THE DEMAND FOR INTERNATIONAL RESERVES: A RESPECIFICATION

Todd Sandler*
Shigeo Minabe

In a recent article in the *American Economic Review*, Michael G. Kelly examined the condition for an optimum level of international reserves. "The government maximizes utility subject to the tradeoff between lower income levels implicit in large reserve holdings and greater income fluctuations generated by exogenous disturbances which cannot be neutralized when reserve holdings are small." (Kelly, p. 655) Kelly formulated his problem into a constrained optimization problem in which the average level of utility (E(U)) was maximized, constrained by a "minimum level of reserves below which it becomes prohibitively costly to pursue stabilization policies." (Kelly, p. 658) Kelly dealt with two variables—exogenous flows that are independent of the external balance (X) and endogenous flows (M) that are under the complete control of the government. These endogenous flows include both imports and capital imports, which affect the income level of the country, unless financed through reduction of international reserves.

In another recent contribution, Peter Clark presented a model that explored optimum international reserves and their relationship to the speed

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*Dr. Sandler is Assistant Professor of Economics at Arizona State University. He expresses thanks to Professor Graves who read an early version of this note. Of course full responsibility rests with the author. The research was financed by a grant from Arizona State University. Dr. Minabe is an Associate Professor of this university.
of adjustment. Both the Kelly and the Clark articles share the same difficulty in that speculative capital flows are inadequately handled within the respective analyses. Kelly chooses to ignore speculative capital flows completely and Clark chooses to treat these flows as a random variable with a zero mean and constant variance. (Clark, p. 360)

This paper extends the previous Kelly model to include a third variable, speculative capital flows (C), that are dependent upon the reserve level (R), and unlike M, and are not subject to government control. The introduction of speculative flows influences the model through its effect on income variance, and is felt within the constraint. The introduction of speculative capital flows allows for the addition of monetary policy whereas the previous two analyses explored adjustment through fiscal policy only. The inclusion of the third variable provides for a much fuller analysis and allows for a more complete determination of the demand for reserves. Not until a full determination of reserve demand is formulated will it be possible to calculate the necessary increases in the supply of international liquidity in the form of SDR's.

This paper is divided into two sections. Section one explores the introduction of speculative flows under the absence of monetary policy and compares the results with Kelly's. In addition, the section examines the model under the alternative assumption that monetary policy is employed by the government in order to influence the effect of speculative flows on the income level. The second section extends the analysis to take a cursory look at the influence of the interest rate. Also, the validity of the constr-

(1) Close readings of the Kelly and Clark article indicate that although on the surface these articles are seemingly quite different, they are really quite similar. This is especially true in terms of the constraint and the degree of control that the government exercises over endogenous flows. A recent survey article by H. Grubel, that appeared in the Journal of Economic Literature, failed to point out the similarities.
A fixed exchange is assumed to exist. The country considered is small so that the import prices remain constant. In addition, fundamental equilibrium holds, which indicates that trends in reserves can be ignored. Variations in the income level \((Y)\) are caused by both the speculative flows and the exogenous element, exports.

The change in reserves \((R)\) is equal to exogenous changes in foreign demand less induced changes in the domestic demand for imports, which includes net capital imports, plus the net change in speculative inflows.

\[ \Delta R = \Delta X - \Delta M + \Delta C \]

Imports are subject to government control in the form of permitted changes to partially offset changes in exports. Variations in speculative flows are viewed by the government as transitory in nature and are not offset. Later, a model is presented in which monetary policy is undertaken in order to offset, partially or totally, the influence of speculative flows. Speculative inflows influence the income level through their effect on the money supply, unless compensatory policy is undertaken, and an increased money supply can lead to a greater income, *ceteris paribus*. Equation 2 expresses the relationship that the change in income is influenced both by the exports change and the speculative flow change.

\[ \Delta Y = \frac{\Delta Y}{\Delta X} \Delta X + \frac{\Delta Y}{\Delta C} \Delta C \]

Following Kelly's analysis (Kelly, P. 657) an import response coefficient; \(f\), equal to \(dM/dX\) is defined, and it indicates the degree to which external

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(2) Fundamental disequilibrium would require an addition of a trend factor of either a linear or nonlinear nature. This trend factor would make for an unstable situation since the constraint would be moving outwards over time unless a nonlinear situation exists.
disturbances are allowed to influence the domestic economy. In addition to f, there is an income response coefficient g, equal to \( \Delta Y/\Delta X \) and a speculative flow response coefficient p, equal to \( \Delta Y/\Delta C \), which reflects the effect that changes in exports and capital flows have on the income level. With the addition of the import propensity (m), a simplifying assumption gives:

\[ f = mg. \]

