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Bolorsuvd BATBOLD
Kentaro KIKUCHI
Koji KUSUDA

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The Institute for Economic and Business Research
Faculty of Economics
SHIGA UNIVERSITY

1-1-1 BANBA, HIKONE,
SHIGA 522-8522, JAPAN

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Bolorsuvd BATBOLD — Faculty of Economics, Shiga University

Kentaro KIKUCHI — Faculty of Economics, Shiga University

Koji KUSUDA† — Faculty of Economics, Shiga University

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Abstract

There exists strong empirical evidence that all inflation rates, interest rates, market price of risk, and return volatilities of assets are stochastic and predictable, which is now a stylized fact. However, to the best of our knowledge, existing models providing solutions to consumption–investment problems do not consider all of the aforementioned stochastic processes, leading to substantially different results depending on the model structure. We consider a consumption–investment problem for a long-term investor with constant relative risk aversion utility, under a quadratic security market model, in which all of the above-mentioned processes are stochastic and predictable. We solve a nonhomogeneous linear partial differential equation for the indirect utility function, and derive a semi-analytical solution. This study obtains the optimal portfolio decomposed into the sum of myopic demand, intertemporal hedging demand, and "inflation hedging demand," and presents that all three types of demand are nonlinear functions of the state vector. The results highlight that the timing aspect is more important than our assumption.

1 Introduction

In this study, we consider a dynamic consumption and investment problem for a long-term investor with constant relative risk aversion (CRRA) utility. There is strong empirical evidence that all inflation rates, interest rates, market price of risk, and return volatilities of assets are stochastic

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†kusuda@biwako.shiga-u.ac.jp

and predictable,[1] which is now a stylized fact. In order to incorporate this fact into security market model, a state vector process is required. Then, the investment opportunity set becomes stochastic owing to the variation in the state vector. Merton [28] constitutes the framework for a dynamic consumption–investment problem with the stochastic investment opportunity set and presents that the optimal portfolio is the sum of myopic demand which is the optimal portfolio for short-term investors, and intertemporal hedging demand specific to long-term investors. Myopic demand does not consider the risk of changes in indirect utility due to changes in state processes and pursues the rewards of investing in risky assets in a myopic manner. In contrast, intertemporal hedging demand insures against the risk of changes in indirect utility. Brennan and Xia [10] then solve a dynamic consumption–investment problem, taking into account the presence of inflation risk, and clarify that the optimal portfolio is decomposed into the sum of the above-mentioned two types of demand and inflation-related demand, which insures against inflation risk. In this paper, we refer to inflation-related demand as 'inflation hedging demand.' We also refer to the optimal portfolio for long-term investors as strategic asset allocation, following the designation of Brennan, Schwartz, and Lagnado [7].

As all the above-mentioned three types of demand in the optimal portfolio generally depend on the stochastic state vector process, the strategic asset allocation emphasizes the importance of the timing aspect. The question is the magnitude of the timing effect. Its importance is hardly recognized in asset management practice. An empirical study by Brinson, Hood, and Beebower [11], which is well known in practitioner circles, analyzes 1974–1983 performance of 91 US pension funds and shows that the timing effect contributes little to returns. Most practitioners seem to recognize that the timing effect is either not large or, even if large, difficult to achieve. It is not surprising that they perceive it that way, because even in academic circles, there is a difference of opinion on the magnitude of the timing effect. Brennan *et al.* [7] and Campbell and Viceira [16], Brandt [5], and Ang and Bekaert [1] consider the optimal portfolio decomposed into myopic demand and intertemporal hedging demand, and test whether the magnitude of the timing effect on intertemporal hedging demand is significant. It is estimated to be large by Brennan *et al.* [7] and Campbell and Viceira [16], whereas, according to Brandt [5] and Ang and Bekaert [1], it is small. Further, Sangvinatsos and Wachter [30] analyze the optimal portfolio decomposed into the above-mentioned three types of demand and present that the magnitude of

[1] It is evident that interest rates are stochastic and mean-reverting. Campbell [12], Campbell and Shiller [15], Fama and French [22], Poterba and Summers [29], and Hodrick [23] present that risk premiums of stocks are stochastic and mean-reverting, and Bollerslev, Chou, and Kroner [4], Campbell, Lo, and MacKinlay [14], Campbell, Lettau, Malkiel, and Xu [13] show that return volatilities of stocks are stochastic and mean-reverting.

the timing effect on the inflation hedging demand is large.

The dynamic consumption–investment problem in the framework of Merton [28] can be attributed to the problem of solving the second-order linear partial differential equation (PDE), which is derived from the Hamilton–Yacobi–Bellman (HJB) equation, for the indirect utility function. A nonhomogeneous term appears in the PDE when the utility is the standard CRRA utility, making it difficult to derive an analytical solution.[2] Incorporating all of the above-mentioned stochastic processes in security markets increases the complexity of the model and the difficulty of deriving analytical solutions. Therefore, in prior studies, a security market model that incorporates only some of the above-mentioned stochastic processes, is assumed, resulting in different empirical results depending on the structure of the model. In many cases, empirical analyses are based on numerical and approximate analytical solutions rather than analytical solutions, and it cannot be denied that the accuracy of numerical and approximate analytical solutions may also affect the results.

Hence, it is desirable to derive analytical solutions for consumption–investment problem under a security market model that incorporates all of the above-mentioned stochastic processes in the security markets and that can be estimated with high accuracy. This study aims to derive such a solution, with all of the above-mentioned stylized facts incorporated into a security market model. Note that, if we can derive an analytical solution for consumption–investment problem, it will also contribute to the practical application of asset management.

The main previous studies on the derivation of analytical solutions to consumption and investment problems for long-term investors with CRRA utility are as follows: Kim and Omberg [25], Brennan [6], Brennan and Xia [8, 9], and Wachter [31] derive semi-analytical solutions. However, these studies assume asset return volatilities as being constant and do not consider inflation.

Campbell and Viceira [17], Brennan and Xia [10], Sangvinatsos and Wachter [30], and Batbold, Kikuchi, and Kusuda [3] emphasize the importance of inflation risk for long-term investors, and consider the case of the stochastic inflation rate. Campbell and Viceira [17] and Batbold *et al.* [3] assume that only instantaneous expected inflation rate is stochastic. Campbell and Viceira [17] derive a loglinear approximate analytical solution, and Batbold *et al.* [3] obtain a semi-analytical solution.[3] Brennan and Xia [10] and Sangvinatsos and Wachter [30] suppose that the inflation rate process is a diffusion process, and they derive semi-analytical solutions. However,

[2]Only when the coefficient of relative risk aversion is one, the nonhomogeneous term vanishes and an analytical solution can be derived, but the results of the empirical analysis show that the coefficient of relative risk aversion exceeds one.

[3]Note that inflation hedging demand does not appear in their optimal portfolio choice, because the volatility of inflation is assumed to be zero in their model.

they assume that asset return volatilities are constant.

Chacko and Viceira [19] and Liu [27] assume that asset return volatilities are stochastic. Chacko and Viceira [19] derive a loglinear approximate solution, and Liu [27] considers a highly general security market model, in which interest rates, market price of risk, and return volatilities of assets are stochastic, and obtain an exact solution. However, they do not consider inflation.

In this study, we analyze the strategic asset allocation of a long-term investor with CRRA utility under the assumption of a quadratic security market model. A class of quadratic security market models is developed by Leippold and Wu [26], to capture the above-mentioned stylized fact in security markets. In our quadratic security market model, all the processes above can be expressed as mean-reverting stochastic processes. In addition, because it is a latent factor model, factors can be estimated objectively, unlike in multi-factor models.

Our quadratic security market model is as follows: We adopt the canonical form of the multidimensional Ornstein-Uhlenbeck process as the state vector process (Dai and Singleton [21]) and assume that instantaneous nominal risk-free rate, instantaneous dividend rate, and instantaneous expected inflation rate are quadratic functions of the state vector and that the market price of risk and inflation volatility are affine functions of the state vector. We also assume that the instantaneously nominal risk-free security, default-free bonds, default-free inflation-indexed bonds, and non-bond indices are traded. Note that as inflation is taken into account, security markets would be incomplete if default-free inflation-indexed bonds were not traded, and that risk-free securities for long-term investors are default-free inflation-indexed bonds, which is emphasized by Campbell and Viceira [17, 18].

The main results of this study are as follows. First, we derive no-arbitrage security return processes, a generalized Fisher equation, and real budget constraint under our quadratic model. We present that instantaneous real risk-free rate is a quadratic function of the state vector, and "real market price of risk," which is the market price of risk minus the volatility on inflation, is an affine function of the state vector. We also show that the real budget constraint is expressed by instantaneous real risk-free rate, real market price of risk, investment control, which is an inner product of security portfolio choices and their volatilities minus inflation volatility, and consumption–wealth ratio.

