gamma-1) \begin{pmatrix} \frac{G_X}{G} \\ \frac{G_Y}{G} \end{pmatrix} \right) \\
&= \gamma J \left\{ \frac{1}{2} \operatorname{tr} \left[\frac{G_{XX}}{G} + \frac{G_{YY}}{G} \right] - \frac{\gamma-1}{2\gamma^2} \left((\Lambda_t^X)' \Lambda_t^X + (\Lambda_t^Y)' \Lambda_t^Y \right) - \frac{\gamma-1}{\gamma} \begin{pmatrix} \Lambda_t^X \\ \Lambda_t^Y \end{pmatrix}' \begin{pmatrix} \frac{G_X}{G} \\ \frac{G_Y}{G} \end{pmatrix} \right\}.
\end{aligned} \tag{A.18}$$

The seventh term in eq. (3.8) is calculated from eq. (3.2) as

$$\frac{\gamma}{1-\gamma} c_t^* J_W = \alpha^{\frac{1}{\gamma}} \frac{W_t^*}{G} \gamma \frac{J}{W_t^*} = \alpha^{\frac{1}{\gamma}} \gamma \frac{J}{G}. \tag{A.19}$$

Substituting eqs. (A.18) and (A.19) into eq. (3.8), and multiplying by $G/(\gamma J)$ yield eq. (3.13).

A.5 Proof of Proposition 2

It is straightforward to see that $a_0(\tau)$ and $a(\tau)$ are expressed as eqs. (3.25) and (3.26). Following Theorem 5.2 in Arimoto [2], we prove that $A(\tau)$ is expressed as eq. (3.27). We consider the following initial value problem of linear differential equation for $N \times N$ matrix-value functions $C_1(\tau)$ and $C_2(\tau)$.

$$\frac{d}{d\tau}\begin{pmatrix} C_1(\tau) \\ C_2(\tau) \end{pmatrix} = \begin{pmatrix} L & -I_{\bar{N}} \\ -\frac{\gamma-1}{\gamma^2}\Lambda'\Lambda & -L' \end{pmatrix}\begin{pmatrix} C_1(\tau) \\ C_2(\tau) \end{pmatrix}, \qquad \begin{pmatrix} C_1(\tau) \\ C_2(\tau) \end{pmatrix} = \begin{pmatrix} I_{\bar{N}} \\ 0_{\bar{N}} \end{pmatrix}. \tag{A.20}$$

A solution to eq. (A.20) is given by eq. (3.28). Since we can prove $C_1(\tau)$ to be regular[3], we define $A(\tau)$ by eq. (3.27). Then noting that

$$\frac{d}{d\tau}C_1^{-1}(\tau) = -C_1^{-1}(\tau)\left\{\frac{d}{d\tau}C_1(\tau)\right\}C_1^{-1}(\tau), \tag{A.21}$$

we can derive

$$\begin{aligned} \frac{d}{d\tau}A(\tau) &= \left\{\frac{d}{d\tau}C_2(\tau)\right\}C_1^{-1}(\tau) + C_2(\tau)\frac{d}{d\tau}C_1^{-1}(\tau) \\ &= \left(-\frac{\gamma-1}{\gamma^2}\Lambda'\Lambda C_1(\tau) - L'C_2(\tau)\right)C_1^{-1}(\tau) - A(\tau)\left(LC_1(\tau) - C_2(\tau)\right)C_1^{-1}(\tau) \\ &= A^2(\tau) - L'A(\tau) - A(\tau)L - \frac{\gamma-1}{\gamma^2}\Lambda'\Lambda, \end{aligned}$$

and thus confirm that $A(\tau)$ satisfies Riccati equation (3.22). For uniqueness of the Riccati equation, see proof in Theorem 5.2 in Arimoto [2]. Finally, for symmetry of $A(\tau)$, taking transposition of Riccati equation (3.22) for $A(\tau)$ yields the same equation for $A(\tau)'$, which implies $A(\tau)' = A(\tau)$ because of uniqueness of the Riccati equation.

[3] See proof in Theorem 5.2 in Arimoto [2].