Taking the variance of income around its mean, equation 2 yields the following results:

\[ V(Y) = g^2 V(X) + p^2 V(C) + 2gp\rho S(X)S(C) \]

Within the above equation, \( \rho \) represents the correlation coefficient and S is the standard deviation. It is reasonable to assume independence of X and C, and therefore equation 3 can be simplified to a form without the correlation term:

\[ V(Y) = g^2 V(X) + p^2 V(C) \]

Speculative capital flows are for the purpose of taking advantage of interest rate differentials and expected changes in the exchange rate. The first model ignores the interest rate influence; however, section 2 explores the additional effect of the interest rate. Unlike the exogenous export flow, speculative flows are dependent on the current state of the balance of payments and are assumed to depend directly on the reserve stock, which is indicative of the resources available to the authorities to forestall an exchange-rate change. More specifically, the change in speculative flows is assumed to be proportional to the change in reserves.

\[ \Delta C = \alpha \Delta R \]

The constant \( \alpha \) is assumed to be between zero and one indicating a less than perfect correlation. A constant value of \( \alpha \) is chosen for analytic simplicity although a variable value of \( \alpha \) in which \( \alpha \) is itself a function of reserves more realistic. The last relationship implies that the variance in
speculative flow is a certain proportion of the variance in reserves.

(6) \[ V(C) = a^2 V(R) \]

Substitute \( f \Delta X \) for \( \Delta M \) in equation 1, and then substitute equation 5 into equation 1. Taking variance around the mean yields the following:

(7) \[ V(R) = \left( \frac{1-f}{1-\alpha} \right)^2 V(X) \]

It is instructive to compare this result with Kelly's equation 3 \( (V(R) = (1-f)^3 V(X)) \). Equation 7 is indicative of a greater degree of reserve variance that results from the "feedback effect" that the reserve variance has on speculative flow variances. As will become evident shortly, the constraint that is operative now is much stronger than the previous constraint put forward by Kelly. As \( \alpha \) approaches one, the variance in reserve increases in value until an unstable situation occurs when \( \alpha \) is one.

Substitution of equation 6 into equation 4 for \( V(C) \), and then substitution of equation 7 into equation 4 for \( V(R) \) gives an expression for income variance in terms of the variance in the exogenous element.

(8) \[ V(Y) = g^2 + p^2 \alpha^2 \left( \frac{1-f}{1-\alpha} \right)^2 V(X) \]

Note that income variance is greater than in Kelly's model (Kelly, p. 658) owing to the additional influence of speculative flows—hence speculative flows are felt within the constraint through its effect on income variance.

Following the analysis of Kelly, use of the Chebychev's inequality for a symmetric distribution yields an expression for the average level of reserves needed to insure that the probability of reserves falling below a target level is a small number \( (e) \). The target level is determined by the confidence level of reserves at which point severe runs on a currency ensue. Equation 9 expresses the value of \( e \) in terms of the variances and average level of reserve.

(9) \[ e = \frac{c V(R)}{E(R)^2} \quad \text{where } c > 0 \]

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Rearranging equation 9, and substitution of equation 7 gives

\[(10)\ E(R) = (c/e) + S(R) = (c/e) + \left(\frac{1-f}{1-\alpha}\right) S(X)\]

Further substitutions for \(f\) and from equation 8 yields the final form of the constraint.

\[(11)\ E(R) = (c/e) + \frac{1}{1-\alpha} \left[ S(X) - \frac{mg}{T} S(Y) \right]\]

where \(T = \left[ g^2 + \bar{p}^2 a^2 \left( \frac{1-mg}{1-\alpha} \right)^2 \right]^{+}\)

Since it can be easily shown that \(g<T\), the constraint in the present model is further to the right than in Kelly’s model (Kelly equation 8, \(E(R) = (c/e) + [S(X) - mS(Y)]\)).