Second, we focus on the solution method developed by Liu [27] and Batbold *et al.* [3]. Liu [27] derive an exact solution, but the parameters of the solution are presented as a solution to a system of ordinary differential equations (ODEs) including the Riccati matrix equation, and the problem of deriving a solution to the system of ODEs remains. Batbold *et al.* [3] consider the dynamic consumption–investment problem under an affine security market model. They derive the same type of exact solution by Liu [27], solve the system of ODEs to parameters of the solution, and reach a semi-analytical solution. We derive a nonhomogeneous linear PDE for indirect utility based on the above-mentioned real budget constraint. Next, we apply the method by Liu [27] and Batbold *et al.* [3], and derive a semi-analytical solution. Finally, we obtain the optimal portfolio decomposed into the sum of myopic demand, intertemporal hedging demand, and inflation hedging demand, as shown by Brennan and Xia [10] and Sangvinatsos and Wachter [30]. In their optimal portfolio, intertemporal hedging demand is a nonlinear function of the state vector, but both types of myopic demand and inflation hedging demand are affine functions because the volatilities of securities are constant in their models. In our optimal portfolio, not only intertemporal demand but also the aforementioned two types of demand are nonlinear functions of the state vector mainly because the inverse matrix of volatilities is a nonlinear function of the state vector. The fact that the optimal portfolio is a nonlinear function of the state vector suggests that achieving the timing effect is not as simple as rebalancing the portfolio weight of a single risky security or a single index based on the business cycle. Conversely, this implies that the timing effect cannot be achieved without dynamically rebalancing the portfolio weights among risky securities in response to various phases created by the variation of the state vector process. As we obtain a semi-analytical formula for optimal portfolio, we can implement the above-mentioned complex portfolio rebalancing to achieve the timing effect as long as we can precisely estimate the parameters and the latent state vector process of our quadratic security market model.

The rest of this paper is organized as follows. In Section 2, we explain the quadratic security market model and real budget constraint. In Section 3, we introduce the investor's consumption–investment problem, derive a semi-analytical solution to this problem, and present the optimal consumption and portfolio choice. In Section 4, we summarize this study and address the remaining issues.

2 Quadratic Security Market Model and Real Budget Constraint

We first introduce the quadratic security market model and present the stochastic differential equations (SDEs) that security's return rate processes satisfy under a no-arbitrage condition. Next, we derive the instantaneous real risk-free rate, real market price of risk, and real budget constraint.

2.1 Quadratic Security Market Model

We consider frictionless US markets over time span $[0, \infty)$. The investors' common subjective probability and information structure are modeled by a complete filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathrm{P})$, where $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,\infty)}$ is the natural filtration generated by an N-dimensional standard Brownian motion B_t. We indicate the expectation operator under P with E, and the conditional expectation operator given $\mathcal{F}_t$ with E_t.

There are markets for consumption commodity and securities at every date $t \in [0, \infty)$, and the consumer price index p_t is observed. The traded securities are the instantaneously nominal risk-free security called the *money market account*, a continuum of zero-coupon bonds whose maturity dates are $(t, t+\tau^*]$, each of which has a one US dollar payoff at maturity, a continuum of zero-coupon inflation-indexed bonds whose maturity dates are $(t, t+\tau^*]$, each of which has a p_T US dollar payoff at maturity T, and J types of non-bond main indices (stock indices, REIT indices, *etc.*).[4]

At every date t, let P_t, P_t^T, P_{It}^T, and S_t^j denote the US dollar prices of the money market account, zero-coupon bond with maturity date T, zero-coupon inflation-indexed bond with maturity date T, and j-th index, respectively. Let A' and I_N denote the transpose of A and the $N\times N$ identity matrix.

We assume the following quadratic latent factor security market model.

Assumption 1. *1. State vector process X_t satisfies the following SDE:*

$$dX_t = -\mathcal{K}X_t\,dt + I_N\,dB_t, \tag{2.1}$$

where $\mathcal{K}$ is an $N \times N$ positive lower triangular constant matrix.

2. The market price Λ_t of risk is an affine function of the state vector, and the instantaneous nominal risk-free rate r_t is a quadratic function of the state vector.

$$\Lambda_t = \lambda + \Lambda X_t, \tag{2.2}$$

$$r_t = \rho_0 + \rho' X_t + \frac{1}{2} X_t' \mathcal{R} X_t, \tag{2.3}$$

where Λ is such that $\mathcal{K} + \Lambda$ is regular, and $\mathcal{R}$ is a positive-definite symmetric matrix.

3. The consumer price index p_t satisfies

$$\frac{dp_t}{p_t} = i_t\,dt + \Lambda_{It}' dB_t, \qquad p_0 = 1, \tag{2.4}$$

where i_t and Λ_{It}' are given by

$$i_t = \iota_0 + \iota' X_t + \frac{1}{2} X_t' \mathcal{I} X_t, \tag{2.5}$$

$$\Lambda_{It} = \lambda_I + \Lambda_I X_t, \tag{2.6}$$

[4] In our models, defaultable bonds can be included into our security market model. In that case, we would model defaultable bond prices based on the quadratic modeling of intensity by Chen, Fillipović, and Poor [20] to retain the consistency with our model. However, we do not consider defaultable bonds to avoid complexity.

where $\mathcal{I}$ is a positive-definite symmetric matrix such that a matrix $\bar{\mathcal{R}}$ defined by

$$\bar{\mathcal{R}} = \mathcal{R} - \mathcal{I} + \Lambda_I'\Lambda + \Lambda'\Lambda_I, \tag{2.7}$$

is positive-definite.

4. *The dividend rate of the j-th index is given by:*

$$D_t^j = \left(\delta_{0j} + \delta_j' X_t + \frac{1}{2} X_t' \Delta_j X_t\right) \exp\left(\sigma_{0j} t + \sigma_j' X_t + \frac{1}{2} X_t' \Sigma_j X_t\right), \tag{2.8}$$

where $(\delta_{0j}, \delta_j, \Delta_j)$ is such that Δ_j is a positive definite symmetric matrix, and[5]

$$\delta_{0j} \geq \frac{1}{2} \delta_j' \Delta_j^{-1} \delta_j, \tag{2.9}$$

Note that $\delta_{0j} + \delta_j' X_t + \frac{1}{2} X_t' \Delta_j X_t$ is the instantaneous rate of dividend.

5. *Markets are complete.*

2.2 No-Arbitrage Rate of Return on Securities

We define $\bar{\Lambda}_t$ and $\bar{r}_t$ by

$$\begin{aligned} \bar{\Lambda}_t &= \Lambda_t - \Lambda_{It}, & (2.10)\\ \bar{r}_t &= r_t - i_t + \Lambda_{It}'\Lambda_t, & (2.11) \end{aligned}$$

where $\bar{\Lambda}_t$ is the real market price of risk.

Note that the real market price of risk is an affine function of X_t and that $\bar{r}_t$ is a quadratic function of X_t:

$$\begin{aligned} \bar{\Lambda}_t &= \bar{\lambda} + \bar{\Lambda} X_t, & (2.12)\\ \bar{r}_t &= \bar{\rho}_0 + \bar{\rho}' X_t + \frac{1}{2} X_t' \bar{\mathcal{R}} X_t, & (2.13) \end{aligned}$$

where $\bar{\mathcal{R}}$ is given by eq.(2.7), and

$$\begin{aligned} \bar{\lambda} &= \lambda - \lambda_I, & (2.14)\\ \bar{\Lambda} &= \Lambda - \Lambda_I, & (2.15)\\ \bar{\rho}_0 &= \rho_0 - \iota_0 + \lambda_I'\lambda, & (2.16)\\ \bar{\rho} &= \rho - \iota + \Lambda'\lambda_I + \Lambda_I'\lambda. & (2.17) \end{aligned}$$

Let $\tau = T - t$ denote the time to maturity of bond P_t^T. First, we show no-arbitrage rate of return on securities.

Lemma 1. *Under Assumption 1, if there is no-arbitrage then security return rate processes satisfy the following:*

[5] Conditions (2.9) ensure that dividend rates are non-negative processes.

