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Émile Borel Versus John Maynard Keynes: The Two Opposing Views on Probability

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Abstract This paper aims to compare French mathematician Émile Borel and British economist John Maynard Keynes with special reference to probability theory. Borel and Keynes were the contemporaries who crossing over the English channel, greatly influenced each other in the twentieth century. In his influential book, Borel (1938) harshly criticized Keynes' position on probability theory. In plain English, Borel regarded probability as a measurable object, thus constituting one important branch of mathematics. In contrast, Keynes thought of probability as a non-measurable item, thereby belonging rather to one branch of logic. Their controversies were rather well-known in the academic world, producing so many papers even after their deaths until today. In hindsight, it seems that differences in their views are very deep-rooted and originated in the critical gulf between the abstract-minded French spirit and the empirical-oriented British tradition. In this paper, I wish to offer a set of fresh angles, thus shedding new light on the French-British probability controversy. The first angle is provided by the rediscovery of Keynes's romantic poem on probability, which can be found at the very end of Keynes's 1921 book but has long been neglected until today. The second angle is given by the reinvestigation of Keynes's original yet almost forgotten concept of "interval-valued probability," and the third angle by the new interpretation of Keynes' strange diagram on non-comparable probabilities. The fourth angle arises from the question of how and to what extent probability is related to non-measurability and ambiguity. There remain so many unsolved problems, waiting for future investigation.

Keywords Émile Borel, John Maynard Keynes, interval-valued probability, ambiguity,
JEL Classification B21・B22・D81・E12 ・

1 The French Spirit Versus the British Spirit: An Introduction

Émile Borel (1871-1956) is an eminent French mathematician who has made great contributions to the fields of probability and hyperbolic geometry. John Maynard Keynes (1883-1946) is a legendary British economist who has also made remarkable achievements to the theory of risk, probability, and uncertainty. It would be safe to say that they were both the contemporaries who, in spite of the existence of the English channel between the European Continent and the British Islands, were in a position to know very well each other's academic works. The purpose of this paper is to compare the works of those two superstars with special reference to the theory of probability. In particular, I would like to offer a set of fresh angles to the famous Borel-Keynes controversy.

It would be a good starting point of my discussion to reexamine the following criticism of Borel against Keynes:

> The above argument may convince us how much the British sprit differs from the French spirit. We should not be overwhelmed by such difference. Nor should we vainly seek to understand what we cannot understand. What we must instead do is that we accept the difference as a matter of fact, thereby making a bridge between the peculiar view of the British people and the unique mind of the French people. It is in this way that we may get a chance to learn how these two spirits are different. The history of sciences can teach us that the cooperation of the two opposing spirits is not only possible but also even leads to many useful results. I myself have exerted all my energy to understand the view of Keynes. As a result, I found the solemn fact that my view was overall far apart from his view. Then by carefully reading his book, I wanted to know more specifically in which way our difference became the greatest.
>
> (Borel. 1938, p. 250)

In his younger days, Keynes had fallen under the influence of the British thinkers such as John Locke, David Hume, G.E. Moore and Bertrand Russell. Since Keynes was a student of King's College at Cambridge University, he spent most of his hours writing a dissertation on probability. After one failure, he eventually received his Fellowship in 1909, but its publication was interrupted by the First World War: in fact, it delayed for a long time until 1921. At the tome when Borel had a chance to read Keynes' final product (1921) *A Treatise on Probability*, Borel was already an established French mathematician with sharp critical eyes and good humor. As was seen in Borel's quoted passage above, he exerted all his energy to understand the unique view of Keynes on probability. However, his effort turned out to be a failure. Borel thought that their different opinions were sprang from a sharp gulf between the abstract-minded French spirit and the empirical-minded British spirit. Those two spirits are fundamentally different and never conform to each other. In his well-known book, Roy Harrod (1951) also referred to such a difference between the British and French spirits:

> In the preface to his *Treatise* Keynes wrote: it may be perceived that Keynes has been much influenced by W.E. Johnson, G.E. Moore and Bertrand Russell, that is to say, by Cambridge, which, with great debts to the writers of Continental Europe, yet continues in direct succession the England tradition of Locke and Berkeley and Hume, of Mill and Sidgwick, who, in spite of their divergences of doctrine, are united in a preference for what is a matter of fact, and have conceived their subject as a branch rather of science than of the creative imagination of prose writers, hoping to be understood.
>
> (Harrod, 1951, Appendix, p. 651)

Keynes seemed to continue in direct succession in the England tradition of empiricism a la Locke, Hume and Mill, so that Keynes placidly thought of probability as a factual branch of logic. In contrast, Borel belonged to the writers of Continental tradition of abstractionism, above all the French tradition of formalism a la Fermat and Pascal, whence Borel rigidly regarded probability as a solid branch of mathematics. Whether probability is reckoned with logic or mathematics is a very important question to ask. Roughly speaking, it is equivalent to ask the question which tradition we should follow, the British or French tradition. Very detailed discussions on this point will be provided throughout this paper.