Using the quadratic utility function

\[(12)\ U = -a[E(Y') - E(Y)]^2 - b[Y - E(Y)]^2, a, b > 0\]

\((Y'\text{ is total output when reserves are zero})\) we get expected utility. Income reduction due to the holding of reserve assets is equal to

\[(13)\ Y' - Y = R_i.\]

In deriving equation (14) we take the expected value of reserves in equation (13) and substitute for \([E(Y') - E(Y)]^2\) within equation (12).

\[(14)\ E(U) = -a\bar{E}(R)^2 - bV(Y)\]

Maximizing expected utility subject to the constraint expressed in equation 11 yields a solution to the optimum level of reserves that accounts for the tradeoff between income variance and lower income levels. Maximization of expected utility, as subject to the constraint, is done with respect to \(E\) \((\bar{R})\) and \(V(Y)\). Equation 14 expresses the optimum level of reserves.

\[(15)\ E(R) = \frac{S(X)}{(c/e) + m^2 (a/b) i^2 g^2 + (e/c) + (1-\alpha)} \frac{1}{1-\alpha}\]

Figure 1 illustrates the result graphically in the first quadrant. The

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(3) Equation 15 is derived in the Appendix.
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indifference curves are concave to the origin, and the constraint is convex to the origin—these shapes insure a unique maximum within the world of "bads" that is dealt within the analysis. The addition of speculative flows does not change the shapes of the curves from previous analyses—this can be verified by checking the signs of the first and second order partials with those of Kelly's. The situation is a maximum since the border Hessian is equal to

\[ |H| = +2az(c/e) \frac{1}{(1-\alpha)^2} \frac{m^2g^2}{T^2} + 2b \]

which is greater than zero.

The final result indicates that the average level of reserves varies directly with \( \alpha \) unless the marginal disutility of income reduction (a) is greater than the marginal disutility of income variance (b) by a large amount. In addition, the income response coefficient (g) plays an expanded role as compared to the Kelly model. Hence, the addition of speculative flows increases the roles of a and b, and focuses a much greater amount of attention on income variations and the repercussions these variations will have on the constraint. Failure to account for the additional influence of speculative flows in the determination of optimum reserves may tend to underestimate the optimum level—a possibility that would impose undesirable costs on a country.

The model can be expanded to include the choice of monetary policy that would control some of the income variation influences of speculative flows. Consider the polar case in which complete sterilization of speculative flows is undertaken so that there is no change in the money supply that results from speculative flows. This assumption implies that changes in income are related to changes in exports only since \( \Delta Y/\Delta C \) is zero.

\[ V(Y) = g^2V(X) \]

Therefore, the variance in income is related to the variance in exogenous exports and the variance in reserve has the same form as before (see equation.
7). This indicates that speculative flows are still felt in the constraint because it increases reserve variance through the "feedback" effect of reserve variances causing speculative flow variance. However, the effect that speculative flows have on the constraint is diminished since it does not affect income variance in addition to the "feedback" effect. The new constraint

\[ E(R) = \frac{c/e}{1-a} - \frac{1}{(1-a)} [S(X) - mS(Y)] \]

lies to the right of Kelly's constraint but it lies to the left of the constraint that is operative when no sterilization is undertaken. Optimum reserves are equal to:

\[ E(R) = \frac{S(X)}{(c/e) + m^2(a/b)^2 + (e/c) + (1-a)} \]

which is smaller than the previous value since \( g \) is between zero and one (see equation 15).

If no policy is established, then the situation would degenerate to the original model presented above. The situation of some sterilization would give a value for optimum reserves somewhere between the two polar cases discussed.

II. Extensions and Problems

Another feasible extension is to include the influence of the interest rate in a more endogenous manner. Previous analyses treat the interest rate as influencing the decision of optimum reserves through its opportunity cost determination only. Since speculative flows are influenced both by expected exchange rate changes and interest variation, which changes portfolio equilibrium and causes stock-adjusting speculative flows, the variance in capital flows can be related to both the interest variance and reserve variance.

(4) For analytic simplicity an additive linear form is assumed.
\( V(C) = \alpha^2 V(R) + \beta^2 V(i) \quad \beta > 0 \)

Substitution of equation 20 into the analysis in place of equation 6 would indicate an additional influence in the constraint, since any interest rate variance would enhance the variance in speculative flows, which, in turn, would increase income variance and shift the constraint to the right. A more complete analysis requires the introduction of a money demand and supply function so that the interest rate can be dealt with in an endogenous fashion.

There are two additional problems that need to be discussed. One concerns the form of the constraint; the other concerns the form of the utility function. The problem that is associated with the constraints is connected with the Chebychev's inequality, which may be too generalized to use within a specific situation.