1. *The money market account:*

$$\frac{dP_t}{P_t} = r_t\,dt, \qquad P_0 = 1. \tag{2.18}$$

2. *The default-free bond with time τ to maturity:*

$$\frac{dP_t^T}{P_t^T} = \big(r_t + (\sigma(\tau) + \Sigma(\tau)X_t)'\Lambda_t\big)\,dt + (\sigma(\tau)+\Sigma(\tau)X_t)'\,dB_t, \qquad P_T^T = 1, \tag{2.19}$$

where

$$\frac{d\Sigma(\tau)}{d\tau} = \Sigma(\tau)^2 - (\mathcal{K}+\Lambda)'\Sigma(\tau) - \Sigma(\tau)(\mathcal{K}+\Lambda) - \mathcal{R}, \qquad \Sigma(0) = 0, \tag{2.20}$$

$$\frac{d\sigma(\tau)}{d\tau} = -(\mathcal{K} + \Lambda - \Sigma(\tau))'\sigma(\tau) - (\Sigma(\tau)\lambda + \rho), \qquad \sigma(0) = 0. \tag{2.21}$$

3. *The default-free inflation-indexed bond with time τ to maturity:*

$$\begin{aligned}\frac{dP_{It}^T}{P_{It}^T} &= \Big(r_t + \Big(\sigma_I(\tau) + \lambda_I + (\Sigma_I(\tau) + \Lambda_I)X_t\Big)'\Lambda_t\Big)\,dt \\ &\quad + \Big(\sigma_I(\tau) + \lambda_I + (\Sigma_I(\tau) + \Lambda_I)X_t\Big)'\,dB_t, \qquad P_{IT}^T = p_T,\end{aligned} \tag{2.22}$$

where

$$\frac{d\Sigma_I(\tau)}{d\tau} = \Sigma_I(\tau)^2 - (\mathcal{K}+\bar{\Lambda})'\Sigma_I(\tau) - \Sigma_I(\tau)(\mathcal{K}+\bar{\Lambda}) - \bar{\mathcal{R}}, \qquad \Sigma_I(0) = 0, \tag{2.23}$$

$$\frac{d\sigma_I(\tau)}{d\tau} = -(\mathcal{K} + \bar{\Lambda} - \Sigma_I(\tau))'\sigma_I(\tau) - (\Sigma_I(\tau)\bar{\lambda} + \bar{\rho}), \qquad \sigma_I(0) = 0. \tag{2.24}$$

4. *The j-th index:*

$$\frac{dS_t^j + D_t^j dt}{S_t^j} = \big(r_t + (\sigma_j + \Sigma_j X_t)'\Lambda_t\big)\,dt + (\sigma_j + \Sigma_j X_t)'\,dB_t, \tag{2.25}$$

where

$$\Sigma_j^2 - (\mathcal{K} + \Lambda)'\Sigma_j - \Sigma_j(\mathcal{K} + \Lambda) + \Delta_j - \mathcal{R}_j = 0,^6 \tag{2.26}$$

$$\sigma_j = (\mathcal{K} + \Lambda - \Sigma_j)'^{-1}(\delta_j - \rho - \Sigma_j\lambda). \tag{2.27}$$

Proof. See Appendix A.1. □

[6] Kikuchi [24] gives a sufficient condition that the unique solution to this Riccatti algebraic equation is positive-definite.

Remark 1. *It is shown in the real budget constraint in eq.(2.29) in Lemma 2 that $\bar{r}_t$ in eq.(2.11) is the instantaneous real risk-free rate, and eq.(2.11) is a generalized Fisher equation. Thus, $(\bar{\rho}_0, \bar{\rho}, \bar{\mathcal{R}})$ is the real rate version of $(\rho_0, \rho, \mathcal{R})$. From this we can observe that $(\Sigma_I(\tau), \sigma_I(\tau))$ in eqs.(2.23) and (2.24) is the real rate version of $(\Sigma(\tau), \sigma(\tau))$ in eqs.(2.20) and (2.21).*

Remark 2. *Note that eqs.(2.20) and (2.23) have the same structure as eq.(3.26). As the analytical solution to eq. (3.26) is presented in eq.(3.34) in Proposition 2, the analytical solutions to eqs.(2.20) and (2.23) are obtained. In the same way, because eqs.(2.21) and (2.24) have the same structure as eq.(3.25) and because the semi-analytical solution to eq.(3.25) is shown in eq.(3.33), the semi-analytical solutions to eqs.(2.21) and (2.24) are derived. Note that, in this paper, we do not present these analytical solutions so that readers can focus their attention on the derivation of analytical solutions to the optimal consumption and investment problem.*

2.3 Real Budget Constraint

Let Φ_t^j denote the portfolio weight on the j-th index. Regarding the default-free bond, let $\varphi_t(\tau)$ and φ_t^I denote the densities of the portfolio weights on the default-free bond and the default-free inflation-indexed bond with τ-time to maturity. We assume that the functional space of the densities of the portfolio weights on the bonds includes the set of distributions.

Let c_t denote a consumption rate and define Ψ_t as:

$$\Psi_t = \int_0^{\tau^*} \left\{ \varphi_t(\tau)(\sigma(\tau)+\Sigma(\tau)X_t)+\varphi_t^I(\tau)(\sigma_I(\tau)+\Sigma_I(\tau)X_t) \right\} d\tau + \sum_{j=1}^{J} \Phi_t^j(\sigma_j+\Sigma_j X_t) - \Lambda_{It}. \tag{2.28}$$

Let $u_t = (c_t, \Psi_t)$ denote a control.

Let W_t denote the real wealth process. Next, the investor's real budget constraint is expressed in the following lemma.

Lemma 2. *Under Assumption 1 and no-arbitrage condition, given a control u_t, the budget constraint satisfies*

$$\frac{dW_t}{W_t} = \left(\bar{r}_t + \Psi_t' \bar{\Lambda}_t - \frac{c_t}{W_t} \right) dt + \Psi_t' \, dB_t. \tag{2.29}$$

Proof. See Appendix A.2. □

Remark 3. *The return rate process on the real wealth in eq.(2.29) shows that $\bar{r}_t$ is the instantaneous real risk-free rate. Note the real budget constraint in eq.(2.29) is expressed by the instantaneous real risk-free rate, the real market price of risk, investment control, and consumption-wealth ratio. The real budget constraint stands for the instantaneous real rate of return on investment. Eq.(2.29) shows that increasing the risky asset investment in* *the measure of Ψ_t increases the risk of the real rate of return on short-term investments while the real expected excess return increases in proportion to the real market price of risk. That is, the real market price of risk is interpreted as the price per unit of investment risk for all investors.*

3 Consumption–Investment Problem and Semi-Analytical Solution

We first introduce the investor's consumption–investment problem, and derive the PDE for an unknown function constituting the indirect utility function from the HJB equation. Next, we derive the semi-analytical solution to the PDE and present the optimal portfolio choice decomposed into the sum of myopic demand, inflation hedging demand, and intertemporal hedging demand.

3.1 Consumption–Investment Problem

Assumption 2. *The investor maximizes the following CRRA utility over a finite time horizon under budget constraint (2.29).*

$$U(c) = \mathrm{E}\left[\int_0^{T^*} \alpha\, e^{-\beta t} \frac{c_t^{1-\gamma}}{1-\gamma} dt + (1-\alpha)\, e^{-\beta T^*} \frac{W_{T^*}^{1-\gamma}}{1-\gamma}\right], \tag{3.1}$$

where β is the subjective discount rate, and γ is the relative risk aversion coefficient. $(\alpha, 1-\alpha)$ are the weights of the intermediate utility and terminal utility, respectively.

Let $\mathbb{X}_t' = (W_t, X_t')$ and let $W_0 > 0$. We call a control satisfying budget constraint (2.29) with initial state $\mathbb{X}_0 = (W_0, X_0')'$ the admissible control and denote the set of admissible controls by $\mathcal{B}(\mathbb{X}_0)$.