The contents of this paper are as follows. The second section will newly introduce a set of new angles or perspectives on the Borel-Keynes controversies. Hopefully, it will shed new light on the question of why and how Borel and Keynes took opposing positions on probability and uncertainty. Several final remarks will be provided in the final third section.

2 A Set of New Angles on the Borel-Keynes Controversy

In this paper, I wish to newly offer a set of three fresh angles on the Borel-Keynes controversy. More specifically, the following four angles or perspectives will newly be given for penetrating analyses:

(1) The first angle: the "romantic poetic expression of probability."
(2) The second angle : the flexible concept of "interval-valued probability."
(3) The third angle : the "diagrammatic expression of non-measurable and non-comparable probabilities."
(4) The fourth angle: the psychological concept of "ignorable probability" and "ambiguity."

Let us take a careful look at Table 1. Then we will see the above four angles which are newly introduced by me. Note that for the sake of convenience, the angles are not well-ordered: for instance, "Angle 1" is intentionally placed at the very end of Table 1, and "Angle 2" between Chapters 15 and 16.

2.1 The First Angle: the "Romantic Poetic Expression of Probability"

The first angle I would like to adopt is the one of romantic poetic expression of probability. As the saying goes, let me take a very close look at the following poem composed by Keynes himself :

O False and treacherous Probability,
Enemy of truth, and friend to wickednesse;
With whose bleare eyes Opinion learns to see,
Truth's feeble party here, and barennesse. (Keynes, 1921, p. 466)

Believe or not, this classical-style poem was put at the very end of Keynes's probability book as if it had intentionally been hidden from the main text: consequently, it has long been neglected until quite recently. When I myself discovered its existence, I was greatly shocked like a thunder out of blue. After a short pose, however, I could recover my spirits, gradually beginning to feel that it must be a blessing to my intensive research on Keynes's true concept of probability. [1)]

As is seen in the above poem, Keynes characteristically regarded probability as a false and treacherous thing. He never thought that the concept of probability was based on a solid foundation. When we toss a perfect coin, the probability of finding "one" is 1/6 because "one" is just one of six possibilities conceivable. That is for sure! Keynes thought however, that this represented no more than a mathematical probability, whereas there should exist many other probabilities affected by imperfect and emotional human factors.

Table 1 Keynes, *A Treatise on Probability* (1921) 【Four Angles Added by Sakai 】

PART I	Chapter 1	The Meaning of Probability
Fundamental	Chapter 2	Probability in relation to the Theory of Knowledge
Ideas	Chapter 3	The Measurement of Probabilities *【Angle 3: Keynes' strange diagram, pp. 38-40】*
	Chapter 4	The Principle of Indifference
	Chapter 5	Other Methods of Determining Probabilities
	Chapter 6	The Weight of Arguments *[Angle 4: Ambiguity and Ignorable probability, p. 82.]*
	Chapter 7	Historical Retrospect
	Chapter 8	The Frequency Theory of Probability
	Chapter 9	The Constructive Theory of Part 1 Summarized
PART II	Chapter 10	Introductory
Fundamental	Chapter11	The Theory of Groups, Logical Consistence, Inference, Logical Priority
Ideas	Chapter 12	The Definitions and Axioms of Inference and Probability
	Chapter 13	The Fundamental Theorems of Necessary Inference
	Chapter 14	The Fundamental Theorems of Probable Inference
	Chapter 15	Approximation of Probabilities *【Angle 2: Interval-valued probability , pp. 176-177 】*
	Chapter 16	Observation of the Theorems of Chapter 14 ....
	Chapter 17	Some Problems of Inverse Probability, Including Averages
PART III	Chapter 18	Introduction
Introduction	Chapter 19	The Nature of Argument by Analogy
and Analogy	Chapter 20	The Value of Multiplication of Instances, or Pure Induction
	Chapter 21	The Nature of Inductive Argument Continued
	Chapter 22	The Justification of These Methods
	Chapter 23	Some Historical Notes of Induction
PART IV	Chapter 24	The Meaning of Objective Chance, and of Randomness
Applications	Chapter 25	Some Problems Arising out of the Discussion of Chance
of Probability	Chance 26	The Application of Probability to Conduct
PART V	Chapter 27	The Nature of Statistical Inference
Foundations	Chapter 28	The Law of Great Numbers
of Statistical	Chapter 29	The Use of *à priori* Probabilities for the Production of Statistical Frequency
Inference	Chapter 30	The Determination of Probability *à posteriori*
	Chapter 31	The Inversion of Bernoulli's Theorems
	Chapter 32	The Inductive Use of Statistical Frequencies
	Chapter 33	Outline of a Constructive Theory
BIBLIOGRAPHY, INDEX		*[Angle 1: The book ends with a romantic poem on probability, p. 466]*

It is noted that Keynes was a successor of the traditional British empiricism. Throughout his life, he respected and even admired David Hume (1711-1776) as the founder of philosophy in Britain. Hume was born in Edinburgh, the capital of Scotland, whose unique and traditional Celtic atmosphere was entirely different from an open-minded and lively Southern England. [2).] .