In specific situations some information regarding the frequency distribution of reserves may be known that would make the use of Chebychev's inequality, which holds true for any distribution, too conservative an estimate. This conservative estimate would tend to bias the optimum reserve level in the upward direction. Therefore, caution must be used in utilizing the Chebychev's inequality when information about the reserve distribution is available.

The second problem concerns the use of the quadratic utility function that appears in both the Clark and the Kelly models. The primary problem of using a quadratic utility function arises from the implicit assumption of increasing absolute risk aversion. Assuming a plausible speed with which the ultimate bliss point is approached, Tsiang has shown that the log form, which assumes decreasing absolute risk aversion and constant relative risk aversion, would be a more appropriate form. (S. Tsiang, pp. 355–61) In addition, Tsiang has shown that in order to ignore third order and higher
terms that result from the Taylor Series expansion of the log form, then $S(Y)/E(Y)$ must be less than one. (S. Tsiang, pp. 358—59) Since it is reasonable to assume that within the situation of international reserves, the variance in income is significantly less than the average income level of the country, third order and higher moments can be ignored. Thus, the skewness and the peakness of the reserve distribution need not enter the utility function, and hence, expected utility, which implies that previous analyses were correct in ignoring skewness. However, as speculative flows increase income variation, the ratio $S(Y)/E(Y)$ increases in value, and skewness can be expected to exert a greater influence.

In summary, this paper has extended previous analyses so that not only exogenous flows, which do not depend on the current balance of payments situation, and imports, and which are controlled by government, are included in the analysis, but also speculative flows, which depend on reserves, are now added. This addition allowed the type of the policy employed in the model to include monetary policy along with fiscal policy. The additional effect of the interest rate was briefly examined. The paper concluded with mention of problems concerning the form of the constraint and the quadratic utility function.

REFERENCES


**MATHEMATICAL APPENDIX**

Maximum: \( E(U) = -at^2 \ E(R)^2 - bV(Y) \)

Subject to:
\[
E(R) = \frac{(c/e) + \left(\frac{1}{1-a}\right) \left[ S(X) - \frac{mg}{T} S(Y) \right]}{S(Y)}
\]

The Lagrangian has the following form:
\[
L = -at^2 \ E(R)^2 - bV(Y) + \lambda \left( \frac{(c/e) + \left(\frac{1}{1-a}\right) \left[ S(X) - \frac{mg}{T} S(Y) \right]}{S(Y)} \right) - E(R)
\]

The first order conditions yield:

1. \[
\frac{\partial L}{\partial E(R)} = -2at^2 \ E(R) - \lambda = 0
\]

2. \[
\frac{\partial L}{\partial S(Y)} = -2bS(Y) - \lambda \left(\frac{1}{1-a}\right) \left(\frac{mg}{T}\right) = 0
\]

3. \[
\frac{\partial L}{\partial \lambda} = (c/e) + \left(\frac{1}{1-a}\right) \left[ S(X) - \frac{mg}{T} S(Y) \right] - E(R) = 0
\]

Using equations (1) and (2) to remove \( \lambda \) and solve for \( E(R) \) yields:

4. \[
E(R) = \frac{bS(Y)}{\frac{(c/e) + \left(\frac{1}{1-a}\right) \left(\frac{mg}{T}\right)}{\frac{S(Y)}}}
\]

Solve for \( S(Y) \) using equation (3) gives:

5. \[
S(Y) = \frac{-E(R)}{\left(\frac{c/e) + \left(\frac{1}{1-a}\right) \left(\frac{mg}{T}\right)}{S(Y)}\right)} + \frac{S(X)}{\frac{mg}{T}}
\]

Substitute equation (5) into equation (4) and solve for \( E(R) \) and then simplify. This gives us optimum reserves

6. \[
E(R) = \frac{S(X)}{\left(\frac{(c/e) + m^2(a/b)^2 g^2 + (c/e) + (1-a)}{(1-a)}\right)}
\]

For second order conditions:

\[
|\bar{H}| = \begin{vmatrix}
-2at^2 & 0 & 1 \\
0 & -2b & -(c/e) + \left(\frac{1}{1-a}\right) \left(\frac{mg}{T}\right) \\
1 & -(c/e) + \left(\frac{1}{1-a}\right) \left(\frac{mg}{T}\right) & 0
\end{vmatrix}
\]

\[
H = -2at^2 \left(\frac{(c/e) \left(\frac{1}{1-a}\right) \left(\frac{mg}{T^2}\right)}{m^2 g^2} \right) - 0 + 1(2b) > 0
\]

Hence it is a. maximum.