The indirect utility function is defined by

$$J(t, \mathbb{X}_t^u) = \mathrm{E}_t\left[\int_t^{T^*} \alpha\, e^{-\beta s} \frac{c_s^{1-\gamma}}{1-\gamma} ds + (1-\alpha)\, e^{-\beta T} \frac{W_{T^*}^{1-\gamma}}{1-\gamma}\right], \qquad \forall t \in [0, T]. \tag{3.2}$$

The investor's consumption–investment problem and the value function are defined by

$$V(\mathbb{X}_0) = \sup_{u \in \mathcal{B}(\mathbb{X}_0)} J(0, \mathbb{X}_0). \tag{3.3}$$

3.2 PDE for the Indirect Utility Function

The HJB equation is expressed as

$$\sup_{u\in\mathcal{B}(\mathbb{X}_0)}\left\{J_t(t,\mathbb{X}^u)+\begin{pmatrix}W_t\left(\bar{r}_t+\Psi_t'\bar{\Lambda}_t\right)-c_t\\ -\mathcal{K}X_t\end{pmatrix}'\begin{pmatrix}J_W(t,\mathbb{X}^u)\\ J_X(t,\mathbb{X}^u)\end{pmatrix}\right.$$
$$\left.+\frac{1}{2}\,\mathrm{tr}\left[\begin{pmatrix}W_t\Psi_t'\\ I_N\end{pmatrix}\begin{pmatrix}W_t\Psi_t'\\ I_N\end{pmatrix}'\begin{pmatrix}J_{WW}(t,\mathbb{X}^u) & J_{WX}(t,\mathbb{X}^u)\\ J_{XW}(t,\mathbb{X}^u) & J_{XX}(t,\mathbb{X}^u)\end{pmatrix}\right]+\alpha\, e^{-\beta t}\frac{c_t^{1-\gamma}}{1-\gamma}\right\}=0, \tag{3.4}$$

$$\text{s.t.}\quad J(T,\mathbb{X}_T^u)=(1-\alpha)\,e^{-\beta T}\frac{W_T^{1-\gamma}}{1-\gamma}.$$

It is straightforward to see that the optimal control $u_t^*=(c_t^*,\Psi_t^*)$ satisfies the following:

$$c_t^* = \alpha^{\frac{1}{\gamma}}e^{-\frac{\beta}{\gamma}t}J_W^{-\frac{1}{\gamma}}, \tag{3.5}$$

$$\Psi_t^* = \frac{\pi_t}{W_t^{*2}J_{WW}}, \tag{3.6}$$

where W_t^* is the optimal wealth process satisfying the budget constraint (2.29) with the optimal control $u_t^*=(c_t^*,\Psi_t^*)$ and π_t is given by

$$\pi_t=-W_t^*\left\{J_W\bar{\Lambda}_t+J_{XW}\right\}. \tag{3.7}$$

When $N=4$, we consider an example of an investor investing in a 10-year default-free bond $P_t(10)$, a 10-year default-free inflation-indexed bond $P_{It}(10)$, a market-capitalization-weighted stock index S_t^1, and a market-capitalization-weighted REIT index S_t^2 in addition to the money market account. We set forth the following notation.

$$\Phi_t=\begin{pmatrix}\Phi_t(10)\\ \Phi_{It}(10)\\ \Phi_t^1\\ \Phi_t^2\end{pmatrix},\qquad \Sigma_t(X_t)=\begin{pmatrix}(\sigma(10)+\Sigma(10)X_t)'\\ (\sigma_I(10)+\Sigma_I(10)X_t)'\\ (\sigma_1+\Sigma_1X_t)'\\ (\sigma_2+\Sigma_2X_t)'\end{pmatrix}. \tag{3.8}$$

Then, eq.(2.28) leads to $\Psi_t=\Sigma_t(X_t)'\Phi_t-\Lambda_{It}$. Thus, it follows from eq.(3.6) that optimal portfolio weights Φ_t^* on risky securities are given by

$$\Phi_t^*=T_t\Sigma_t(X_t)'^{-1}\bar{\Lambda}_t+T_t\Sigma_t(X_t)'^{-1}\frac{\partial}{\partial X_t}\log J_W+\Sigma_t(X_t)'^{-1}\Lambda_{It}, \tag{3.9}$$

where T_t is a reciprocal of relative risk aversion given by

$$T_t=\left(-\frac{W_t^*J_{WW}}{J_W}\right)^{-1}. \tag{3.10}$$

Remark 4. *The optimal portfolio is decomposed into the sum of three terms, as shown by Brennan and Xia [10] and Sangvinatsos and Wachter [30]. The first term in eq.(3.9) myopically pursues the market price of risk, which is the reward for investing in risky assets, without considering the risk of changes in the indirect utility owing to changes in the state process. It is called myopic demand. The derivative in the second term is the rate of the increase in the marginal indirect utility per unit of the increase in the state process. Considering that the marginal indirect utility is diminishing, and therefore an increase in the marginal indirect utility indicates a decrease in the indirect utility, the second term can be interpreted as representing the demand for insurance against the risk of changes in the indirect utility due to changes in the state process. It is called intertemporal hedging demand.*[7] *The third term insures the inflation risk. It is called inflation hedging demand in this paper.*

The consumption-related terms in HJB equation (3.4) are computed as

$$-c_t^* J_W + \alpha e^{-\beta t} \frac{c_t^{*1-\gamma}}{1-\gamma} = \frac{c_t^*}{1-\gamma}\left\{(\gamma-1)J_W + \alpha e^{-\beta t} c_t^{*-\gamma}\right\} = \frac{\gamma}{1-\gamma} c_t^* J_W. \tag{3.11}$$

The investment-related terms in HJB equation (3.4) are computed as

$$W_t^* J_W \bar{\Lambda}_t' \Psi_t^* + \frac{1}{2}\operatorname{tr}\left[\begin{pmatrix} W_t^*(\Psi_t^*)' \\ I_N \end{pmatrix}\begin{pmatrix} W_t^*(\Psi_t^*)' \\ I_N \end{pmatrix}'\begin{pmatrix} J_{WW} & J_{WX} \\ J_{XW} & J_{XX} \end{pmatrix}\right] = \frac{1}{2}\operatorname{tr}\left[J_{XX}\right] - \frac{\pi_t'\pi_t}{2W_t^{*2}J_{WW}}. \tag{3.12}$$

By substituting optimal control (3.5) and (3.6) into HJB equation (3.4) and by using equations (3.11) and (3.12), the following PDE for J is obtained.

$$J_t + \frac{1}{2}\operatorname{tr}\left[J_{XX}\right] - \frac{\pi_t'\pi_t}{2W_t^{*2}J_{WW}} + W_t^* \bar{r}_t J_W - (\mathcal{K}X_t)' J_X + \frac{\gamma}{1-\gamma} c_t^* J_W = 0. \tag{3.13}$$

From the PDE above, we conjecture that the indirect utility function takes the following form:

$$J(t, \mathbb{X}_t) = e^{-\beta t} \frac{W_t^{1-\gamma}}{1-\gamma} \big(G(t, X_t)\big)^{\gamma}, \tag{3.14}$$

where G is a function of (t, X_t).

By inserting equations (3.5) and (3.6) and the partial derivatives of J into PDE (3.13), we obtain the following proposition:

Proposition 1. *Under Assumptions 1 and 2 and no-arbitrage condition, the indirect utility function, optimal consumption, and optimal investment*

[7] This interpretation is pointed out by Wachter [31].

for problem (3.3) satisfy equations (3.14), (3.15), and (3.17), respectively. Function $G(t, X_t)$ constituting the indirect utility function is a solution to PDE (3.18).

$$c_t^* = \alpha^{\frac{1}{\gamma}} \frac{W_t^*}{G}, \tag{3.15}$$

where

$$W_t^* = W_0 \exp\left(\int_0^t \left(\bar{r}_s + (\Psi_s^*)' \bar{\Lambda}_s - \frac{\alpha^{\frac{1}{\gamma}}}{G(X_s)} - \frac{1}{2} (\Psi_s^*)' \Psi_s^* \right) ds + (\Psi_s^*)' \, dB_s \right), \tag{3.16}$$

$$\Psi_t^* = \frac{1}{\gamma} \bar{\Lambda}_t + \frac{G_X}{G}, \tag{3.17}$$

$$G_t + \mathcal{L}G + \alpha^{\frac{1}{\gamma}} = 0, \qquad G(T, X_T) = (1-\alpha)^{\frac{1}{\gamma}}, \tag{3.18}$$

where $\mathcal{L}$ is a linear differential operator defined by

$$\begin{aligned} \mathcal{L}G = {} & \frac{1}{2} \operatorname{tr}[G_{XX}] + \left(-\mathcal{K}X - \frac{\gamma - 1}{\gamma} (\bar{\lambda} + \bar{\Lambda} X) \right)' G_X \\ & - \left\{ \frac{\gamma - 1}{2\gamma^2} (\bar{\lambda} + \bar{\Lambda} X)' (\bar{\lambda} + \bar{\Lambda} X) + \frac{\gamma - 1}{\gamma} \left(\bar{\rho}_0 + \bar{\rho}' X + \frac{1}{2} X' \bar{\mathcal{R}} X \right) + \frac{\beta}{\gamma} \right\} G, \end{aligned} \tag{3.19}$$

where $\bar{\rho}_0, \bar{\rho}, \mathcal{R}, \bar{\lambda}, \bar{\Lambda}$ are given by eqs.(2.16)–(2.15).