By writing a very influential book, Hume (1739-40) planned to discover the fundamental principles in the nature and diversity of human behaviors. For instance, he remarked:

> It is evident that all the sciences have a relation, greater or less, to human nature; and however wide any of them may seem to run from it, they still return back by one passage or another. Even mathematics, natural philosophy, and natural religion, are in some measure dependent on the science of Man; since they lie under the cognizance of men, and are judged of by their powers and faculties.
>
> (Hume, 1739-40, Introduction, p. xv)

Being impressed by this Hume's strong message, Keynes strongly believed that probability theory must belong to a branch of the science of man. We should think of probability not as a mechanical abstract concept, but rather as a more empirical and emotional concept.

Whereas Hume and Keynes by and large followed the humanistic-pluralistic British tradition of probability, Borel took a different approach to probability theory. Borel characteristically followed the abstract-minded French tradition of probability which originated in the classical works of Pascal and Fermat. It is recalled that being motivated by the question from his friend Mele, a professional gambler, Pascal began to systematically study the games of rolling dice. The mathematics of dice rolling and coin tossing was a starting point of investigation into probability theory, so that Pascal's probabilities should be measurable and comparable. To sum up, Borel simply followed such French tradition of exact and numerical probabilities. .

Keynes as a practical man followed a different path from Borel as an abstract-minded man. Keynes found much more interest in legal trials, thus appeared to be impressed by the principle of "benefit of doubt." No jury can have a perfect information over the guiltiness of a criminal suspect in question. A jury with a limited amount of evidence has to decide his verdict. The less the amount of disadvantageous evidences for a suspect, the more likely is he going to receive a judgment of "not guilty." After all, whether he is guilty or not is a very practical, not mathematical, matter to decide. It is possible that at a later date, a new serious evidence for the criminal act can be found. Then, the probability that the suspect is declared guilty will surely be greater. After all, in Keynes's opinion, the concept of probability is neither objective nor mathematical, but merely subjective and logical, depending on the amount of information available.

2.2 The Second Angle: the Flexible Concept of "Interval-Valued Probability"

We are ready to turn to the second angle which is placed between Chapters 15 and 16 in Table 1. Keynes was not happy to give probability a specific numerical value, but rather adopted a more flexible concept of "interval-valued probability," or "the interval between upper and lower values." In other words, he preferred to discuss "approximation of probabilities" or "inexact numerical comparison." Unfortunately, however, this point has been long ignored by so many researchers until the recent restoration by Brady (2004) and its development by Sakai (2016, 2019a, 2019b). [3)]

Unquestionably, Keynes's probability book (1921) was a well-organized yet very difficult book. Although Borel attempted to comprehend its Part II, its extreme toughness appeared to be too much for him:

> When I [Borel] read Mr. Keynes's new probability book, I found that Part II entitled " Fundamental Theories" was regarded by him as one of his most important contributions to probability theory. I would like to beg forgiveness, however, for my bold decision of completely omitting that part. Under the strong influence of Bertrand Russell, Keynes attempted to develop his ardent argument of mathematics as one branch of logic. I must confess, however, that I have no interest whatever in such a reckless plan. (Borel, 1938, p. 250)

Borel, one of Keynes's contemporaries, did his best to understand Keynes's new probability book. But alas, his effort proved all in vain: he made a wrong decision of omitting Part II, presumably the best part of Keynes's new theory. In recent times, such wrong judgment was correctly pointed out by Brady (2004). On this point, I myself am now in general agreement with Brady, and have published several papers on the basis of the Keynes-Brady theory.

As far as I can see, the best yet neglected part of Part II may be found in the following paragraph Chapter 15:

> The sphere of inexact numerical comparison is not, however, quite so limited. Many probabilities, which are incapable of numerical measurement, can be placed nevertheless *between* numerical limits. And by taking particular non-numerical probabilities as standards a great number of comparisons or approximate measurements become possible. If we can place a probability in an order of magnitude with some standard probability, we can obtain its approximate measure by comparison.
>
> (Keynes, 1921, pp. 176-177)

In hindsight, the critical importance of this paragraph should not be overestimated. I am afraid, however, that even the young Keynes himself was not fully aware of it. For those one hundred years, not only Borel but also almost all scholars around the world did not pay a special attention to it at all. In what follows, let me try to make its true meaning very clear. To put it simply, there are many probabilities which cannot be described by single numerical values. However, as was already mentioned by Keynes (1921), some of those can nevertheless be placed between the two numerical limits. For example, let me consider the "interval-valued probability," or more simply, "interval probability" of the form $[\alpha_2, \alpha_1]$ in which α_1 and α_2 respectively indicate the upper and lower limits of the interval. Given the two interval probabilities, $[\alpha_2, \alpha_1]$ and $[\beta_2, \beta_1]$, we are interested in determining which one is "more probable." Under some possible situations, one may be "more probable" than another. Under other situations, however, no one can be "more probable" than the other: it is impossible to make such a comparison. More generally, as is seen in Fig. 2, we will have to take care of five possibilities of comparisons, that is, ⑴,⑵,⑶.⑷ and ⑸