Proof. See Appendix A.3. □

3.3 Semi-Analytical Solution

A nonhomogeneous term $\alpha^{\frac{1}{\gamma}}$ appears in PDE (3.18), making it difficult to derive an analytical solution. Batbold *et al.* [3] develop a method by Liu [27] further, and presents a method to derive a semi-analytical solution by exploiting an analytical solution to a homogeneous PDE that abandons the nonhomogeneous term. Following their method, we examine homogeneous PDE (3.20).

$$\frac{\partial}{\partial \tau} g(\tau, X) = \mathcal{L} g(\tau, X), \qquad g(0, X) = 1. \tag{3.20}$$

An analytical solution to PDE (3.20) is expressed as:

$$g(\tau, X) = \exp\left(a_0(\tau) + a(\tau)' X + \frac{1}{2} X' A(\tau) X \right), \tag{3.21}$$

where $A(\tau)$ is a symmetric matrix.

Then, it follows from the linearlity of $\mathcal{L}$ that, under the interchange of the differentiation and integration operators, the semi-analytical solution to PDE (3.18) is expressed as

$$G(t,X) = \alpha^{\frac{1}{\gamma}} \int_0^{T-t} g(s,X)\,ds + (1-\alpha)^{\frac{1}{\gamma}} g(T-t,X), \tag{3.22}$$

By substituting g and its derivatives into PDE (3.20) and by paying attention to $A' = A$ and

$$X'\left(\mathcal{K} + \frac{\gamma-1}{\gamma}\bar{\Lambda}\right)' AX = X'A\left(\mathcal{K} + \frac{\gamma-1}{\gamma}\bar{\Lambda}\right)X,$$

we obtain

$$\begin{aligned}
\frac{d}{d\tau}a_0(\tau) + X'\frac{d}{d\tau}a(\tau) + \frac{1}{2}X'\frac{d}{d\tau}A(\tau)X &= \frac{1}{2}\operatorname{tr}\big[aa' + A + aX'A + AXa' + AXX'A\big] \\
&+ \left\{-\frac{\gamma-1}{\gamma}\bar{\lambda} - \left(\mathcal{K} + \frac{\gamma-1}{\gamma}\bar{\Lambda}\right)X\right\}' a - \frac{\gamma-1}{\gamma}X'A\bar{\lambda} \\
&- \frac{1}{2}X'\left(\mathcal{K} + \frac{\gamma-1}{\gamma}\bar{\Lambda}\right)' AX - \frac{1}{2}X'A\left(\mathcal{K} + \frac{\gamma-1}{\gamma}\bar{\Lambda}\right)X \\
-\left\{\frac{\gamma-1}{2\gamma^2}(\bar{\lambda}'\bar{\lambda} + 2\bar{\lambda}'\bar{\Lambda}X + X'\bar{\Lambda}'\bar{\Lambda}X) + \frac{\gamma-1}{\gamma}\left(\bar{\rho}_0 + \bar{\rho}'X + \frac{1}{2}X'\bar{\mathcal{R}}X\right) + \frac{\beta}{\gamma}\right\}&.
\end{aligned} \tag{3.23}$$

As the equation above is identical on X, the following system of ODEs for (a_0, a, A) is derived.

$$\begin{aligned}
\frac{d}{d\tau}a_0(\tau) &= \frac{1}{2}a(\tau)'a(\tau) + \frac{1}{2}\operatorname{tr}[A(\tau)] - \frac{\gamma-1}{\gamma}\lambda' a(\tau) - \left(\frac{\gamma-1}{2\gamma^2}\bar{\lambda}'\bar{\lambda} + \frac{\gamma-1}{\gamma}\bar{\rho}_0 + \frac{\beta}{\gamma}\right), \\
a_0(0) &= 0.
\end{aligned} \tag{3.24}$$

$$\begin{aligned}
\frac{d}{d\tau}a(\tau) &= \left\{A(\tau) - \left(\mathcal{K} + \frac{\gamma-1}{\gamma}\bar{\Lambda}\right)'\right\}a(\tau) - \frac{\gamma-1}{\gamma}A(\tau)\bar{\lambda} - \left(\frac{\gamma-1}{\gamma^2}\bar{\Lambda}'\bar{\lambda} + \frac{\gamma-1}{\gamma}\bar{\rho}\right), \\
a(0) &= 0,
\end{aligned} \tag{3.25}$$

$$\begin{aligned}
\frac{d}{d\tau}A(\tau) &= A(\tau)^2 - \left(\mathcal{K} + \frac{\gamma-1}{\gamma}\bar{\Lambda}\right)' A(\tau) - A(\tau)\left(\mathcal{K} + \frac{\gamma-1}{\gamma}\bar{\Lambda}\right) - \left(\frac{\gamma-1}{\gamma^2}\bar{\Lambda}'\bar{\Lambda} + \frac{\gamma-1}{\gamma}\bar{\mathcal{R}}\right), \\
A(0) &= 0.
\end{aligned} \tag{3.26}$$

We should also note that

$$G_X(t,X) = G(t,X)\big(\bar{a}_t(X) + \bar{A}_t(X)X_t\big), \tag{3.27}$$

where $\bar{a}_t(X)$ and $\bar{A}_t(X)$ are the weighted averages of $a(\tau)$ and $A(\tau)$ weighted by $g(\tau, X_t)$, respectively, and defined as follows:

$$\bar{a}_t(X_t) = \frac{\int_0^{T^*-t} \alpha^{\frac{1}{\gamma}} g(s, X_t) a(s)\, ds + (1-\alpha)^{\frac{1}{\gamma}} g(T^*-t, X_t) a(T^*-t)}{\alpha^{\frac{1}{\gamma}} \int_0^{T^*-t} g(s, X_t)\, ds + (1-\alpha)^{\frac{1}{\gamma}} g(T^*-t, X_t)} \tag{3.28}$$

$$\bar{A}_t(X_t) = \frac{\int_0^{T^*-t} \alpha^{\frac{1}{\gamma}} g(s, X_t) A(s)\, ds + (1-\alpha)^{\frac{1}{\gamma}} g(T^*-t, X_t) A(T^*-t)}{\alpha^{\frac{1}{\gamma}} \int_0^{T^*-t} g(s, X_t)\, ds + (1-\alpha)^{\frac{1}{\gamma}} g(T^*-t, X_t)} \tag{3.29}$$

Then, we have Proposition 2.

Proposition 2. *Under Assumptions 1 and 2 and no-arbitrage condition, the optimal consumption and the optimal investment for problem (3.3) satisfy equations (3.30) and (3.31), respectively.*

$$c_t^* = \frac{\alpha^{\frac{1}{\gamma}} W_t^*}{\alpha^{\frac{1}{\gamma}} \int_0^{T^*-t} g(s, X_s)\, ds + (1-\alpha)^{\frac{1}{\gamma}} g(T^*-t, X_{T^*-t})}, \tag{3.30}$$

where W_t^* is given by equation (3.16) and

$$\Psi_t^* = \frac{1}{\gamma}(\bar{\lambda} + \bar{\Lambda} X_t) + \Big(\bar{a}_t(X_t) + \bar{A}_t(X_t) X_t\Big), \tag{3.31}$$

where $(\bar{a}, \bar{A})$ are given by equations (3.28) and (3.29), respectively, and $(\bar{a}, a, A)$ is given by equations (3.32)–(3.34).

$$a_0(\tau) = \int_0^\tau \left\{ \frac{1}{2} a(s)' a(s) + \frac{1}{2}\operatorname{tr}[A(s)] - \frac{\gamma-1}{\gamma} \bar{\lambda}' a(s) - \left\{ \frac{\gamma-1}{\gamma} \left(\frac{1}{2\gamma} \bar{\lambda}' \bar{\lambda} + \bar{\rho}_0 \right) + \frac{\beta}{\gamma} \right\} \right\} ds, \tag{3.32}$$

$$a(\tau) = \int_0^\tau e^{\int_s^\tau \left\{ A(t) - \left(\mathcal{K} + \frac{\gamma-1}{\gamma} \bar{\Lambda} \right) \right\} dt} \left\{ -\frac{\gamma-1}{\gamma} \left(A(s) \bar{\lambda} + \left(\frac{1}{\gamma} \bar{\Lambda}' \bar{\lambda} + \bar{\rho} \right) \right) \right\} ds, \tag{3.33}$$

$$A(\tau) = C_2(\tau) C_1^{-1}(\tau), \tag{3.34}$$

where

$$\begin{pmatrix} C_1(\tau) \\ C_2(\tau) \end{pmatrix} = \exp\left(\tau \begin{pmatrix} \mathcal{K} + \frac{\gamma-1}{\gamma} \bar{\Lambda} & -I_N \\ -\frac{\gamma-1}{\gamma} \left(\frac{1}{\gamma} \bar{\Lambda}' \bar{\Lambda} + \bar{\mathcal{R}} \right) & -\left(\mathcal{K} + \frac{\gamma-1}{\gamma} \bar{\Lambda} \right)' \end{pmatrix} \right) \begin{pmatrix} I_N \\ 0_N \end{pmatrix}, \tag{3.35}$$

where 0_N is an $N \times N$ zero matrix, and $\bar{\rho}_0, \bar{\rho}, \mathcal{R}, \bar{\lambda}, \bar{\Lambda}$ are given by eqs.(2.16)–(2.15).