In Case ⑴, the two intervals, $[\alpha_2, \alpha_1]$ and $[\beta_2, \beta_1]$, are not overlapped at all. Since α_1 is greater than β_1, and α_2 is greater than β_2, it should be quite natural to say that the interval $[\alpha_2, \alpha_1]$ is more probable than the one $[\beta_2, \beta_1]$. In Case ⑵, the two intervals are overlapped. Since we still have $\alpha_1 > \beta_1$ and $\alpha_2 > \beta_2$, it would equally be safe to say that $[\alpha_2, \alpha_1]$ is more probable than $[\beta_2, \beta_1]$. In Cases ⑶ and ⑷, we face the new situations in which either the lower limits or the upper limits are just equal. In Case ⑶ where we have $\alpha_1 > \beta_1$ but $\alpha_2 = \beta_2$, it would still be reasonable to declare that as a whole, $[\alpha_2, \alpha_1]$ is more probable than $[\beta_2, \beta_1]$. In case ⑷ where $\alpha_1 = \beta_1$ but $\alpha_2 < \beta_2$, we may rightly say that $[\beta_2, \beta_1]$ is now more probable than $[\alpha_2, \alpha_1]$. In Case ⑸, we see that one interval completely contains another interval since we have $\alpha_1 > \beta_1$ but $\alpha_2 < \beta_2$. Under such a new situation, it is not possible to say that one interval is more probable than another: indeed, they are no longer comparable.

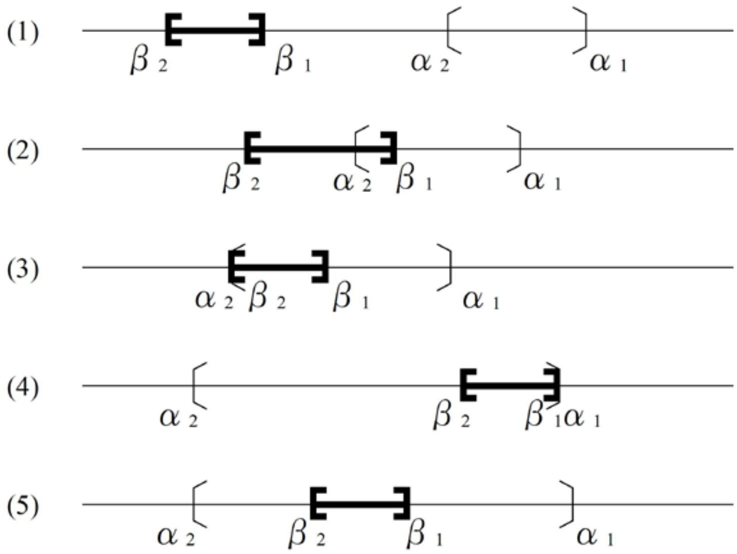
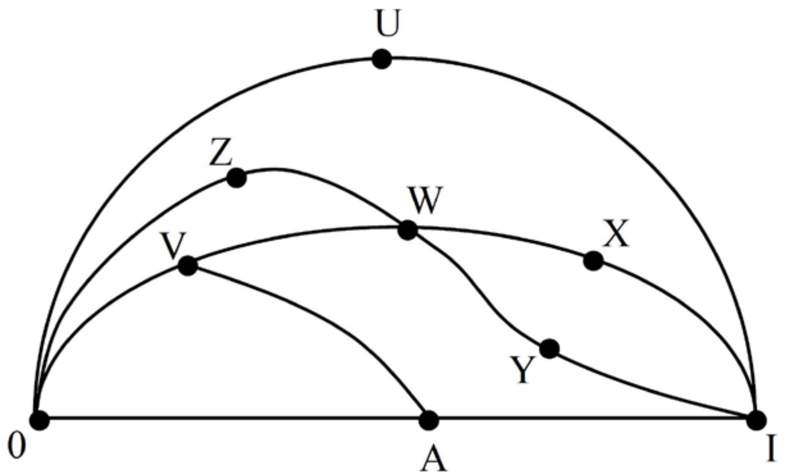


Fig. 1　Interval Probabilities Compared：　Many Possible Cases

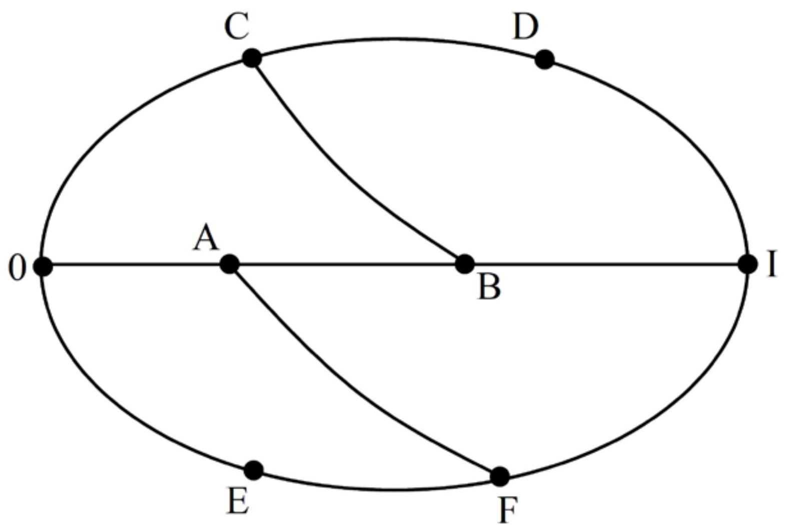
Although the unique idea of interval probability was already introduced by Keynes (1921) , it has by and large been neglected for a long time until Brady (2004).　While on the theoretical level, Brady's restoration was a very brilliant job, its practical applicability has been rather unclear until today.　It is Sakai (2016, 2019a, 2019b) who has dared to　apply Brady's idea to successfully tracing the "Keynes's Charming Chart" and giving an alternative simple solution to the "Daniel Ellsberg Paradox."　This will be shown in the following subsection.

2.3　The Third Angle：　The "Diagrammatic Expression of Non-Measurable and Non-Comparable Probabilities"

As the saying goes, seeing is believing.　Let us take careful looks at the two charming diagrams in Fig. 2.　At the first appearance, they look like the two different kinds of treasure hunting maps.　Of course, they should have much deeper implications. The first diagram (A) was originally drawn by Keynes (1921) himself.　Although since then, so many books and papers have published to discuss Keynes theory of probability, it seems to be quite strange that almost all of them have been contented with the mere



(A)　Keynes's Charming Chart



(B)　Borel's Charming Chart

Fig. 2　The Two Charming Charts:　Keynes Versus Borel

reproduction of Chart (A), with no intention whatever to go beyond into an unexplored virgin soil.[4]

In order to break through the hard crust of convention, I published a pair of papers (2016, 2018a) in which I attempted to give numerical examples for all the points in Chart (A) above:

Point O = [0, 0], Point A = [0.6, 0.6], Point I = [1, 1],
Point U = [0, 1], Point V = [0.3, 0.4], Point W = [0.45, 0.65],
Point X = [0.5, 0.8], Point Y = [0.5, 0.9], Point Z = [0.1, 0.65].

If we adopt the ordering rule of "more probable to", then we can or cannot order those nine points rather easily. For instance, X is more probable than W, which is in turn more probable than V, which also in turn more probable than O. Y is more probable than W, which is more probable than Z, which is more probable than O. However, X and Y are not comparable, nor are V and Z.

It was Keynes's outstanding contribution to indicate that the ordering of probabilities based on interval intervals must not be total, but merely partial. It seemed that Borel as one of Keynes's rivals did not like the flexible idea of "interval probabilities," thus focusing only on the rigid concept of "point probabilities." Besides, Borel did not stop walking here, and even went further toward criticizing Keynes. Such twisted mind of Borel could clearly be seen in Chart (B) since it was sarcastically drawn by Borel himself. .

While the standard straight line OAI, which connects Points O and I, is placed on the very basis of Keynes's Charming Chart, the corresponding standard line OABI is now located on the center of Borel's Charming Chart. How different are the two Charts ! I guess that this would show Borel's sarcastic stance on Keynes.

I am now ready to provide numerical illustrations for all the points in Borel's Charming Chart:

Point O = [0, 0], Point A = [0.3, 0.3], Point B = [0.5, 0.5], Point I = [1, 1],
Point C = [0.05, 0.65], Point D = [0.15, 0.85],
Point E = [0.2, 0.4], Point F = [0.4, 0.8] .

. As can easily be seen, I is more probable than D, which is more probable than C, which is more probable than O. Besides, I is more probable than F, which is more probable than E, which is more probable than O. While B is more probable than C, F is more probable than A. However, C and E are not comparable, nor are D and F.

To sum up, I believe that those comparisons may clearly show the effectiveness of the concept of interval probabilities. Surely, comparison of interval probabilities is more powerful than comparison of point probabilities.

2.4 The Fourth Angle: the Psychological Concept of "Ambiguity" and "Ignorable Probability"

In the history of economic thought, the economics of risk has an old history, dating back to the expected utility theory of Daniel Bernoulli (1700-1782). Remarkably, Bernoulli was twenty years senior to Adam Smith (1723-1790), well-known as the Father of Economics.