Proof. See Appendix A.4. □

3.4 Optimal Portfolio with Inflation Risk and Stochastic Volatility

Finally, we show the optimal portfolio for the same example in Section 3.2. As $\varPsi_t = \Sigma_t(X_t)'\varPhi_t - \Lambda_{It}$, it follows from eq.(3.31) that the optimal portfolio choice $\varPhi_t^*$ is given by

$$\varPhi_t^* = \frac{1}{\gamma}\Sigma_t(X_t)'^{-1}\Big(\bar{\lambda}+\bar{\Lambda}X_t\Big)+\Sigma_t(X_t)'^{-1}\Big(\bar{a}_t(X_t)+\bar{A}_t(X_t)X_t\Big)+\Sigma_t(X_t)'^{-1}\Big(\lambda_I+\Lambda_I X_t\Big). \tag{3.36}$$

The optimal portfolio choice for the money market account is $1-(\varPhi_t(10)+\varPhi_{It}(10)+\varPhi_t^1+\varPhi_t^2)$.

The optimal portfolio choice is decomposed into the sum of myopic demand, intertemporal hedging demand, and inflation hedging demand, as shown by Brennan and Xia [10] and Sangvinatsos and Wachter [30]. In their optimal portfolio, intertemporal hedging demand is a nonlinear function of the state vector, but both myopic demand and inflation hedging demand are affine functions because the volatilities of securities are constant in their models. In our optimal portfolio in eq.(3.36), not only intertemporal demand but also the above-mentioned two types of demand are nonlinear functions of the state vector because the inverse matrix of volatilities is a nonlinear function of the state vector, and all three terms in parentheses are also functions of the state vector. The fact that the optimal portfolio is a nonlinear function of the state vector suggests that achieving the timing effect is not as simple as rebalancing the portfolio weight of a single risky security or a single index based on the business cycle. On the contrary, this implies that the timing effect cannot be achieved without dynamically rebalancing the portfolio weights among risky securities in response to various phases created by the variation of the state vector process. As we obtain a semi-analytical formula for optimal portfolio, we can implement the above-mentioned complex portfolio rebalancing to achieve the timing effect as long as we can precisely estimate the parameters and the latent state vector process of our quadratic security market model.

Remark 5. *As already expressed in eqs.(3.28) and (3.29), $\bar{a}_t(X)$ and $\bar{A}_t(X)$ are the weighted averages of $a(\tau)$ and $A(\tau)$ weighted by $g(\tau, X_t)$, respectively. Note that g depends on (a_0, a, A) and the state vector X_t and (a_0, a, A) in eqs.(3.32)–(3.35) depend on the parameters related to the market price of risk, interest rate risk, inflation risk, and relative risk aversion of the investor. Moreover, as $\Sigma_t(X_t)$ depends not only on X_t but also on $\sigma, \Sigma, \sigma_I, \Sigma_I, \sigma_j, \Sigma_j$, the set of these parameters depends on all the parameters of the model. This implies that intertemporal hedging demand is the most complicated among the three types of demand.*

Remark 6. *Inflation risk is characterized by the set $(\iota, \mathcal{I}, \lambda_I, \Lambda_I)$ of the parameters. Note that the elements (σ_I, Σ_I) of the volatility matrix $\Sigma_t(X_t)$*

depend on the set $(\iota, \mathcal{I}, \lambda_I, \Lambda_I)$. The set (λ_I, Λ_I) of the parameters is included in the real market price $\bar{\Lambda}_t$ of risk in myopic demand, and the set $(\iota, \mathcal{I}, \lambda_I, \Lambda_I)$ of the parameters is included in intertemporal hedging demand. Thus, the inflation risk is managed not only by inflation hedging demand, but also by myopic demand and intertemporal hedging demand.

4 Conclusion

This study considers a consumption–investment problem for a long-term investor with CRRA utility, under a quadratic security market model, in which all inflation rates, interest rates, market price of risk, and return volatilities of assets are stochastic and predictable. We apply the method by Liu [27] and Batbold *et al.* [3] to the nonhomogeneous PDE for the investor's indirect utility function, and derive a semi-analytical solution. We obtain the optimal portfolio decomposed into the sum of myopic demand, intertemporal hedging demand, and inflation hedging demand and present that all of the three types of demand are nonlinear functions of the state vector. This suggests that the timing aspect is more important than we assumed.

As we obtain a semi-analytical formula for optimal portfolio, we can implement the above-mentioned complex portfolio rebalancing to achieve the timing effect as long as we can precisely estimate the parameters and the latent state vector process of our quadratic security market model. If the security market model is affine rather than quadratic, then the affine model can be interpreted as a linear state-space model; therefore, the parameters and the state vector process of the model can be estimated with high accuracy by the maximum likelihood method based on the Kalman filtering. However, in the case of our quadratic model, the state-space model is nonlinear, so some type of pseudo maximum method based on nonlinear filtering is necessary. The estimation of our research remains the future plan.

A Proofs

A.1 Proof of Lemma 1

As there is no arbitrage, there exists a unique risk-neutral measure P^*. It follows from Girsanov's theorem that process B_t^* defined by

$$B_t^* = B_t + \int_0^t \Lambda_s \, ds, \tag{A.1}$$

is a standard Brownian motion under P^*. Then, the SDE for X_t under P^* is rewritten as

$$\begin{aligned} dX_t &= -(\mathcal{K}X_t + \Lambda_t)\,dt + I_N\,dB_t^* \\ &= -\big(\lambda + (\mathcal{K}+\Lambda)X_t\big)\,dt + I_N\,dB_t^*. \end{aligned}$$

First, we consider the case of default-free bond P_t^T. As $P_T^T = 1$, P_t^T is written as

$$P_t^T = \mathrm{E}_t^*\left[\exp\left(-\int_t^T r_s\,ds\right)\right], \tag{A.2}$$

where E^* is the expectation under P^*.

Since r_t is a function of X_t, P_t^T is expressed as a smooth function f of (X_t, t).

$$P_t^T = f(X_t, t), \tag{A.3}$$

and it follows from Feynman–Kac's formula that f is a solution to the following PDE:

$$f_t + \big(-\lambda - (\mathcal{K}+\Lambda)X_t\big)' f_X + \frac{1}{2}\mathrm{tr}[f_{XX}] - \left(\rho_0 + \rho' X_t + \frac{1}{2}X_t'\mathcal{R}X_t\right) f = 0, \qquad f(X_T, T) = 1. \tag{A.4}$$

As the PDE above is a second-order linear equation, the solution f is expressed as

$$f(X_t, t) = \exp\left(\bar{\sigma}(\tau) + \sigma(\tau)' X_t + \frac{1}{2}X_t'\Sigma(\tau)X_t\right), \qquad (\bar{\sigma}(0), \sigma(0), \Sigma(\tau)) = (0,0,0), \tag{A.5}$$

where $\bar{\sigma}(\tau), \sigma(\tau)$, and $\Sigma(\tau)$ are smooth functions of $\tau = T - t$ and $\Sigma(\tau)$ is a symmetric matrix. It should be noted that $\Sigma(\tau)' = \Sigma(\tau)$ and

$$X_t'(\mathcal{K}+\Lambda)'\Sigma(\tau)X_t = X_t'\Sigma(\tau)(\mathcal{K}+\Lambda)X_t.$$

By differentiating eq.(A.5) and by inserting the result into eq.(A.4), we have

$$\begin{aligned} &-\frac{d\bar{\sigma}(\tau)}{d\tau} - X_t'\frac{d\sigma(\tau)}{d\tau} - \frac{1}{2}X_t'\frac{d\Sigma(\tau)}{d\tau}X_t - \lambda'(\sigma(\tau) + \Sigma(\tau)X_t) - X_t'(\mathcal{K}+\Lambda)'\sigma(\tau) \\ &- \frac{1}{2}X_t'(\mathcal{K}+\Lambda)'\Sigma(\tau)X_t - \frac{1}{2}X_t'\Sigma(\tau)(\mathcal{K}+\Lambda)X_t + \frac{1}{2}\big(\sigma(\tau)'\sigma(\tau) + \mathrm{tr}[\Sigma(\tau)]\big) \\ &\quad + X_t'\Sigma(\tau)\sigma(\tau) + \frac{1}{2}X_t'\Sigma(\tau)^2 X_t - \left(\rho_0 + \rho' X_t + \frac{1}{2}X_t'\mathcal{R}X_t\right) = 0. \end{aligned} \tag{A.6}$$

Since the equation above is identical on X_t, eqs.(2.20) and (2.21) are obtained. By differentiating eq.(A.5), we obtain SDE (2.19).