Bernoulli was originally born as a celebrated son of the Bernoulli family of mathematical geniuses in Switzerland. When he was invited as a professor of pure and applied mathematics at the newly established St. Petersburg Academy in Russia, he found much interest in the economics of gambling as one branch of general decision making under risk. More specifically, he considered the following game of coin tossing. Any perfect coin has two sides, namely "Head" and "Tail." Let a man named Peter pay 1 million yen to obtain the right to toss a series of coins until "Head" appears on stage. Peter is promised to obtain 2 thousand yen when "Head" appears at the first trial, 4 thousand yen at the second trial, 8 thousand yen at the third trial, and generally,. 2^N thousand yen at the N-th trial. The question of interest is whether or not Peter is willing to participate in such coin tossing game.

Let us simply obey the expected pay-off rule. Then the amount of expected pay-off of the game G is provided as follows:

$$\begin{aligned} E(G) &= (1/2)(2) + (1/4)(4) + (1/8)(8) + \dots + (1/2^N)(2^N)+\dots \\ &= 1 + 1 + 1 + \dots + 1 + \dots = +\infty \qquad (1) \end{aligned}$$

Apparently, this amount is greater than 1 million yen or the expensive admission fee. Consequently, we conclude that Peter should participate in the game. Such a conclusion, however, would apparently contradict everyman's common sense.

Is there a good device by which we can escape from the paradox ? It is Bernoulli who has boldly introduced the new principle of expected utility. In order to show the effectiveness of the new rule, let U(x) be the utility function of x, Peter's income. Then, when Peter participates in the game G, the amount of expected utility he can obtain is given as follows:

$$EU(G) = (1/2)U(2) + (1/4)U(4) + (1/8)U(8) + \dots + (1/2^N)U(2^N) +\dots \qquad (2)$$

Let us specify U(x) = log x. Then Eq. (2) can be transformed to the following equation:

$$
\begin{aligned}
EU(G) &= (1/2)(\log 2) + (1/4)(\log 4) + (1/8)(\log 8) + \ldots + (1/2^N)(\log 2^N) + .. \\
&= (1/2 + 2/4 + 3/8 + \ldots + N/2^N + \ldots) \log 2 \\
&= 2 \log 2 \quad = 4 \qquad (3)
\end{aligned}
$$

Note that EU(G) is a finite value of 4, being definitely less than 1 million yen. Therefore, we come to the reasonable conclusion that we should not pay a very expensive admission fee for the coin tossing game. This sounds very reasonable, so that the St. Petersburg paradox is now nicely resolved !

Having made careful preparations so far, we are now ready to thoroughly discuss the fourth angle of psychological connections. Looking back to Fig. 1, we can see that this last angle is placed between Chapters 6 and 7. Remarkably, Keynes (1921) once wrote:

> The typical case, in which there may be a *practical* connection between weight and probable error, may be illustrated by the two cases following of balls drawn from an urn. In each case we require the probability of a white ball; in the first case we know that the urn contains black and white in equal proportions; in the second case the proportion of ach color is unknown, and each ball is as likely to be black as white. It is evident the in either case the probability of drawing a white ball is 1/2, but the weight of the argument in favor of this conclusion is greater in the first case.
>
> (Keynes, 1921, p. 75)

The question at issue can be well-illustrated in Fig. 3. At present, it represents the problem of decision making under ambiguity, which has developed very well by Ellsberg (1961, 1962, 2001) and several others since then. Keynes himself has given his own interpretation by which the "weight of argument" in favor of drawing W is greater in Case Ⅰ than Case Ⅱ. Alternatively, we could say that people prefer the numerically clear case Ⅰ to the vague case Ⅱ because they indicate "ambiguity aversion."

It should be noted that the two-color urn example clearly tells us that the expected utility rule of Daniel Bernoulli is no longer valid. To show this, let us calculate and compare the expected utilities of Cases Ⅰ and Ⅱ. Then, we will immediately obtain the following results:

Let us draw balls (White W or Black B) from an urn. We are then required to find the probability of drawing W :.

Case (Ⅰ) (Bet on W) We know that each urn contains W and B in equal proportions:

TOTAL 60 = [W 30 , B 30]

Case (Ⅱ) (Bet on W) The proportion of each color is unknown:

TOTAL 60 = [W ? , B ?]

Fig. 3 Keynes on the urn problem: decision making under ambiguity

$$EU(\text{Case Ⅰ}) = EU(\text{Case Ⅱ}) = (1/2)U(W) + (1/2)U(B) \tag{4}$$

It is recalled that the indifference principle of Keynes (1921) is adopted here, meaning that the unknown proportions of W and B can positively be interpreted as just equal proportions, namely half-and-half. Therefore, if we employ the expected utility rule, Case Ⅰ and Case Ⅱ should give us exactly the same amount of utility. Clearly, this contradicts people's attitude toward ambiguity aversion. In other words, if we stick to the common sense against ambiguity, we have to abandon the existing expected utility theory.

Keynes' attitude against mere mathematical expectations becomes stronger in his main work *General Theory* (1936). In fact, he strongly remarked:

> Human decision affecting the future whether personal or political or economic, cannot depend on strict mathematical expectation, since the basis for making such calculations does not exist; and that is our innate urge to activity which makes the wheels go around, our rational selves choosing between the alternatives as best we are able, calculating where we can, but often falling back for our motive on whim or sentiment or chance. (Keynes 1936, pp. 162-163)

We are now entering the world of uncertainty in which the basis for making mathematical expectations does not exist, and often falling back for our motive on whim, sentiment and pure chance. In a sense, this is the new world that is called by Keynes "animal sprits" or people's spontaneous urge to action rather than inaction. It is in this way that Keynes has departed from mathematical calculations, arguing against the naive application of the idea of expected utility to economic theory. [5)]

In contrast to Keynes, the Frenchman Borel has taken a more conservative stance, relying more heavily on mathematical calculations. While Keynes followed the British tradition of empirical philosophy and multilayer humanistic view, Borel was apparently the faithful follower of the French tradition of abstract philosophy and single-minded mechanical view. Although both scholars were fond of pure and applied mathematics, Keynes took a more humanistic and quality-oriented approach, but Borel a more mechanical and quantitative approach. On appearance, the difference between the two characters seems to be small enough to find a reconciliation. On deeper psychology and culture, however, it is so wide distance apart that the two views has long been irreconcilable. The gulf between the British and French tradition looks so large like a real distancing channel between the British islands and the Continent ! .

Let me reexamine Borel's position on the St. Petersberg paradox in details. While in a popular book, Borel (1938) spared so many pages to scrutinize this famous paradox, it appeared that he intentionally kept some distance from the excitement of the paradox bustling. In fact, he took a very careful look at the Bernoulli calculation with coolheaded eyes. Looking far back to Eq. (2), we can easily see a series of troublesome calculations such as the fraction $1/2^{N}$ for N = 2,4,8, 16, 32, 64, 128, 256, 512, 1024, ... , noting that N will continue forever! Unquestionably, N is likely to exceed one million, one billion, or even one trillion. According to Borel's common sense and conscience, such extra big numbers would not belong to the limited field of human capabilities, but rather the unlimited world of Almighty God. In this respect, Borel once remarked:

> It is only the Almighty God who can carry out such extremely complex calculations dealing with 10^{300000} numbers and thus thoroughly enjoy the St. Petersburg coin-tossing game. Besides, we would need to suppose one more condition under which His opponent is also another Almighty God. (Borel 1938, p. 118)

It is noted that 10^{6} or the 6th power of ten is one million, a very big number. Characteristically, Borel would not stop here, but go further and further to the 300000th power of ten. In my opinion, this would possibly demonstrate his sarcastic way of putting his own remark.

Apart from such a reference to "Almighty God," I think that there should be least one good point worthy of serious discussion. This is Borel's original idea of "ignorable probabilities" in human measures, world measures, space measures and so on.

In what follows, I would like to focus on the most important concept of "ignorable probabilities in human measures" and its applicability to decision making under risk. For that purpose, let me consider the following passage of Borel (1955):

> Let us consider the peace time statistics in Paris where several million residents live. In a big city like Paris, the critical traffic accidents caused by trains, street cars and the like, on average, occur by the rate of "one accident in a single day." This implies that the probability of the accident by which a Parisian using a civil traffic loses his life in one day could safely be estimated as "one millionth." Suppose that in order to avoid such a very small risk, a man gives up his working activities and decides to stay at home all day long. Then, he will perhaps be ridiculed as "a fool."
>
> (Borel, 1955, pp. 54-55)

Borel considered the question of whether or not a resident in Paris should stay at home to avoid the severe traffic accident with its mortality being one millionth. If a Parisian decides to stay home, he would be extremely timid, thus even being labeled as a fool. According to Borel, the probability of one millionth is very small, so that it should be regarded as an "ignorable probability."

Now, let me try to rewrite Eq. (1) in much greater detail so that we may have more than twenty long terms:

$$
\begin{aligned}
E(G) = {} & (1/2)(2) + (1/4)(4) + (1/8)(8) + (1/16)(16) + (1/32)(32) + \\
& (1/64)(64) + (1/128)(128) + (1/256)(256) + (1/512)(512) + (1/1024)(1024) + \\
& (1/2048)(2048) + (1/4096)(4096) + (1/8192)(8192) + (1/16384)(16384) + (1/32768)(32768) + \\
& (1/65536)(65536) + (1/131072)(131072) + (1/262144)(262144) + (1/524288)(524288) + \\
& (1/1048376)(1048376) + (1/10^{21})(10^{21}) + \dots\dots\dots + (1/2^N)(2^N) + \dots\dots \qquad (5)
\end{aligned}
$$

It is noted here that the 20th power of 10 is 1,048,376, already exceeding one million. So, if we follow Borel's principle of ignorable probability in human measures, all the terms following after the 20th may be regarded as ignorable or zero. For this "adjusted game AG," the "adjusted expected value" E(AG) may be calculated as follows:

$$
\begin{aligned}
E(AG) &= 1 + 1 + \dots + 1 \text{ (all one until 20th term)} + 0 + 0\dots \text{ (all zero hereafter)} \\
&= 20\,! \qquad (6)
\end{aligned}
$$

The new type of question to ask now is whether or not we should participate in the adjusted coin- tossing game with ignorable probabilities. The game participating fee, or simply the admission fee, is as expensive as 1 million yen, which is far greater than 20 thousand yen, or the amount of the expected adjusted prize. As a result, we should by all means engage in the game ! So, our common sense is reasonably recovered. Unfortunately, however, Borel himself did not know such powerful recovery of common sense. I must say that Borel's idea of "ignorable probability in human measures" is still alive, and indeed very much alive even today.