In the case of default-free inflation-indexed bond P_{It}^T, we define an equivalent probability measure $\bar{\mathrm{P}}$ by the following Radon-Nikodym derivative with respect to P^*:

$$\frac{d\bar{\mathrm{P}}}{d\mathrm{P}^*} = \exp\left(-\frac{1}{2}\int_0^{T^*}(-\Lambda_{Is})'(-\Lambda_{Is})\,ds - \int_0^{T^*}(-\Lambda_{Is}')\,dB_s^*\right).$$

Then, it follows by Girsanov's theorem that a process $\bar{B}_t$ defined by

$$\bar{B}_t = B_t^* - \int_0^t \Lambda_{Is}\, ds, \tag{A.7}$$

is a standard Brownian motion under $\bar{\mathrm{P}}$, and the SDE for X_t under $\bar{\mathrm{P}}$ is rewritten as

$$\begin{aligned} dX_t &= -(\mathcal{K}X_t + \Lambda_t - \Lambda_{It})\, dt + I_N\, d\bar{B}_t \\ &= -\big(\bar{\lambda} + (\mathcal{K} + \bar{\Lambda})X_t\big)\, dt + I_N\, d\bar{B}_t. \end{aligned}$$

Thus, P_{It}^T is calculated as

$$\begin{aligned} P_{It}^T &= E_t^* \left[\exp\left(-\int_t^T r_s\, ds \right) p_T \right] \\ &= p_t \mathrm{E}_t^* \left[\exp\left(-\int_t^T \left(r_s - i_s + \frac{1}{2}\Lambda_{Is}'\Lambda_{Is} \right) ds + \int_t^T \Lambda_{Is}' dB_s \right) \right] \\ &= p_t \mathrm{E}_t^* \left[\exp\left(-\int_t^T \left(r_s - i_s + \frac{1}{2}\Lambda_{Is}'\Lambda_{Is} + \Lambda_{Is}'\Lambda_s \right) ds + \int_t^T \Lambda_{Is}' dB_s^* \right) \right] \\ &= p_t \mathrm{E}_t^* \left[\exp\left(-\int_t^T \left(r_s - i_s + \Lambda_{Is}'\Lambda_s \right) ds \right) \left(\frac{d\bar{\mathrm{P}}}{d\mathrm{P}^*} \right)_t \right] = p_t \bar{\mathrm{E}}_t \left[\exp\left(-\int_t^T \bar{r}_s\, ds \right) \right], \end{aligned} \tag{A.8}$$

where $\bar{\mathrm{E}}$ is the expectation under $\bar{\mathrm{P}}$. Since all of the processes r_t, i_t, Λ_{It}', and Λ_t are functions of X_t, the real price of P_{It}^T is expressed as a smooth function $f(X_t, t)$

$$\frac{P_{It}^T}{p_t} = f(X_t, t), \tag{A.9}$$

and it follows from Feynman–Kac's formula that f is a solution to the following PDE:

$$f_t - \big(\bar{\lambda} + (\mathcal{K} + \bar{\Lambda})X_t\big)' f_X + \frac{1}{2}\mathrm{tr}[f_{XX}] - \left(\bar{\rho}_0 + \bar{\rho}' X_t + \frac{1}{2} X_t' \bar{\mathcal{R}} X_t \right) f = 0, \qquad f(X_T, T) = 1. \tag{A.10}$$

Hence, f is expressed as

$$f(X_t, t) = \exp\left(\bar{\sigma}_I(\tau) + \sigma_I(\tau)' X_t + \frac{1}{2} X_t' \Sigma_I(\tau) X_t \right), \qquad (\bar{\sigma}_I(0), \sigma_I(0), \Sigma_I(\tau)) = (0, 0, 0), \tag{A.11}$$

where $\bar{\sigma}_I(\tau), \sigma_I(\tau)$ and $\Sigma_I(\tau)$ are smooth functions of τ and $\Sigma_I(\tau)$ is a symmetric matrix. It should be noted that $\Sigma_I(\tau)' = \Sigma_I(\tau)$ and

$$X_t'(\mathcal{K} + \bar{\Lambda})' \Sigma_I(\tau) X_t = X_t' \Sigma_I(\tau)(\mathcal{K} + \bar{\Lambda}) X_t.$$

By differentiating eq.(A.11) and by inserting the result into eq.(A.10), the following equation is obtained:

$$
\begin{aligned}
&-\frac{d\bar{\sigma}_I(\tau)}{d\tau}-X_t'\frac{d\sigma_I(\tau)}{d\tau}-\frac{1}{2}X_t'\frac{d\Sigma_I(\tau)}{d\tau}X_t-\bar{\lambda}'(\sigma_I(\tau)+\Sigma_I(\tau)X_t)-X_t'(\mathcal{K}+\bar{\Lambda})'\sigma_I(\tau)\\
&-\frac{1}{2}X_t'(\mathcal{K}+\bar{\Lambda})'\Sigma_I(\tau)X_t-\frac{1}{2}X_t'\Sigma_I(\tau)(\mathcal{K}+\bar{\Lambda})X_t+\frac{1}{2}\big(\sigma_I(\tau)'\sigma_I(\tau)+\mathrm{tr}[\Sigma_I(\tau)]\big)\\
&+X_t'\Sigma_I(\tau)\sigma_I(\tau)+\frac{1}{2}X_t'\Sigma_I(\tau)^2X_t-\left(\bar{\rho}_0+\bar{\rho}'X_t+\frac{1}{2}X_t'\bar{\mathcal{R}}X_t\right)=0. \quad \text{(A.12)}
\end{aligned}
$$

Note the following equation:

$$
\frac{dP_{It}^T}{P_{It}^T}=\frac{dp_t}{p_t}+\frac{df(X_t,t)}{f(X_t,t)}+\frac{dp_t}{p_t}\frac{df(X_t,t)}{f(X_t,t)}. \tag{A.13}
$$

Therefore, we obtain eqs.(2.22)–(2.24).

On the j-th index, Kikuchi [24] proves that S_t^j is given by

$$
S_t^j=\exp\left(\sigma_{0j}t+\sigma_j'X_t+\frac{1}{2}X_t'\Sigma_jX_t\right). \tag{A.14}
$$

Hence, the instantaneous dividend rate process is

$$
\frac{D_t^j}{S_t^j}=\delta_{0j}+\delta_j'X_t+\frac{1}{2}X_t'\Delta_jX_t. \tag{A.15}
$$

In a similar way, the following identical equation on X_t is obtained from eqs.(A.14) and (A.15).

$$
\begin{aligned}
&\sigma_{0j}-\lambda'(\sigma_j+\Sigma_jX_t)-X_t'(\mathcal{K}+\Lambda)'\sigma_j-\frac{1}{2}X_t'(\mathcal{K}+\Lambda)'\Sigma_jX_t-\frac{1}{2}X_t'\Sigma_j(\mathcal{K}+\Lambda)X_t\\
&+\frac{1}{2}\big(\sigma_j'\sigma_j+\mathrm{tr}[\Sigma_j]\big)+X_t'\Sigma_j\sigma_j+\frac{1}{2}X_t'\Sigma_j^2X_t+\left(\delta_{0j}-\rho_0+(\delta_j-\rho)'X_t+\frac{1}{2}X_t'(\Delta_j-\mathcal{R})X_t\right)=0. \quad \text{(A.16)}
\end{aligned}
$$

Therefore, we obtain eqs.(2.25)–(2.27).