3 The French and British Spirits Revisited : Final Remarks

In the above, I have carefully compared Ēmile Borel and John Maynard Keynes with special reference to probability theory. Borel was a representative French mathematician in the sense that he regarded probability as a mathematical concept, thus emphasizing a quantitative and even axiomatic structure of probability theory. In contrast, Keynes was a typical British mathematician/economist in the sense that he thought of probability as a philosophical/ logical concept, thereby adopting a qualitative and subjective approach to probability theory. It is recalled that the young Keynes was very fond of the following passage: "O false and treacherous probability, enemy of truth, and friend to wickedness." According to Keynes, probability is a double-edged sword, possessing both sunny and shady sides. Surely, it may contain a piece of truth. Whether it is a big piece or a small piece, no one knows. However, it might be an enemy of truth, betraying a honest observer. [6)]

It was the high-spirited Borel who greatly disliked such kind of indecisiveness or vagueness. While there has been a seemingly unbridgeable discrepancy between Borel and Keynes, I would firmly believe that these two approaches are not substitutive but rather complementary. In short, they could be good rivals, not bad enemies !

To sum up, I can say that there have been two competitive schools of probability, namely the French School including Borel, and the British School containing Keynes. It would a very good question to ask the reason why the aforementioned differences between the two schools have emerged. I am inclined to think that they are deeply rooted, probably far deeply rooted in cultural and ethnic discrepancies between the two nations. In this respect, I would like to refer to the following remarks made by Hiroo Mita, a mathematician and a Japanese translator of Borel (1942):

> It is often said that Differential and Integral Calculus was established by Newton and Leibnitz. Most Frenchmen, however, have objections against such vulgar opinion. They would instead think that even before Newton and Leibnitz, the main body of Calculus was already finished by the two great French mathematicians, Fermat and Pascal. ...... Undoubtedly, Calculus and Probability represent one great tradition of French mathematics. Émile Borel is surely one of the greatest mathematicians the French tradition has ever produced. (Mita, 1942, p. 307)

Mita's remark was made a long time ago. I would strongly believe, however, that the remark is of the utmost importance and very much alive even today. As the saying goes, life is short but art is long! And science is very, very long!

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Footnotes

1) For details, see Sakai (2016), p. 4. In retropect, Keynes's romantic poem may be deeply rooted in the following passage of John Locke (1632-1704). "Probability is to us the guide of life since in the greatest part of our concernment, God has afforded only the Twilight of Probability to that state of Mediocrity and Probationership He has been pleased to place us in here." This impressive passage was fondly quoted by Keynes (1921), p. 323. We can see how much Keynes's philosophy was influenced by Locke, well-known as the "Farther of Liberalism".

2) Hume's influential book *A Treatise on Human Nature* was published in the period 1739-1740. Perhaps, being impressed by the compactness of the book, Keynes's highly-motivated book (1921) appeared to have a similar title: *A Treatise on Probability.* It is noted that around 180 years had passed between those two books. As the saying goes, life is short but art is long !

3) Also see Brady & Arthmar (2012).

4) For instance, Gillies (2000) was contented to merely reproduce Keynes's charming chart without any attempt to give numerical explorations.

5) In his follow-up paper, Keynes (1937) made the following remark on uncertainty :

"By 'uncertain' knowledge, let me explain, I do not explain merely to distinguish what is known for certain from what is only possible. The game of roulette is not subject, in this sense, to uncertainty, nor is the prospect of a Victory bond being drawn. ... The sense in which I am using term is that in which the prospect of a European war is uncertain, or the price of copper and the rate of interest twenty years later hence... . About these matters, there is no scientific basis on which to form any calculable probability whatever. We simply do not know. " (Keynes 1937, pp. 209-223) Thus, Keynes's argument in favor of uncertainty and animal spirits has preserved until his death. In my opinion, Keynes without uncertainty looks like Hamlet without prince. Also see Braithwaite (1973) and Skidelsky (1997).

6) John R. Hicks was one of the greatest economists after the death of John. Maynard Keynes. Remarkably, Hicks has supported Keynes's position on probability, arguing the following: " Economics is a leading example of uncertain; it is knowledge, yet it is evidently uncertain," (Hicks, 1979, p. 2) It is also noted that Hicks developed his own idea on "interval probability." For details, see Sakai (2016).