A.2 Proof of Lemma 2

Let $(\vartheta,(\vartheta(\tau)),(\vartheta^I(\tau)),(\vartheta^j))$ denote a portfolio. The nominal wealth p_tW_t is given by

$$
p_tW_t=\vartheta_tP_t+\int_0^{\tau^*}\big(\vartheta_t(\tau)P_t(\tau)+\vartheta_t^I(\tau)P_{It}(\tau)\big)d\tau+\sum_{j=1}^J\vartheta_t^jS_t^j.
$$

Then, given c_t, the self-financing portfolio $(\vartheta, (\vartheta(\tau)), (\vartheta^I(\tau)), (\vartheta^j))$ satisfies

$$\begin{aligned}
\frac{d(p_tW_t)}{p_tW_t} &= \frac{1}{p_tW_t}\left\{\vartheta_t dP_t + \int_0^{\tau^*}\Big(\vartheta_t(\tau)dP_t(\tau) + \vartheta_t^I(\tau)dP_{It}(\tau)\Big)d\tau + \sum_{j=1}^J \vartheta_t^j\Big(dS_t^j + D_t^j dt\Big) - p_tc_tdt\right\}\\
&= \frac{\vartheta_tP_t}{p_tW_t}\frac{dP_t}{P_t} + \int_0^{\tau^*}\left(\frac{\vartheta_t(\tau)P_t(\tau)}{p_tW_t}\frac{dP_t(\tau)}{P_t(\tau)} + \frac{\vartheta_t^I(\tau)P_{It}(\tau)}{p_tW_t}\frac{dP_{It}(\tau)}{P_{It}(\tau)}\right)d\tau\\
&\quad + \sum_{j=1}^J \frac{\vartheta_t^jS_t^j}{p_tW_t}\frac{dS_t^j + D_t^jdt}{S_t^j} - \frac{c_t}{W_t}dt\\
&= \left(1 - \int_0^{\tau^*}\big(\varphi_t(\tau) + \varphi_t^I(\tau)\big)d\tau - \sum_{j=1}^J \varPhi_t^j\right)\frac{dP_t}{P_t} + \int_0^{\tau^*}\left(\varphi_t(\tau)\frac{dP_t(\tau)}{P_t(\tau)} + \varphi_t^I(\tau)\frac{dP_{It}(\tau)}{P_{It}(\tau)}\right)d\tau\\
&\quad + \sum_{j=1}^J \varPhi_t^j\frac{dS_t^j + D_t^jdt}{S_t^j} - \frac{c_t}{W_t}dt.
\end{aligned}$$

Substituting equations (2.18), (2.19), (2.22), and (2.25) into the equation above and using the investment control $\varPsi_t$ yield

$$\frac{d(p_tW_t)}{p_tW_t} = \left(r_t + (\varPsi_t + \Lambda_{It})'\Lambda_t - \frac{c_t}{W_t}\right)dt + (\varPsi_t + \Lambda_{It})'\,dB_t. \tag{A.17}$$

Noting that

$$\frac{d(p_tW_t)}{p_tW_t} = \frac{dW_t}{W_t} + \frac{dp_t}{p_t} + \frac{dW_t}{W_t}\frac{dp_t}{p_t},$$

and that the volatility of W_t is equal to $\varPsi_t$, we get

$$\frac{dW_t}{W_t} = \frac{d(p_tW_t)}{p_tW_t} - i_tdt - \Lambda_{It}'dB_t - \varPsi_t'\Lambda_{It}\,dt.$$

By inserting eq.(A.17) into the equation above, we obtain eq.(2.29).

A.3 Proof of Proposition 1

First, optimal consumption control (3.15) is obtained as follows:

$$c_t^* = \alpha^{\frac{1}{\gamma}}e^{-\frac{\beta}{\gamma}t}J_W^{-\frac{1}{\gamma}} = \alpha^{\frac{1}{\gamma}}e^{-\frac{\beta}{\gamma}t}\left\{e^{-\beta t}(W_t^*)^{-\gamma}G^{\gamma}\right\}^{-\frac{1}{\gamma}} = \alpha^{\frac{1}{\gamma}}\frac{W_t^*}{G}.$$

Then, by inserting c_t^* into budget constraint (2.29) and by solving the SDE, we obtain equation (3.16).

Second, the derivatives of J are given by

$$J_t = -\beta J + \gamma J\frac{G_t}{G}, \qquad WJ_W = (1-\gamma)J, \qquad J_X = \gamma\,J\frac{G_X}{G},$$

$$W^2J_{WW} = -\gamma(1-\gamma)J, \qquad WJ_{XW} = \gamma(1-\gamma)J\frac{G_X}{G},$$

$$J_{XX} = \gamma\,J\left\{(\gamma-1)\frac{G_X}{G}\frac{G_X'}{G} + \frac{G_{XX}}{G}\right\}.$$

Then, the numerator and the denominator on the right-hand side of equation (3.6) are rewritten as:

$$\pi_t = (\gamma - 1)J\left(\bar{\Lambda}_t + \gamma\frac{G_X}{G}\right), \tag{A.18}$$

$$W_t^2 J_{WW} = \gamma(\gamma - 1)J. \tag{A.19}$$

Therefore, by inserting equations (A.18) and (A.19) into equation (3.6), we obtain equation (3.17). The second and third terms in PDE (3.13) are calculated from equations (A.18) and (A.19) as

$$\begin{aligned}
&\frac{1}{2}\operatorname{tr}[J_{XX}] - \frac{\pi_t'\pi_t}{2W_t^2 J_{WW}} \\
&= \frac{\gamma}{2}J\operatorname{tr}\left[\left\{(\gamma-1)\frac{G_X}{G}\frac{G_X'}{G} + \frac{G_{XX}}{G}\right\}\right] - \frac{\gamma-1}{2\gamma}J\left(\bar{\Lambda}_t + \gamma\frac{G_X}{G}\right)'\left(\bar{\Lambda}_t + \gamma\frac{G_X}{G}\right) \\
&= J\left\{\frac{\gamma}{2}\operatorname{tr}\left[\frac{G_{XX}}{G}\right] - \frac{\gamma-1}{2\gamma}\bar{\Lambda}_t'\bar{\Lambda}_t - (\gamma-1)\bar{\Lambda}_t'\frac{G_X}{G}\right\}.
\end{aligned} \tag{A.20}$$

The sixth term in PDE (3.13) is calculated from equation (3.5) as

$$\frac{\gamma}{1-\gamma}c_t^* J_W = \frac{\gamma}{1-\gamma}\alpha^{\frac{1}{\gamma}}\frac{W_t^*}{G}(1-\gamma)\frac{J}{G} = \gamma\alpha^{\frac{1}{\gamma}}\frac{J}{G}. \tag{A.21}$$

Substituting equations (A.20) and (A.21) into equation (3.13), and dividing by $\gamma J/G$ yield equation (3.18).

A.4 Proof of Proposition 2

By substituting eqs.(3.22) and (3.27) into eqs.(3.15) and (3.17), respectively, we obtain eqs.(3.30) and (3.31). It is straightforward to see that $a_0(\tau)$ and $a(\tau)$ are expressed as eqs.(3.32) and (3.33), respectively.

Following Theorem 5.2 in Arimoto [2], we prove that $A(\tau)$ is a unique symmetric solution to matrix differential Riccati equation (3.34). We consider the following initial value problem of the linear ODE for the $N \times N$ matrix-value functions $C_1(\tau)$ and $C_2(\tau)$:

$$\frac{d}{d\tau}\begin{pmatrix} C_1(\tau) \\ C_2(\tau) \end{pmatrix} = \begin{pmatrix} L & -I_N \\ -Q & -L' \end{pmatrix}\begin{pmatrix} C_1(\tau) \\ C_2(\tau) \end{pmatrix}, \qquad \begin{pmatrix} C_1(0) \\ C_2(0) \end{pmatrix} = \begin{pmatrix} I_N \\ 0_N \end{pmatrix}, \tag{A.22}$$

where

$$L = \mathcal{K} + \frac{\gamma-1}{\gamma}\bar{\Lambda}, \qquad Q = \frac{\gamma-1}{\gamma}\left(\frac{1}{\gamma}\bar{\Lambda}'\bar{\Lambda} + \bar{\mathcal{R}}\right).$$

A solution to equation (A.22) is given by equation (3.35). As we can prove that $C_1(\tau)$ is regular,[8] we define $A(\tau)$ by equation (3.34). Subsequently, considering that

$$\frac{d}{d\tau}C_1^{-1}(\tau) = -C_1^{-1}(\tau)\left\{\frac{d}{d\tau}C_1(\tau)\right\}C_1^{-1}(\tau), \tag{A.23}$$

[8]See the proof for Theorem 5.2 in Arimoto [2].

we can derive

$$\begin{aligned}\frac{d}{d\tau}A(\tau) &= \left\{\frac{d}{d\tau}C_2(\tau)\right\}C_1^{-1}(\tau) + C_2(\tau)\frac{d}{d\tau}C_1^{-1}(\tau)\\ &= \left(-QC_1(\tau) - L'C_2(\tau)\right)C_1^{-1}(\tau) - A(\tau)\left(LC_1(\tau) - C_2(\tau)\right)C_1^{-1}(\tau)\\ &= A(\tau)^2 - L'A(\tau) - A(\tau)L - Q,\end{aligned}$$

and thus confirm that $A(\tau)$ satisfies matrix differential Riccati equation (3.26). For the uniqueness of the Riccati equation, see the proof of Theorem 5.2 in Arimoto [2]. Finally, for the symmetry of $A(\tau)$, taking the transposition of Riccati equation (3.26) for $A(\tau)$ yields the same equation for $A(\tau)'$, which implies that $A(\tau)' = A(\tau)$ because of the uniqueness of the Riccati equation.

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